Corollary
For each λ, $H^{2d_\lambda}(Fl_n^\lambda)$ is an irrep of S_n and all the irreps arise this way.

Proof
$\mathbb{C}[S_n] = \operatorname{End}(\bigoplus_{\lambda} H^{2d_\lambda}(Fl_n^\lambda) \otimes IC_{\mathcal{O}_\lambda})$

$= \bigoplus_\lambda \operatorname{End}_{\mathbb{C}}(H^{2d_\lambda}(Fl_n^\lambda))$ ← these are distinct simple objects. $\operatorname{End}(IC_{\mathcal{O}_\lambda}) = \mathbb{C}$

Thus the map $\mathbb{C}[S_n] \to \operatorname{End}_{\mathbb{C}}(H^{2d_\lambda}(Fl_n^\lambda))$ is surjective $\Rightarrow$ irrep. □

Actually, we want to show that $H^{2d_\lambda}(Fl_n^\lambda)$ is the irrep associated to λ. Not sure how to do this -- later using geometric Satake.

We can also use sheaf theory to give an alternate proof of the iso. $H(Z) \cong \mathbb{C}S_n$

First consider $\tilde{g} = \{(X, V_\bullet) : X \in M_n, V_\bullet \in Fl_n, XV_i \subseteq V_i\}$

We have $f: \tilde{g} \to M_n$

$\tilde{g} \supseteq \tilde{g}^{rs} \to M_n^{rs} \subseteq M_n$ — diagonalizable with distinct eigenvalues; $\tilde{g}^{rs} \to M_n^{rs}$ an $n!$-1 covering.

$f_{\tilde{g}}$ is semismall, in fact small, so $f_* \mathbb{C}_{\tilde{g}}[\,] = IC_{M_n, L}$

where L is just the local system $\mathbb{C}[S_n]$ coming from the $n!$ to 1 cover.

Now $f_* \mathbb{C}_{\tilde{N}}[\,] = i^* IC_{M_n, L}[\,]$ by base change

$i: N \hookrightarrow M_n$

We have an action of S_n on $f_* \underline{\mathbb{C}}_{\tilde{g}}$
and so on $f_* \underline{\mathbb{C}}_{\tilde{N}}$

categorical framework for Springer theory.

We have $\mathbb{C}[S_n] \to \mathrm{End}(f_* \underline{\mathbb{C}}_{\tilde{N}})$.

We define a functor:

$$\mathrm{Perv}_{GL_n}(\mathcal{N}) \longrightarrow \mathrm{Rep}(S_n)$$

category of
perverse sheaves
on $\mathcal{N}$ constructible
w.r.t. the strat
by GL_n-orbits

this functor is an equivalence
($\exists$ a modular version as well)

Affine Grassmannian

Now we will study

$$Gr = \{\mathcal{O}\text{-lattices in } K^n\} \qquad K = \mathbb{C}((z)),\ \mathcal{O} = \mathbb{C}[[z]].$$
$$= \{ L \subset K^n : L = \mathrm{span}_{\mathcal{O}}(v_1, \dots, v_n) \text{ s.t. } v_1, \dots, v_n \text{ is a } K\text{-basis for } K^n \}$$

Gr is analogous to $\{\mathbb{Z}\text{-lattices in } \mathbb{R}^2\}$

We have $GL_n(K) \circlearrowright Gr$ transitively
and $Gr = GL_n(K)/GL_n(\mathcal{O})$.

Gr has the structure of an "ind-scheme".

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, get $L_\lambda = \mathrm{span}_{\mathcal{O}}(z^{\lambda_1}e_1, \dots, z^{\lambda_n}e_n)$

Let $Gr^\lambda = GL_n(\mathcal{O}) L_\lambda$

Prop $Gr = \bigcup_{\lambda \in \mathbb{Z}^n_+} Gr^\lambda$ **Proof** row and column operations.

if $\lambda_1, \dots, \lambda_n \geq 0$, then $L_\lambda \subset L_0 = \mathcal{O}^n$

and $z|_{L_0/L_\lambda}$ is a nilpotent operator of Jordan type λ.

Lemma

$Gr^\lambda = \{ L_0 \supset L : z|_{L_0/L}$ is a nilpo— $\lambda \}$

Proof

If $L \in Gr^\lambda$, then $L \subset L_0$ and get Jordan type $\}$
since have an iso $L_0/L \to L_0/L_\lambda$

if $z|_{L_0/L}$ has type λ, then $L \in Gr^\mu$ for some μ
since $L \subset L_0$ $\mu_i \geq 0$ for all i $\square$.

Note that if $L \in Gr^\lambda$ then $L_0 \supset L \supset z^{\lambda_1} L_0$
(since for all $v \in L_0$, $z^{\lambda_1} v \in L$).

So L can be thought of as a point
in a Grassmannian.

$\Rightarrow$ Gr^λ is a f.d. quasi-projective variety.

For any λ

$\overline{Gr^\lambda} = \bigcup_{\mu \leq \lambda} Gr^\mu$ where $\mu \leq \lambda$ means
$\lambda - \mu \in \{\sum_{i=1}^{n-1} p_i \alpha_i : p_i \in \mathbb{N}\}$.

Cor.

$\overline{Gr^\lambda}$ is a finite dimensional projective variety.

Gr has connected components labelled by integers.

For each $k \in \mathbb{Z}$

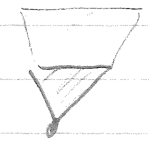
$Gr(k) = \bigcup_{\substack{\lambda_1 + \dots + \lambda_n = k \\ \lambda \in \mathbb{Z}^n_+}} \overline{Gr^{(\lambda_1, \dots, \lambda_n)}}$

Eg

GL_2, $k=0$

$Gr^{(0,0)} = pt$

$\overline{Gr^{(1,-1)}}$ = weighted $\mathbb{P}^2$



$\overline{Gr^{(2,-2)}}$ = 4-dim.

Prop

$$\dim \overline{Gr^\lambda} = (n-1)\lambda_1 + (n-3)\lambda_2 + \dots + (1-n)\lambda_n$$

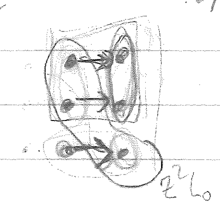
Proof

$Gr^\lambda \supset GL_n L_\lambda \cong GL_n/P_\lambda$ and we have a map $Gr^\lambda \to GL_n/P_\lambda$ which is a vector bundle.

eg

$\lambda = (2,0,0)$

$\{L \subset L_0 : z|_{L_0/L}$ has Jordan type $(2,0,0)\}$



$L \subset \mathbb{C}^6 = L_0/z^2L_0$

$\dim L = 4$

s.t. $z|_{L_0/L}$ has Jordan type $(2,0,0)$

$Gr^{(2,0,0)} \to$

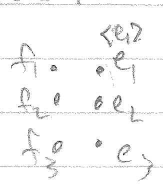
$$L \mapsto L \cap zL_0/z^2L_0 \subset zL_0/z^2L_0$$

2-dim.

get any 2-dim subspace this way.

$Gr^{(2,0,0)} \to \mathbb{P}^2$

fibre is $\mathbb{C}^2$



$$\langle f_1 + ae_2 + be_3 \rangle$$

□

Recall $Gr =$ $\mathcal{O}$-lattices in K^n

$\lambda \in \mathbb{Z}^n$ have $L_\lambda = \langle z^{\lambda_1} e_1, \ldots, z^{\lambda_n} e_n \rangle$

$Gr^\lambda = GL_n(\mathcal{O}) L_\lambda \quad \lambda \in \mathbb{Z}^n_+$.

If $\lambda_n \geq 0$, $Gr^\lambda = \{ L \subset L_0 : z|_{L_0/L}$ has Jordan type $\lambda \}$.

get Gr^λ locally closed in some Grassmannian.

eg $\lambda = \omega_k = (1, \ldots, 1, 0, \ldots 0)$ then $Gr^\lambda = G(k,n)$ and is closed

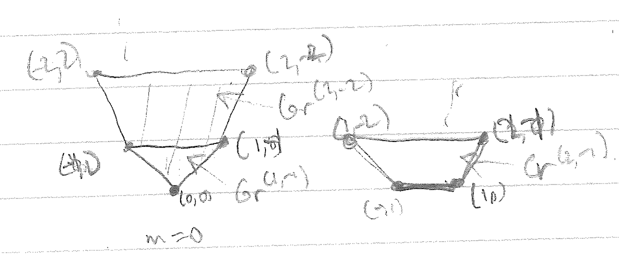
we also saw that $\overline{Gr^\lambda} = \bigcup_{\mu \leq \lambda} Gr^\mu$

recall $\mu \leq \lambda \Rightarrow \sum \mu_i = \sum \lambda_i$.

In fact Gr breaks into connected components $Gr(m) = \bigcup_{\lambda_1 + \cdots + \lambda_m = n} Gr^\lambda$

$Gr(m) = \{ L : \dim L_0/L \cap L_0 - \dim L/L \cap L_0 = m \} = \{ [g] : \operatorname{val}(\det g) = m \}$ $g \in GL_n(K)$

all connected components are isomorphic $L \mapsto \begin{bmatrix} z & 0 \\ & \ddots \end{bmatrix} L$



$|\lambda| = 2$

$z^{\lambda_i} e_i \;\text{---}\; z^{\lambda_i - k} e_i$

$z^{\lambda_j} e_j \;\text{---}\; z^{\lambda_j + k} e_j$

$L_\lambda \;\text{---}\; L_\mu$

$\mathbb{P}$-line $\mathbb{P}^1$

iff $\lambda - \mu = k\alpha$.

for some $\alpha = (0 \cdots 1 \, 0 \cdots -1 \cdots 0)$, $k \in \mathbb{N}$.

Prop

$\dim \overline{Gr^\lambda} = (n-1)\lambda_1 + \cdots + (1-n)\lambda_n$

$\dim = 1$

$L \cap z^{-1} L_0 / L_0$

$L_0 \subset z^{-1} L_0 \subset z^{-2} L_0$

Proof

eg $\lambda = (2, 0, -2)$

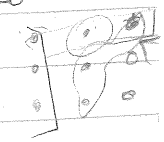
We have a map $Gr^\lambda \to GL_n/P_\lambda \leftarrow$ partial flag variety

The fibres are vector bundles

$L \subset L_0 \qquad L_0 \subset L \qquad L \subset z^{-m} L_0$

$z^k L_0 \subset \cdots \subset z L_0 \subset L_0 \qquad L_0 \subset z^{-1} L_0 \subset z^{-2} L_0 \subset \cdots$

$L \cap z^k L_0 / L \cap z^{-(k-1)} L_0 \subset z^k L_0$



π:

L_0

L

$Gr(\mathcal{O}) \rightsquigarrow B$

$Gr^\lambda \leftrightsquigarrow X_W$

$Gr \leftrightsquigarrow Fl_n$

analogy $Gr \leftrightsquigarrow \mathcal{N}$

in fact, we can consider

Proposition

$$\left\{ L \subset L_0 : [e_1], \dots, [e_n] \text{ forms a basis of } L_0/L \right\} \xleftrightarrow{\sim} \mathcal{N}$$

Proof

$$L \longmapsto \left[z|_{L_0/L} \right]_{[e_1] \dots [e_n]}$$

$$\mathrm{Span}(z e_1 + X(e_1), \dots) \longleftarrow X$$

then $z[e_1] = [z e_1]$
$= [X(e_1)]$

Moreover: $Gr^\lambda \cap \{L \subset L_0 \dots\} = \mathcal{O}_\lambda$

We define
$d : Gr \times Gr \to X_+$
by $d(L_1, L_2) = \lambda$ if $L_2 \subset L_1$ and $z|_{L_1/L_2}$ has type λ
$d(L_1, zL_2) = d(L_1, L_2) - (1, \dots, 1)$

• parametrization of $GL_n(K)$ orbits on Gr

first

relative position of two lattices

analog of Bott-Samelson

T^*Fl_n

Given a sequence $\underline{\lambda} = \lambda_1, \dots, \lambda_m \in X_+$ we define

$$Gr^{\underline{\lambda}} = \{ (L_0, L_1, \dots, L_m) : d(L_i, L_{i+1}) = \lambda_i \}$$

often we will use $\lambda_i = \omega_{k_i}$

We have $m_{\underline{\lambda}} : Gr^{\underline{\lambda}} \to Gr$

eg $n=2$ $\underline{\lambda} = (\omega_1, \omega_1, \omega_1, \omega_1)$

get

$$Gr^{\underline{\lambda}} = \{ L_0 \supset L_1 \supset \dots \supset L_4 : \dim L_i/L_{i+1} = 1 \} \longrightarrow Gr$$

$$L_\circ \longmapsto L_4$$

So. $m_{\underline{\lambda}}^{-1}(L) = \{ L_0 \supset L_1 \supset \dots \supset L_4 = L : \cancel{z} L_i \subset L_i \}$

$= \text{Springer fibre for } z|_{L_0/L}$

eg $L = z^2 L_0 = L_{(2,2)}$

So $m_{\underline{\lambda}}^{-1}(L_{(2,2)}) = (2,2)$ Springer fibre

Prop

More generally if $\underline{\lambda} = (\underbrace{\omega_1, \dots, \omega_1}_{m})$

Then we have

$$m_{\underline{\lambda}} : Gr^{\underline{\lambda}} = \{ L_0 \supset L_1 \supset \dots \supset L_m \} \to Gr$$

and for any $\mu = (\mu_1, \mu_n)$

$m_{\underline{\lambda}}^{-1}(L_\mu) = Fl_m^X$ where X is a nilpotent of type μ and $m = \Sigma \mu_i$

if $\underline{\lambda} = (\omega_{k_1}, \dots, \omega_{k_m})$

then $m_{\underline{\lambda}}^{-1}(L_\mu) \cong \{ 0 \subseteq V_1 \subseteq \dots \subseteq V_m = \mathbb{C}^N : \dim V_i = \dim V_{i-1} + k_i, \; X V_i \subset V_{i-1} \}$

where X is a nilpotent of type μ.

$\underline{\lambda} = (\underbrace{\omega_1, \dots, \omega_1}_{n})$

~~Prop~~

We have $m_{\underline{\lambda}}^{-1}\left(\{ L \subset L_0 : (z) \text{ is a } \text{Fr: } \text{basis } (\text{of } L_0/L) \} \right) \cong T^* Fl_n$

$$\downarrow \qquad\qquad \downarrow$$

$$\{ \quad \} \cong \mathcal{N}$$

Theorem

$$Gr \cong \{(E, \varphi) : E \text{ is a vector bundle on } \mathbb{P}^1,\ \varphi : E|_{\mathbb{P}^1 \setminus 0} \longrightarrow E_0|_{\mathbb{P}^1 \setminus 0} \text{ is a triv away from } 0\}$$

Proof

Given (E, φ) we let $L = \{ s \in \underbrace{\Gamma(\mathbb{P}^1 \setminus \{0,\infty\}, E_0)}_{= \mathbb{C}[z, z^{-1}]^n} : \varphi^*(s) \text{ extends to a section over } 0 \}$

i.e. we get $L = \Gamma(\mathbb{P}^1 \setminus \infty, E)$ which is embedded into $\Gamma(\mathbb{P}^1 \setminus \{0,\infty\}, E_0)$.

Then L is a $\mathbb{C}[z]$ $\underset{\Updownarrow \text{ invertible}}{\text{lattice}}$ in $\mathbb{C}[z, z^{-1}]^n$

$\rightsquigarrow L \otimes_{\mathbb{C}[z]} \mathcal{O}$ is an $\mathcal{O}$-lattice in $\mathbb{C}((z))^n$ $\square$

From this perspective, $L_\lambda = \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i \cdot 0)$ with canonical trivialization

where $\mathcal{O}(n \cdot 0)$ = rational functions allowed to have a pole of order n at 0.

$$Gr(m) = \{(E, \varphi) : \deg(E) = -m\}$$

$$\{L_0 \subset L : [e_1], \dots, [e_n] \text{ is a basis}\} = \overline{Gr^{(n,0,\dots,0)}} \cap \{(E, \varphi) : E \cong \mathcal{O}(-1)^{\oplus n}\}$$

$\overline{Gr^{(n,0,\dots,0)}} = \{(E, \varphi) :$ $\deg(E) = n$ and the trivialization $E \dashrightarrow E_0$ extends to an inclusion of coherent sheaves $E \subset E_0 \}$

$$\lambda_i \geq 0 \quad Gr^\lambda = \{(E, \varphi) : E \subset E_0 \text{ and } E_0/E \cong \bigoplus_i \mathbb{C}[z]/z^{\lambda_i}\}$$

Last time I described $Gr^\lambda \to$ partial flag variety.
Simpler description:

a $\mathbb{Z}$-flag in $\mathbb{C}^n$ is a sequence $0 \dots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \dots \mathbb{C}^n$
eventually 0 eventually $\mathbb{C}^n$

$Fl_{\mathbb{Z}}(\mathbb{C}^n) = \bigsqcup_{\lambda \in \mathbb{Z}} Fl_\lambda(\mathbb{C}^n)$

where $Fl_\lambda(\mathbb{C}^n) = \{V_\bullet : \dim V_j - \dim V_{j-1} = \text{mult of } j \text{ in } \lambda\}$

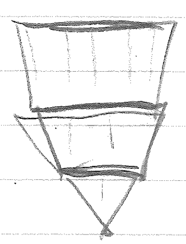
eg $Fl_0(\mathbb{C}^n) = \dots 0 = V_{-1} \subset V_0 = \mathbb{C}^n \dots$

$Fl_{(3,2,1)}(\mathbb{C}^3) = \quad 0 = V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3$

So $Fl_\lambda(\mathbb{C}^n)$ is a partial flag variety.

Define $\pi : Gr \to Fl_{\mathbb{Z}}(\mathbb{C}^n)$
by $L \mapsto \{ V_j = \{ v \in \mathbb{C}^n : z^j v \in L \} \}$
(π is not a morphism).
we get $\pi : Gr^\lambda \to Fl_\lambda(\mathbb{C}^n)$ a vector bundle.

We have $i : Fl_{\mathbb{Z}}(\mathbb{C}^n) \to Gr$
by $V_\bullet \longmapsto \bigoplus_{k \in \mathbb{Z}} V_k z^k$



In fact $\mathbb{C}^\times \curvearrowright Gr$ by "loop rotation"
and $i(Fl_{\mathbb{Z}}(\mathbb{C}^n))$ is the set of fixed p