If $m(w)_{ij} \geq m(v)_{ij} \quad \forall i,j$

then $v \leq w$

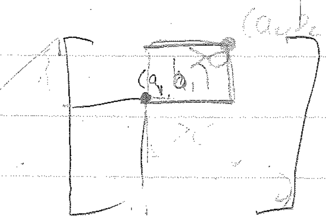
i.e. $\exists$ a sequence of trans $t_1 \ldots t_r$

s.t. $v t_1 \ldots t_r = w$

w/ length increasing each time.

$M_{ij} = m(w)_{ij} - m(v)_{ij} \geq 0$

choose a_1, b_1 s.t. $M_{a_1 b_1} > 0$ and



then $w(a_1) = b_1 \quad v(a_1) < b_1$

then take so a_0 and (a_1, a_0)

to reduce the matrix

Define:

$v \leq w$ if $\exists$ a chain of transposition $t_1 \ldots t_n$

s.t. ~~$l(v) \leq l(w)$~~ $v t_1 \ldots t_n = w$

and $l(v t_1 \ldots t_i) > l(v t_1 \ldots t_{i-1})$

$l(v s_i) = \begin{cases} l(v)+1 \\ l(v)-1 \end{cases}$

Lemma

$v \leq w$ iff $m(w)_{ij} \leq m(v)_{ij}$

Note $l(v t_{(a,b)}) > l(v) = l(w)$ $\leq$ number of pairs to write it

iff $v(a) < v(b)$

Theorem

$v \leq w$ iff $v \leq_g w$

Theorem (strong exchange)

Suppose $l(tw) < l(w)$

and $w = s_{i_1} \ldots s_{i_k}$

then $tw = s_{i_1} \ldots \hat{s}_{i_j} \ldots s_{i_k}$

Proof

If $X_v \subset \overline{X_w}$, then $m(w) \leq m(v)_{ij}$

Conversely if ~~$\neq$~~ $\overline{}$

$X_v \leq X_w$

Corollary

If $v \leq w$, then

v can be obtained

Last time

Given two flags $V_\bullet, W_\bullet$, we say they are in relative position $w \in S_n$ if

$$\dim V_i \cap W_j = \#\text{ of } r \leq j \text{ s.t. } w(r) \leq i \quad \forall i,j$$

In k^n, we have the standard flag $E_\bullet$ with $E_i = \operatorname{span}(e_1, \dots, e_i)$
and w-standard flag $E_\bullet^w$ with $E_i^w = \operatorname{span}(e_{w(1)}, \dots, e_{w(i)})$

$X_w = \{ V_\bullet \in Fl_n : E_\bullet \text{ and } V_\bullet \text{ are in rel position } w \}$

$E_\bullet^w \in X_w$ and $X_w = B E_\bullet^w$, where $B = \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix}$

If $V_\bullet \in X_w$, then $\exists!$ basis $v_1, \dots, v_n$ s.t.

(i) $V_i = \operatorname{span}(v_1, \dots, v_i)$ "$(v_1, \dots, v_n)$ represents $V_\bullet$"

(ii) $v_i = e_{w(i)} + \sum_{\substack{j > i \\ w(j) < w(i)}} a_{ij} e_{w(j)}$ $\qquad \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\ell(w) = \#$ of $i < j$ s.t. $w(i) > w(j)$

Then $X_w \to k^{\ell(w)}$ a bijection.

Later we will give Fl_n the structure of a variety and this will be an isomorphism.

Another perspective on X_w.

$T = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \subset GL_n$, $T = (k^\times)^n$ torus

Then T acts on Fl_n.

Lemma

The fixed points of T acting on Fl_n are $\{ E_\bullet^w : w \in S_n \}$

Proof

If $V_\bullet \subset k^n$ is fixed by T,
Then each V_i is an invariant subspace for each matrix $\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}$.

If the t_i are distinct, then the only invariant subspaces are the coordinate subspaces.

group	GL_n	B	T
# of orbits	1	$n!$	∞
# of fixed pts	0	$1_{E_\bullet}$	$n!$

Let X be any complex variety with $\mathbb{C}^\times \curvearrowright X$, $x \in X$
We say $\lim_{t \to 0} t \cdot x = x_0 \in X$ if the map

$$\mathbb{C}^\times \longrightarrow X$$
$$t \longmapsto t \cdot x$$

extends to a map $\mathbb{C} \longrightarrow X$
$$0 \longmapsto x_0$$

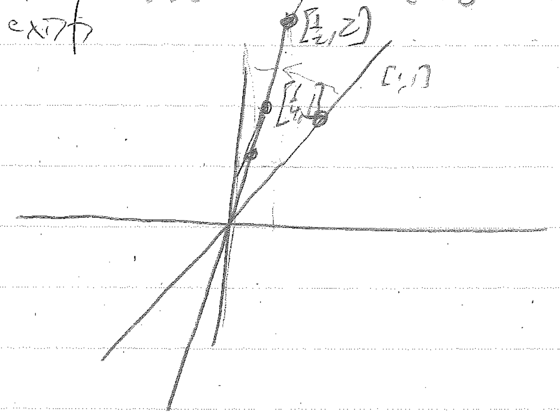
Facts from algebraic geometry: (0) x_0 is a T-fixed point
(1) If X is seperated, then if the limit exists, it is unique (Hausdorf)
(2) If X is proper, then the limit always exists (compact)
(3) Every projective variety is proper and separated

Eg
$\mathbb{C}^\times \curvearrowright \mathbb{C}^2$ by $t \cdot (a,b) = (ta, t^{-1} b)$

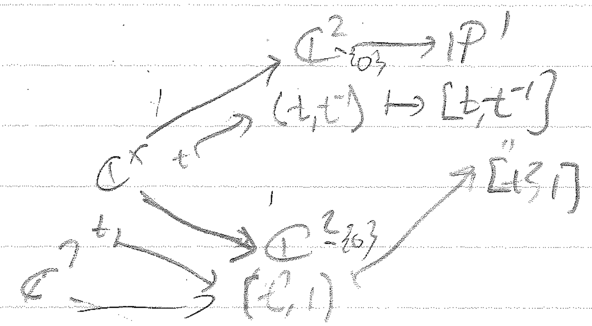
Then $\lim_{t \to 0} t \cdot (1,1) = \lim_{t \to 0} (t, t^{-1})$ does not exist

$\mathbb{C}^\times \curvearrowright \mathbb{P}^1 =$ 1-dimensional subspaces of $\mathbb{C}^2$
$= \{[a,b] : (a,b) \neq (0,0)\}$

Then $\lim_{t\to 0} t\cdot[1,1] = \lim_{t\to 0}[t,t^{-1}] = \lim_{t\to 0}[t^2,1] = [0,1]$
exists



More formally I have



Now consider $\mathbb{C}^\times = \{(t^n, t^{n-1}, \dots, t) : t\in\mathbb{C}^\times\} \subset T$
and $\mathbb{C}^\times \curvearrowright Fl_n$.

The fixed points of $\mathbb{C}^\times$ are still just $E_\bullet^w$, $w\in S_n$

Prop
$X_w = \{V_\bullet \in Fl_n : \lim_{t\to 0} t\cdot V_\bullet = E_\bullet^w\}$

Proof
Let $V_\bullet\in Fl_n$, then $\exists$ a basis $v_1,\dots,v_n$ as
Y above so
smallest power of t
$$t\cdot v_i = t^{n+1-w(i)} e_{w(i)} + \sum_{\substack{j>i \\ w(j)<w(i)}} t^{n+1-w(j)} e_{w(j)}$$
$\Rightarrow$ in the limit this goes away and get

Moment map image.

Let $\lambda = (\lambda_1, .., \lambda_n)$ be distinct real numbers $\lambda_1 > ... > \lambda_n$

Let $\mathcal{H}_\lambda = \{$Hermitian matrices with eigenvalues $\lambda_1, .., \lambda_n\}$ $(A^\dagger = A)$

Such a matrix $A \in \mathcal{H}_\lambda$, is diagonalizable, so if L_i = eigenspace of λ_i, then $\mathbb{C}^n = L_1 \oplus ... \oplus L_n$ and $L_i \perp L_j$ (wrt $\langle , \rangle$)

Lemma

There are bijections

$$Fl(\mathbb{C}^n) \longrightarrow \left\{\begin{matrix}\text{decompositions of } \mathbb{C}^n \text{ into} \\ \text{pairwise orthogonal} \\ \text{lines}\end{matrix}\right\} \longleftarrow \mathcal{H}_\lambda$$

$$V_\bullet \longmapsto (L_1 = V_1, L_2 = \text{orth comp of } V_1 \text{ in } V_2, ...)$$

$$\{(L_i)\} \longleftarrow A$$

$\mathcal{H}_\lambda$ has a transitive action of U_n since every Hermitian matrix is orthogonally diagonalizable

So we can identify $\mathcal{H}_\lambda = U_n / T_c$

$T_c = (S^1)^n \subseteq U_n \cap T$

Now we have the map

$$\underset{\substack{\text{Hermitian} \\ \text{matrices}}}{\mathfrak{u}_n^*} \longrightarrow \mathbb{R}^n$$

given by taking the diagonal entries

This is the moment map for the coadjoint

To see this.

$u_n \cong u_n^* \longrightarrow t_{\mathbb{C}}^* = \mathbb{R}^n$

given by the bilinear form $(A, B) \longmapsto tr(AB)$

The moment map image $\Phi(\mathcal{H}_\lambda)$ is the image of $\mathcal{H}_\lambda$ under this map.

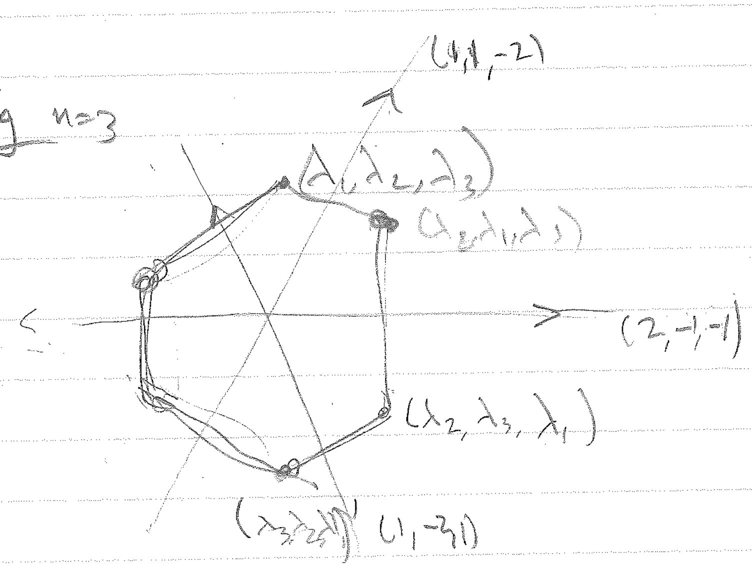
$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \in \mathcal{H}_\lambda \qquad \begin{bmatrix} \lambda_{w(1)} & & \\ & \ddots & \\ & & \lambda_{w(n)} \end{bmatrix} \in \mathcal{H}_\lambda$$

So $(\lambda_{w(1)}, \dots, \lambda_{w(n)}) \in \Phi(\mathcal{H}_\lambda)$ for all w.

Lemma

$\Phi(\mathcal{H}_\lambda)$ is the convex hull of the set $\{(\lambda_{w(1)}, \dots, \lambda_{w(n)}) : w \in S_n\}$

Eg $n=3$



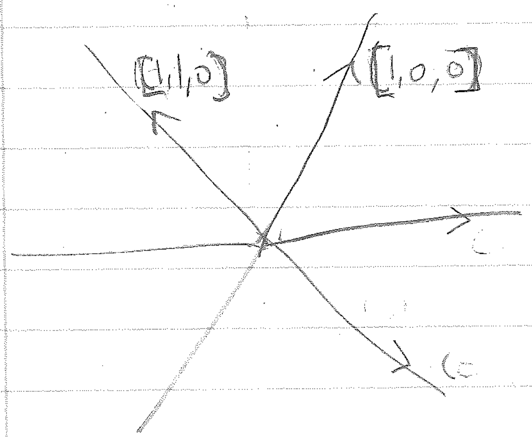
$\lambda_1 + \lambda_2 + \lambda_3 = 0$

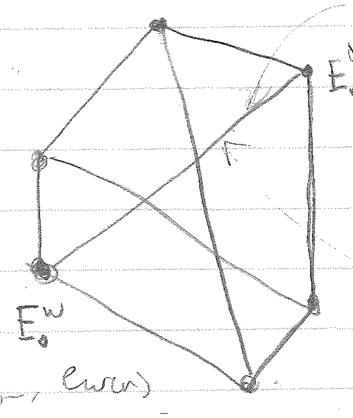
~~Later we~~ will see that $\Phi(\mathcal{H}_\lambda)$ is the polytope of a generic torus orbit in $Fl(\mathbb{C}^n)$

More precisely given any point $V_\bullet \in Fl_n$ we can form the orbit closure
$$\overline{T \cdot (V_\bullet)} \subset Fl_n$$

Then $\overline{T \cdot V_\bullet}$ is a projective toric variety so it is described by a complete fan

This fan is the dual fan of the polytope $\Phi(\mathcal{O}_\lambda)$ for any λ.





these are the polytopes of 1-dim torus orbits

$e_{w(1)}, \dots, e_{w(n)}$

$(e_{w(1)}, \dots, a e_{w(i)} + b e_{w(j)}, e_{w(i+1)}, \dots, d e_{w(j)}, \dots)$

Given w and (i,j)

$V_\bullet$ s.t.

$$\begin{array}{ccccc} V_1 \subset V_2 & V_{i-1} \subset V_i \subset V_{i+1} \\ \| & \cap & \| \\ E_1^w & E_i^w + E_j & V_i + E_i^w \end{array}$$

E_2^w

Embedding:

① Veronese

V vector space $\qquad \mathrm{Sym}^k V$

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(\mathrm{Sym}^k V)$$
$$[v] \longmapsto [v \cdots v]$$

in coordinates:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[x,y] \longmapsto [x^3, x^2y, xy^2, y^3]$$

② Segre

V, W vector spaces

$$\mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow \mathbb{P}(V \otimes W)$$
$$[v], [w] \longmapsto [v \otimes w]$$

in coordinate $\mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$$[a,b]\ [x,y,z] \longmapsto [xa, xb, ya, yb, \ldots]$$

③ Plucker emb.

V vector

$$G(k,V) \longrightarrow \mathbb{P}(\Lambda^k V)$$
$$\mathrm{span}(v_1, .., v_k) \longmapsto [v_1 \wedge \cdots \wedge v_k]$$

well-defined

trust

all of these are closed embeddings

Given $f \in \mathbb{C}[x_1, .. x_n]$ hom. $\qquad$ Def A proj variety

$f([a_0 \ldots a_N]) = 0$ makes sense $\qquad$ is the van

If V is a vector space, $\mathbb{P}(V)$ = 1-dim subspaces of V.
If $f \in \mathrm{Sym}^k V^*$, then we have
$V(f) \subset \mathbb{P}(V)$.
If $f_1, \dots, f_r \in \mathrm{Sym}\, V^*$ homogeneous
$X = V(f_1, \dots, f_r) \subset \mathbb{P}(V)$ a proj. variety.

$I(X) \subset \mathrm{Sym}\, V^*$ is a radical homogeneous ideal.

(1) Veronese: $\mathbb{P}(V) \longrightarrow \mathbb{P}(\mathrm{Sym}^k V)$

$[v] \mapsto [v \cdots v]$ $\quad (\alpha\alpha\beta)(\alpha\beta\beta)$

$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ $\quad a \; b \; c \; d \quad bc-ad$

$[x, y] \mapsto [x^3, x^2y, xy^2, y^3]$

the ideal of the Veronese embedding is generated by

$\mathrm{Sym}^2(\mathrm{Sym}^k V)^* \ni (\alpha_1 \cdots \alpha_k)(\beta_1 \cdots \beta_k) - (\beta_1 \alpha_2 \cdots \alpha_k)(\alpha_1 \beta_2 \cdots \beta_k) \quad \alpha_1, \dots, \alpha_k \in V^*$

$\alpha_1 \cdots \alpha_k \in (\mathrm{Sym}^k V)^*$

$(\alpha_1 \cdots \alpha_k)(\beta_1 \cdots \beta_k)(v \circ \cdots \circ v) = \alpha_1(v) \cdots \alpha_k(v)\, \beta_1(v) \cdots \beta_k(v)$

$\mathrm{Sym}^{2k}(\mathbb{C}^n)^* \subset \mathrm{Sym}^2(\mathrm{Sym}^k \mathbb{C}^n)$

$(\mathrm{Sym}^{2k} \mathbb{C}^n)^\perp \subset (\mathrm{Sym}^2(\mathrm{Sym}^k \mathbb{C}^n))^*$

$\mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$[x, y, z] \longmapsto [x^2, xy, y^2, xz, yz, z^2]$
$\qquad\qquad a \; b \; c \; d \; e \; f$

$\alpha, \beta, \gamma \in (\mathbb{C}^3)^*$

$af - d^2, \; ac - b^2, \; bd - ae,$

$(\alpha\alpha)(\gamma\gamma) - (\alpha\gamma)(\alpha\gamma), \quad (\alpha\beta)(\alpha\gamma) - (\alpha\alpha)(\beta\gamma)$

(1) Segre $\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$

$[v], [w] \longmapsto [v \otimes w]$

$\mathbb{P}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$[x_1, x_2], [y_1, y_2, y_3] \mapsto [x_1y_1, x_2y_1, x_1y_2, x_2y_2, x_1y_3, x_2y_3]$
$\qquad a \; b \; c \; d \; e \; f$

$\alpha_1, \alpha_2 \in V^*$, $\beta_1, \beta_2 \in W^*$ $\quad \alpha_1 \otimes \beta_1 \in (V \otimes W)^*$

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) - (\alpha_1 \otimes \beta_2)(\alpha_2 \otimes \beta_1) \in \mathrm{Sym}^2 (V \otimes W)^*$$

the ideal of segre

(1) Segre embedding

If X, Y are proj varieties then $X \times Y$ is also a projective variety

(2) Plucker embedding

V vector space

$$G(k, V) \longrightarrow \mathbb{P}(\Lambda^k V)$$

$$\mathrm{span}(v_1, \ldots, v_k) \longmapsto [v_1 \wedge \cdots \wedge v_k]$$

in coordinates

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \longmapsto [\underbrace{x_1 y_2 - x_2 y_1}_{p_{12}}, \underbrace{x_1 y_3 - x_3 y_1}_{p_{13}}, \ldots]$$

$\dim \Lambda^2 \mathbb{C}^4 = 6 \qquad p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0$

There is a map

$$\Lambda^{k+1} V^* \otimes \Lambda^{k-1} V^* \longrightarrow \mathrm{Sym}^2((\Lambda^k V)^*)$$

and the image of this map generates the ideal of the Plucker embedding.

Lemma

Given $x \in \Lambda^k V$, get

$$\varphi_x : V \longrightarrow \Lambda^{k+1} V$$

then $\dim \ker \varphi_x \leq k$.

Moreover if $\dim \ker \varphi_x = k$, then $x = v_1 \wedge \cdots \wedge v_k$

where $\ker \varphi_x = \mathrm{span}(v_1, \ldots, v_k)$

Now it is well-known that

$\{\phi : V \to W : rk\,\phi \le k\} \subset Hom(V,W) = V^* \otimes W$

is a variety with ideal generated by "$k+1 \times k+1$" minors

$\Lambda^{k+1}\phi : \Lambda^{k+1}V \to \Lambda^{k+1}W$ should vanish

$\alpha \in \Lambda^{k+1}W$ $\quad \alpha_1, \dots, \alpha_k \in W$

$\varphi_x : V \to \Lambda^{k+1}V$ $\quad rk\,\varphi_x \le n-k \Rightarrow \dim\ker\varphi_x \ge k$

$\Rightarrow \dim\ker\varphi_x = k$

so I will choose $\quad \Rightarrow x = v_1 \wedge \dots \wedge v_k$

Given $x \in \Lambda^k V$ and $\alpha \in \Lambda^{k-1}V^*$ we can construct an element $\iota_\alpha(x) \in V$

$\iota_\alpha(v \wedge w) = \alpha(v)w - \alpha(w)v \quad (k=2)$

<u>Lemma</u>

x is decomposable as $x = v_1 \wedge \dots \wedge v_k$
iff $\iota_\alpha(x) \wedge x = 0$ for all $\alpha \in \Lambda^{k-1}V^*$
$(\Leftarrow \iota_\alpha(x) \in span(v_1, \dots, v_k))$

Thus given $\alpha \in \Lambda^{k-1}V^*$ and $\beta \in \Lambda^{k+1}V^*$ we get a map

$x \longmapsto \beta(\iota_\alpha(x) \wedge x)$

$\psi_{\alpha\beta} : \Lambda^k V \longrightarrow \mathbb{C}$.

This gives an element $\psi_{\alpha\beta} \in Sym^2(\Lambda^k V)^*$

Thus we get a map $\Lambda^{k-1}V^* \otimes \Lambda^{k+1}V^* \longrightarrow Sym^2(\Lambda^k V)^*$ whose image is precisely the ideal of the Plücker embedding.

Now given $a_1, \ldots, a_d \in \mathbb{Z}$

$$Fl(V) \longrightarrow \prod_k G(k,V) \longrightarrow \prod_k \mathbb{P}(\Lambda^k V) \to \prod_k \mathbb{P}(Sym^{a_k}(\Lambda^k V)) \longrightarrow \mathbb{P}\left(\bigotimes_k Sym^{a_k} \Lambda^k\right)$$

We will still need to check that this is a closed embedding.

In fact, the ideal of this embedding is defined by quadratic generators!

$Sym^{\bullet} V(\lambda) \longrightarrow \bigoplus \Gamma(G/P, \mathcal{O}(\lambda))^{*} = \bigoplus V(k\lambda)$

kernel is generated in degree 2

$\ker(Sym^2 V(\lambda) \to V(2\lambda))$

A line bundle on a variety X is $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ which is Zariski locally trivial, $X \times \mathbb{C}$, $L_1 \otimes L_2$

[illegible] $\mathcal{O}(1)$ $(\mathbb{P}^n, \mathcal{O}(1))$

A linear system is (L, i, V) where

$i : V^* \to \Gamma(X, L)$ base point free: for all $x \in X$ $\exists\, \alpha \in V^*$ s.t. $i(\alpha)(x) \neq 0$

Theorem

There is a bijection between base point free linear systems on X and maps $\Phi : X \to \mathbb{P}(V)$.

Proof

Given a map $f : X \to \mathbb{P}(V)$, we can [illegible]

$i : \Gamma(\mathbb{P}(V), \mathcal{O}(1)) \longrightarrow \Gamma(X, f^* \mathcal{O}(1))$

$\quad \| $

$\quad V^*$

Given (L, i, V) get

$X \longrightarrow$ hyperplanes in $\Gamma(X, L) \longrightarrow$ hyperplanes in $V^* \cong \mathbb{P}(V)$

$x \longmapsto \{ s : s(x) = 0 \} \longmapsto \{ \alpha \in V^* : i(\alpha)(x) = 0 \}$

eg

$L = \mathcal{O}(3)$ on $\mathbb{P}^1$

$\Gamma(\mathbb{P}^1, \mathcal{O}(3)) = \mathbb{C}[x,y]_3 = \mathrm{Span}(x^3, \ldots, y^3)$

Veronese map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, $V = \mathbb{C}[x,y]_3^*$

$L = \mathcal{O}(n)$ on $\mathbb{P}(V)$

$\Gamma(\mathbb{P}(V), \mathcal{O}(n)) = Sym^n V^*$

$(L, i, Sym^n V)$

$\mathbb{P}(V) \to \mathbb{P}(Sym^n V)$ is Veronese

[illegible] $\Phi(X)$ [illegible] $\mathbb{P}(V)$

Def of equiv line bundles

GL_n-equiv vector bundle on Fl_n $\Longleftrightarrow$ f.d. reps of B

$\{[g,w]\} = GL_n \times_B W \longleftarrow W$

$L \longrightarrow L_{E_0}$

[illegible]

[illegible] B

[illegible] T, since $T = B/[B,B]$

and [illegible] $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$

[illegible] $B \to T \to \mathbb{C}^*$

$(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n}$

[illegible]

$\mathcal{O}(\lambda) = GL_n \times_B \mathbb{C}_\lambda$ [illegible]

we over $V_\bullet$ [illegible] $V_1^{-\lambda_1} \otimes (V_2/V_1)^{-\lambda_2} \otimes \cdots \otimes (V_n/V_{n-1})^{-\lambda_n}$

Claim: $\mathcal{O}(\lambda)$ is the pullback of $\mathcal{O}(1)$ under $Fl \to \mathbb{P}\left(\bigotimes_k \mathrm{Sym}^{m_k}(\Lambda^k V)^*\right)$

The map $Fl \to \mathbb{P}(\bigotimes_k \mathrm{Sym}^{m_k}(\Lambda^k V))$

is [illegible]

$\bigotimes_k \mathrm{Sym}^{m_k} (\Lambda^k V^*) \to \Gamma(Fl_n, \mathcal{O}(\lambda))$

For [illegible] λ [illegible] $\Gamma(Fl, \mathcal{O}(\lambda))$

Thm

$V(\lambda) =$ an irreducible rep of GL_n [illegible]

0 otherwise

We get $\bigotimes_k \mathrm{Sym}^{m_k}$ [illegible] $GL(V)$ equiv

[illegible] $\mathbb{P}(V) \to \mathbb{P}($ [illegible] $V)$

later [illegible] $V^*) \to \bigoplus \mathrm{Sym}^{kr} V^*$

[illegible] Sym^k [illegible] $V^*) \to \mathrm{Sym}^{2k} V^*$

[illegible]

$\mathrm{Sym}^k V_{(m,0,\ldots)} \to V_{(km,0,\ldots,0)}$

$Fl \to \prod_k G(k,V) \to \prod_k \mathbb{P}(\Lambda^k V) \to \prod_k \mathbb{P}(\mathrm{Sym}^{m_k} \Lambda^k V) \to \mathbb{P}\left(\bigotimes_k \mathrm{Sym}^{m_k} \Lambda^k V\right)$

$\mathbb{Z}^{n-1} = \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ iso if $H^1(X, \mathcal{O}), H^2(X, \mathcal{O}) = 0 \Leftarrow X$ has a paving by affine spaces

$\uparrow \qquad \qquad \uparrow \overset{\shortparallel}{\mathbb{Z}^{n-1}}$ basis given by Schubert divisors

$\mathbb{Z}^n = \mathrm{Pic}^G(X) \to H^2_G(X, \mathbb{Z}) = \mathbb{Z}^n$

Some basic representation theory

A rep of GL_n is a vector space V and a (alg) group hom map $GL_n \to GL(V)$ (ie matrices are polynomial in the entries)

A rep is called irreducible if $\nexists\, 0 \neq W \subsetneq V$ s.t. W is invariant under the action of GL_n

Eg

$\mathbb{C}^n$ standard rep, $\Lambda^k \mathbb{C}^n$, $\mathrm{Sym}^k \mathbb{C}^n$ are all irreducible reps of GL_n

Theorem

If V is a rep of GL_n, then $\exists$ irreducible subreps $W_1, \dots, W_k$ st $V = W_1 \oplus \dots \oplus W_k$

Now consider $T = (\mathbb{C}^\times)^n$

It also satisfies complete reducibility.

Moreover all irreducible reps are 1 dimensional, since T is abelian.

These 1-dim reps are group hom

$$T \to \mathbb{C}^\times$$

and they are of the form $t = (t_1, \dots, t_n) \mapsto t_1^{\mu_1} \cdots t_n^{\mu_n} = t^\mu$ for some integers $(\mu_1, \dots, \mu_n) = \mu$

$\mathbb{Z}^n$ = wt lattice of T.

If V is a rep of GL_n then we can restrict V to T and we get

$$V = \bigoplus_{\mu \in \mathbb{Z}^n} V_\mu$$

$V_\mu = \{ v : t \cdot v = t^\mu v \quad \forall t \in T \}$

Eg

$\mathrm{Sym}^2\mathbb{C}^3$ has a basis $e_1^2, e_1e_2, e_2^2, e_1e_3$

and $(t_1,t_2,t_3)\cdot e_1^2 = (t_1e_1)^2 = t_1^2 e_1^2$

$(t_1,t_2,t_3)\cdot e_1e_3 = t_1t_3\, e_1e_3)$

So $\mathrm{Sym}^2\mathbb{C}^3 = (\mathrm{Sym}^2\mathbb{C}^3)_{(2,0,0)} \oplus \mathrm{Sym}^2\mathbb{C}^3)_{(1,0,1)} \ldots$

all 1-dimensional

Define a partial order on $\mathbb{Z}^n$ by

$\mu \geq \nu$ if $\mu - \nu \in Q_+ = \{\sum_{i=1}^{n-1} n_i\alpha_i : n_i \in \mathbb{N}\}$ $\alpha_i = (\ldots 1 \ldots -1 \ldots)$

If $b \in N$, and $v \in V_\nu$ then $bv \in \bigoplus_{\mu \geq \nu} V_\mu$, actually $bv = v(b) + \bigoplus_{\mu > \nu} V_\mu$

A vector v is called a <u>highest weight vector</u> if

$bv \in \mathbb{C}v \quad \forall\, b \in B$

So $b \in V^N$ and b is a wt vector.

<u>Lemma</u>

If V is any rep, Then it has a highest wt vector

L any equiv line bundle on Fl

<u>Lemma</u>

Then $\dim \Gamma(\mathrm{Fl}, L)^N \leq 1$

<u>Proof</u>

$\Gamma(\mathrm{Fl}, L)^N \to L_{F_0^{w_0}}$

$s \mapsto s(F_0^{w_0})$

is injective, since if $s(F_0^{w_0}) = 0$

$\Rightarrow s(nF_0^{w_0}) = 0 \quad \forall\, n \in N$

$\Rightarrow s = 0$ $\square$

Corollary

If $\Gamma(Fl_n, L) \neq 0$, then $\Gamma(Fl_n, L)$ is irreducible

Theorem

$\Gamma(Fl, \mathcal{O}(\lambda)) \neq 0$ iff $\lambda \in \mathbb{Z}^n_+ = \{(\lambda_1, \ldots, \lambda_n) : \lambda_1 \geq \ldots \geq \lambda_n\}$

Proof

Suppose $\lambda \in \mathbb{Z}^n_+$

Over the big cell $= X_{w_0} = BE_\bullet^{w_0} = \{V_\bullet : V_\bullet$ and $E_\bullet$ are transverse$\}$

we have a free action of N.

So we can define a non-zero section s of $\mathcal{O}(\lambda)$, which is N-invariant.

Explicitly, we choose a basis

$v_1 = e_n + \ldots, \quad v_2 = e_{n-1} + \ldots$

for $V_\bullet$

and then we identify $V_i / V_{i-1} = \mathbb{C}$ since $[v_i]$ gives a basis for each quotient

$$Fl_n = X^{w_0} \cup \bigcup_{i=1}^{n-1} X^{w_0 s_i} \cup \underbrace{\bigcup_{\substack{w \\ \ell(w) \leq \binom{n}{2} - 1}} X_w}_{= Y}$$

Hartogs's lemma

Let X be an irreducible normal variety, $Y \subset X$

s.t. $\dim X - \dim Y \geq 2$

Let L be a line bundle on X and s a section

defined on $X \setminus Y$. Then s extends to a section on X

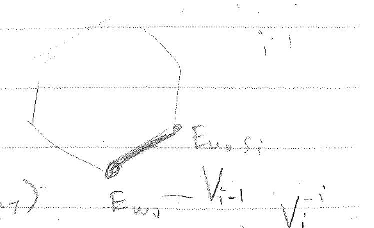
So we need to check extension to

$X_{w_0} \cup X_{w_0 s_i}$

for each i

Consider the copy of $\mathbb{P}^1 \subset Fl_n$

given by $[a,b] \mapsto \langle e_n \rangle' \subset \langle e_n, e_{n-1} \rangle$



Then $\mathcal{O}(\lambda)$ restricts to the line bundle $\mathcal{O}(\lambda_i, \lambda_{i+1})$.

$$V_i / V_{i+1} = \mathcal{O}(-1,0) \quad V_{i+1} / V_i = \mathcal{O}(0,-1)$$

Now, to extend our section over the $\mathbb{P}^1$ we just do a standard computation

$\Gamma(\mathbb{P}^1, \mathcal{O}(\lambda_i, \lambda_{i+1})) = \mathrm{Sym}^{\lambda_i - \lambda_{i+1}} \mathbb{C}^2$
and we have the section $x^{\lambda_i - \lambda_{i+1}}$ on $\mathbb{P}^1 \setminus \infty$)

So it extends over $\mathbb{P}^1$

We have $X_{w_0} \cup X_{w_0 s_i} = (N \times \mathbb{P}^1) / N_{s_i}$
$$nx \leftarrow [n, x]$$
where N_{s_i} is the subgroup $\begin{bmatrix} 1 & & 0 \\ & * & \\ & & 1 \end{bmatrix}$

$\square$

To complete our chain, suppose V is a rep of GL_n and let $v \in V$ be a highest wt vector.

Then we define
$$\psi : \mathcal{F}l_n \longrightarrow \mathbb{P}(V)$$
$$[g] \longmapsto [gv]$$

To summarize, we have the following chain:

maps to proj space	line bundles		reps
	GL_n-equiv line bundles on $\mathcal{F}l_n$ with sections	$\longrightarrow$	irreducible reps of GL_n
	$\mathcal{O}(\lambda)$	$\longmapsto$	$V(\lambda)$
	$\cap$		
	GL_n-equiv line bundles on $\mathcal{F}l_n$		

equiv maps ~~to projective space~~ $\longleftarrow$ linear series $\longleftrightarrow$ reps of GL_n with hw vector

$V^* \to \Gamma(Fl, \mathcal{O}(\lambda)) \longrightarrow$ map $V^* \longrightarrow \Gamma(Fl, \mathcal{O}(\lambda))$

eg

$Fl_n \xrightarrow{\psi} \mathbb{P}(\Lambda^k \mathbb{C}^n)$

gives the linear series

$(\Lambda^k \mathbb{C}^n)^* \longrightarrow \Gamma(Fl_n, \psi^* \mathcal{O}(1)) = \Gamma(Fl, \mathcal{O}(\overbrace{1,\dots,1,0,\dots,0}^{\omega_k}))$

$\psi^* \mathcal{O}(1)$ is the line bundle with fibre $(\det V_k)^*$

so corresponds to $V_1^{-1} \otimes V_2/V_1^{-1} \cdots \otimes (V_k/V_{k-1})^{-1}$

This map is an isom since $(\Lambda^k \mathbb{C}^n)^*$ and $\Gamma(\quad)$ are irreducible reps.

$V_\bullet \mapsto \det V_k \otimes \det V_\ell$

$Fl_n \longrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n \otimes \Lambda^\ell \mathbb{C}^n)$

corresponds to

$(\Lambda^k \mathbb{C}^n \otimes \Lambda^\ell \mathbb{C}^n)^* \longrightarrow \Gamma(Fl, \mathcal{O}(\omega_k + \omega_\ell))$

This is surj, not inj.

Continuing this analysis

$Fl_n \longrightarrow \mathbb{P}(\bigotimes_k \mathrm{Sym}^{a_k} \Lambda^k \mathbb{C}^n)$ | actually $Fl_n \to \mathbb{P}(\bigotimes_{k=1}^{n} \mathrm{Sym}^{a_n} \Lambda^n \mathbb{C}^n)$ $a_n \geq 0$ $k<n$ $a_n \in \mathbb{Z}$

corresponds to

$\bigotimes_k \mathrm{Sym}^{a_k} (\Lambda^k \mathbb{C}^n)^* \longrightarrow \Gamma(Fl, \mathcal{O}(\lambda))$

where $\lambda = \sum a_k \omega_k$

or equiv

$V(\lambda) \hookrightarrow \bigotimes_K \mathrm{Sym}^{a_k} (\Lambda^k \mathbb{C}^n)$

$Fl_n \to \mathbb{P}(V)$ $v \in V$

equiv maps to projective space	$\Leftrightarrow$ equiv line series	$\Leftrightarrow$ rep along with a h.w. vector	= rep along with $W \to V$ irrep

$Fl_n \to \mathbb{P}(V)$ $\Rightarrow$ $V^* \to \Gamma(Fl_n, v^*\mathcal{O}(1))$ $\Leftrightarrow$ $\Gamma(Fl_n, v^*\mathcal{O}(1))^* \to V$

$V(\lambda)$ - w/ a unique h.w. vector

equiv (expanding) maps to proj space	$\Longleftrightarrow$ equiv line bundles	$\Longleftrightarrow$ irreducible reps.

If $X \subset \mathbb{P}(V)$ is any projective variety, we can form

$\bigoplus_k \Gamma(X, \mathcal{O}(k))$

which is a graded ring, called the homogeneous coordinate ring.

eg $X = \mathbb{P}^1 \subset \mathbb{P}^1 \rightsquigarrow \mathbb{C}[x,y]$

$X = \mathbb{P}(V) \rightsquigarrow \mathrm{Sym}\, V^*$

For $X \subset \mathbb{P}(V)$, we have $\bigoplus_k \Gamma(\mathbb{P}(V), \mathcal{O}(k)) = \mathrm{Sym}\, V^* \to \bigoplus_k \Gamma(X, \mathcal{O}(k))$

Consider $Fl_n \subset \mathbb{P}(V(\lambda)^*)$

so $\mathcal{O}(1)$ restricts to $\mathcal{O}(\lambda)$

We get $\mathrm{Sym}\, V(\lambda) \to \Gamma(Fl_n, \mathcal{O}(k\lambda)) = \bigoplus_k V(k\lambda)$

Theorem (Kostant, Ramanathan)
The kernel of $\mathrm{Sym}\, V(\lambda) \to \bigoplus_k V(k\lambda)$ is generated in degree 2

i.e. $\mathrm{Sym}^2 V(\lambda) = V(2\lambda) \oplus$ ___ other reps of GL_n.

eg
$Fl_n \to Gr_k\mathbb{C}^n \to \mathbb{P}(\Lambda^k \mathbb{C}^n)$

we got

$\mathrm{Sym}\, \Lambda^k\mathbb{C}^n \to \bigoplus_r \Gamma(Gr_k\mathbb{C}^n, \mathcal{O}(r)) = \bigoplus_r V(r\omega_k)$

and the kernel is generated by the kernel of

$\mathrm{Sym}^2(\Lambda^k \mathbb{C}^n) \to V(2\omega_k)$

eg
$k=2$:
i.e. $G(2,n)$ then
$$\mathrm{Sym}^2(\Lambda^2\mathbb{C}^n) = V(2\omega_2) \oplus \Lambda^4\mathbb{C}^n$$

and so the ideal of $G(2,n)$ in the Plucker embedding is gen by $\Lambda^4\mathbb{C}^n$

i.e. $\gamma \in \Lambda^2 V$ can be written as $v \wedge v$
iff $\gamma \wedge \gamma = 0$
i.e. the equations come from $\Lambda^4 V^*$

In terms of usual coordinates

$$[\quad] \longmapsto [P_{12}, P_{13}, \cdots \quad]$$

$2 \times n$ matrix $\qquad \binom{n}{2}$ 2×2 minors

and then the ideal is gen by $\Lambda^4\mathbb{C}^n$
which is $P_{ab}P_{cd} - P_{ac}P_{bd} + P_{ad}P_{bc} = 0$

for any $a<b<c<d$

General k:

We have for any V,

$$\Lambda^{k+1}V^* \otimes \Lambda^{k-1}V^* \longrightarrow \mathrm{Sym}^2(\Lambda^k V)^*$$

and the image of this is precisely the kernel.

Schubert varieties

Recall, we define Schubert cells

$$X_w^\circ = \{ V_\bullet : \dim E_i \cap V_j = m(w)_{ij} \}$$

<u>Lemma</u>

$$\overline{X_w} = \{ V_\bullet : \dim E_i \cap V_j \geq m(w)_{ij} \}$$

$$= \bigcup_{v \leq w} X_v$$

∃ a reduced word for w s.t. v is obtained by deleting some

$\Leftrightarrow$ where $v \leq w$ means $m(v)_{ij} \geq m(w)_{ij} \ \forall i,j$

$\boxed{\text{Bruhat order}}$

eg $w = id$, $m(id)_{ij} = \min(i,j) \geq m(w)_{ij} \ \forall w$.

$$id \leq w.$$

$usv = uvv^{-1}sv$

<u>Proof</u>

By B-invariance

$$\overline{X_w} = \bigcup_{v \leq_g w} X_v$$

for some partial order $\leq$.

Suppose $s = (ij)$ is any transposition, then there is a $\mathbb{P}^1$ connecting E^{ws} and E^w

use the flags

$\langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \ldots$ $\quad$ $c\,a\,e_{w(i)} + b\,e_{w(j)}$ instead of $w(i)$

E^{ws} —— E^w

if $w(i) < w(j)$, then this $\mathbb{P}^1$ is mostly in X_{ws}

$\Rightarrow E^w \in \overline{X_{ws}}$

$\Rightarrow w \leq_g ws$

in fact