6 Toric Manifolds

Definition 6.1 (M^{2n}, ω) *is a toric manifold if it is equipped with the (effective) Hamiltonian action of a torus T^n for which the dimension of the torus is half the dimension of the manifold.*

$$\Phi : M^{2n} \to \mathbb{R}^n$$

Comparison with integrable systems: An *integrable system* is a symplectic manifold M^{2n} equipped with n linearly independent Poisson commuting functions $f_1, \ldots, f_n$ (in other words the corresponding Hamiltonian vector fields are linearly independent almost everywhere).

So a toric manifold is an integrable system for which the functions $f_1, \ldots, f_n$ may be chosen in such a way that the Hamiltonian flows of the Poisson commuting functions are periodic with period 1 almost everywhere.

The image of the moment map $\Phi(M)$ is a convex polyhedron $B \subset \mathbb{R}^n$ (the Newton polytope of M).

All polytopes arising from toric manifolds satisfy the following:

Proposition 6.2

1. *For each vertex p there are exactly n edges leaving it*
2. *The edges are of the form $p + tv_j$ $(j = 1, \ldots, n)$ where $v_j \in (\mathbb{Z}^n)^*(= \Lambda^W)$).*
3. *The weights $v_1, \ldots, v_n$ form a basis of the weight lattice Λ^W, for each vertex p.*

Remark: M is a toric orbifold (rather than a smooth manifold) iff only (1) and (2) are satisfied. (A reference on orbifolds is [23].)

We shall see below that

Theorem 6.3 *(see e.g. Audin Chap. VII or Guillemin Chap. 1) If B is a convex polytope satisfying (1), (2), (3) then there is a toric manifold M such that $\Phi(M) = B$.*

Theorem 6.4 *(Delzant) Toric manifolds are classified by their moment polytopes: in other words, if M_1, M_2 are two toric manifolds with moment maps Φ_1 and Φ_2 and $\Phi_1(M_1) = \Phi_2(M_2)$, then there is a T^n-equivariant symplectic diffeomorphism between M_1 and M_2.*

To see Theorem **??**, given a polytope B satisfying (1)-(3) we exhibit a toric manifold M with $\Phi(M) = B$.

Write $B = \cap_{j=1}^{d} \{x \in \mathbb{R}^n :< x, u_j > \leq \lambda_j\}$ for $u_j \in \mathbb{R}^n$ and $\lambda_j \in \mathbb{R}$.

Definition 6.5 *If B is an n-dimensional polyhedron in $\mathbb{R}^n$, then (a) F_i is an i-dimensional face of B if F_i is an i-simplex (b)* $\mathrm{Int} F_i$ *is congruent to the interior of the i-simplex. (c) Every point in B is in the interior of exactly one face.*

Definition 6.6 *A facet of an n-dimensional polytope is an $(n-1)$-dimensional face. The number of facets in B is d: they are indexed by j, and have normals $u_j \in \mathbb{Z}^n$. The u_j are assumed to be* primitive *(in other words they are not given by an integer multiple of another element of $\mathbb{Z}^n$).*

We have a short exact sequence of vector spaces

$$0 \to \mathbf{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0$$

where $\pi : e_j \mapsto u_j$. Because $u_j \in \Lambda^W = \mathrm{Hom}(\mathbb{Z}^n, 2\pi\mathbb{Z})$, this exponentiates to

$$1 \to N \xrightarrow{i} U(1)^d \xrightarrow{\pi} U(1)^n \to 1$$

so $N = \mathrm{Ker}(\pi)$ is a torus.

We know the moment map for the action of $U(1)^d$ on $\mathbb{C}^d$ is

$$J : (z_1, \ldots, z_d) \mapsto -\frac{1}{2}\left(|z_1|^2, \ldots, |z_d|^2\right) + c.$$

Set $c = (\lambda_1, \ldots, \lambda_d)$.

For the action of N on $\mathbb{C}^d$ the moment map is $i^* \circ J$ where

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathbf{n}^* \to 0$$

(for $\mathbf{n} = \mathrm{Lie}(N) \cong \mathbb{R}^{n-d}$). Reduce $\mathbb{C}^d$ with respect to the action of N :

Proposition 6.7 *(a)$(i^* \circ J)^{-1}(0)/N$ is a symplectic manifold M (b) M is equipped with the Hamiltonian action of T^n with moment map Φ and $\Phi(M) = B$.*

Example 6.8

$$\mathbb{C}P^2$$

The moment polytope is the right triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Let u_i be the normal vector to the i-th face. $u_1 = (0,-1)$, $u_2 = (-1,0)$ and $u_3 = \frac{1}{2}(1,1)$.

$$\mathbf{n} \to \mathbb{R}^3 \xrightarrow{\pi} \mathbb{R}^2$$

$$\pi : e_i \mapsto u_i$$

$$(1,0,0) \mapsto (0,-1)$$

$$(0,1,0) \mapsto (-1,0)$$

$$(0,0,1) \mapsto (1,1)$$

$\lambda_1 = \lambda_2 = 0$*;* $\lambda_3 = \frac{1}{2}$

$$\pi = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{n} = \mathbb{R}(1,1,1) \subset \mathbb{R}^3 = \mathrm{Ker}(\pi)$$

$$N \cong U(1)$$

Reduce $\mathbb{C}^3$ with respect to action of N:

$$i : (1,1,1) \mapsto \mathbb{R}^3$$

$$i^* : \mathbb{R}^3 \to \mathbb{R}$$

$$i^*(v) =< v, (1,1,1) >$$

$$J(z_1, z_2, z_3) = -\frac{1}{2}(|z_1|^2, |z_2|^2, |z_3|^2) + (\lambda_1, \lambda_2, \lambda_3)$$

$$i^* \circ J(z_1, z_2, z_3) = -\frac{1}{2}\sum_j |z_j|^2 + (\lambda_1 + \lambda_2 + \lambda_3)$$

$$= -\frac{1}{2}\sum_j |z_j|^2 + \frac{1}{2}$$

$$(i^* \circ J)^{-1}(0)/N = \mathbb{C}P^2$$

Proof of Proposition ??: breaks down into
Lemma 1 N acts freely on $(i^* \circ J)^{-1}(0)$ (by section on symplectic quotients)
Define $B \subset (\mathbb{R}^n)^*$; $B' = \pi^*(B) \subset (\mathbb{R}^d)^*$
Lemma 2: Claim

$$(i^* \circ J)^{-1}(0) = J^{-1}(B') = \{z \in \mathbb{R}^d : i^* \circ J(z) = 0\}$$

Proof:

$$J(\mathbb{R}^d) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d :< e_i, x > \leq \lambda_i, i = 1, \ldots, d\}.$$

$i^*(x) = 0$ iff $x = \pi^*(y)$ for $y \in (\mathbb{R}^n)^*$; in other words $i^*(J(z)) = 0$ iff $J(z) = \pi^*(y)$ for some $y \in (\mathbb{R}^n)^*$ and

$$< e_i, \pi^*(y) > \leq \lambda_i, \quad i = 1, \ldots, d$$

iff

$$< \pi(e_i), y > \leq \lambda_i, i = 1, \ldots, d$$

iff

$$< u_i, y > \leq \lambda_i, \quad i = 1, \ldots, d$$

iff $y \in B$ iff $\pi^*(y) \in B'$ so $J(z) \in B'$.

Lemma 3: If $z \in \mathbb{C}^d$, define $I \subset \{1, \ldots, d\}$ by $z_i = 0$ iff $i \in I$.

$$\mathrm{Stab}(z) = T_I = \{(u_1, \ldots, u_d) : i \in I \Rightarrow u_i = 1\}.$$

This is

$$U(1)^{d-|I|}.$$

Lemma 4: For any $z \in J^{-1}(\Delta')$, $\mathrm{Stab}(z)$ is transverse to N, which acts freely at z.
Proof Faces F_I of Δ (or of its image Δ') are determined by choosing a subset

$$< u_i, y >= \lambda_i, \quad y \in \mathbb{R}^{n*}$$

$$< e_i, \pi^*(y) >= \lambda_i$$

These determine a torus $T_I \subset T^d$.

Note that the condition

$$I \subset I'$$

is equivalent to

$$F_{I'} \subset F_I.$$

Also that the largest sets I (corresponding to vertices $F_I = p$) have n elements because of the hypothesis that each vertex has n edges leaving from it, or equivalently it is the intersection of n facets.

If a vertex is the intersection of m facets then each edge is the choice of one facet to omit, in other words the number of edges is the number of facets.

So for any I (corresponding to a facet) it sits in several maximal $I_{\max}$ with $|I_{\max}| = h$ corresponding to the vertices in the face for which $F_{I_{\max}} = p$ (where p is a vertex). In other words $\{u_i, i \in I_{\max}\}$ forms a basis of the integer lattice $\mathbb{Z}^n = \Lambda^I \subset \mathbf{t}$.

The edges leaving the vertex p correspond to a basis of $\Lambda^W \subset \mathbf{t}^*$ which is dual to the basis $\{u_i, i \in I_{\max}\}$ since

$$T_I = \{(z_1, \ldots, z_d) \in U(1)^d : z_j = 1 \text{ if } j \in I\} = \{exp(\sum_j \theta_j e_j), \theta_j \in \mathbb{R}, e_j \text{ basis of } \mathbb{R}^d\}$$

under

$$\exp(\pi)(T_I) \overset{\exp \pi}{=} \exp \sum_{j \neq I} \theta_j u_j.$$

So

$$T_{I_{\max}} \cong T^n.$$

So

$$T_I \to T_{I_{\max}} \cong T^n.$$

So since $N = \mathrm{Ker}(T^d \to T^n)$, $N \cap T_I = \{1\}$ for any I corresponding to a face F_I of the polyhedron $B' = \pi^*(B)$. This happens iff N acts freely on $J^{-1}(F_I)$ so N acts freely on $J^{-1}(B') = (i^* \circ J)^{-1}(0)$.

By our earlier results on reduction in stages, $T^n = T^d/N$ acts on $(i^* \circ J)^{-1}(0)/N = M$ in a Hamiltonian way.

The above results show:

$$0 \to N \to T^d \to T^n \to 0$$

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi} (\mathbb{R}^d)^* \to \mathbf{n}^* \to 0$$

If

$$z \in J^{-1}(B') = (i^* \circ J)^{-1}(0)$$

then

$$J(z) \in \mathrm{Im}(\mathbb{R}^n)^*$$

and so

$$\Phi : M \to (\mathbb{R}^n)^*$$
$$(\mathbb{R}^n)^* \to (\mathbb{R}^d)^*$$
$$\Phi(m) \in \mathrm{Im}(J)$$

Claim

$$m \in \Phi^{-1}(F_I) \longleftrightarrow \mathrm{Stab}(m) = T_I \subset T^n$$

so

$$\mathrm{Stab}(m) \cong T_I.$$

So $\Phi(m) \in \mathrm{Int}(B)$ iff T^n acts freely at m.

$\Phi(m)$ is in exactly one facet iff T^n acts with 1-dimensional stabilizer.

$\Phi(n)$ is in intersection of exactly 2 facets iff T^n acts with 2-dimensional stabilizer, etc.

$\Phi(m)$ is a vertex iff T^n fixes m.

Remark 6.9 *$B \subset \mathbf{t}^* = (R^n)^*$: normals to facets are $u_j \in \Lambda^I \subset \mathbf{t}$, if F_I is intersection of $< x, u_i >= \lambda_i$. For $i \in I$. the u_i $(i \in I)$ generate the stabilizer at any point in $\Phi^{-1}(F_I)$.*

Theorem 6.10 *$\Phi^{-1}(b) \cong T^n/T_I$ if $b \in \mathrm{Int}(F_I)$. In particular, the symplectic quotient of M at any point $b \in (\mathbb{R}^n)^*$ is a point $\Phi^{-1}(b)/T^n$.*

3. Fans and alternative description of toric manifolds (Ref: Audin Chap. VII)

Definition 6.11 *A* fan Σ *is the specification of a family of convex cones in $\mathbb{R}^n$ with origin* 0 *generated by elements $u_i \in \Lambda^I$ and for which*

(a) every face of a cone is a cone

(b) if C_1 and C_2 are cones then $C_1 \cap C_2$ is a face of C_1 and of C_2.

The data in a fan is "dual" to the data in the polyhedron B.

1-dimensional cones C_i correspond to rays $\mathbb{R}u_i$ through the normals u_i to the hyperplanes cutting out B

An indexing set $I \subset \{1, \ldots, d\}$ of order r determines a cone $C_I = C(U_{i_1}, \ldots, U_{i_r})$ of dimension r which corresponds to the face $F_i = \{x :< u_i, x >= \lambda_i \text{ for } i \in I\}$ in B of codimension r (dimension $n - r$).

The origin 0 (which is a 0 dimensional cone) corresponds to the face of dimension n.

However, when you pass from polyhedron B to fan Σ, you lose the information λ_i $(i = 1, \ldots, d)$ specifying the distance of hyperplanes in B from the origin.

Proposition 6.12 *Fans classify toric manifolds up to diffeomorphism.*

Newton polytopes classify toric manifolds up to symplectic diffeomorphism. For example, spheres S^2 of different radius but the same centre have the same fan but different Newton polytopes $[-r, r]$ where r is the radius of the sphere.

Construction of toric manifold starting from a fan

Note that for any indexing set $I \subset \{1, \ldots, d\}$ of order r, the cone C_I may or may not be present in the fan Σ.

(depending on whether or not the intersection of the hyperplanes $\cap_i < u_i, y >= \lambda_i\}$ is nonempty).

We have, as previously,

$$0 \to \mathbf{n} \to \mathbb{R}^d \to \mathbb{R}^n \to 0$$

$$1 \to N \to U(1)^d \to U(1)^n \to 0$$

$$1 \to N_{\mathbb{C}} \to (\mathbb{C}^*)^d \to (\mathbb{C}^*)^n \to 0$$

$N_{\mathbb{C}} \cong (\mathbb{C}^*)^{d-n}$ is the (complex) Lie group whose Lie algebra is $\mathbf{n} \otimes \mathbb{C}$: it is called the complexification of N.

Definition 6.13 $e_I = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : z_j = 0 \text{ if } j \notin I\}$ *has dimension* $|I|$. *In particular* $e_\emptyset = 0$. *The toric manifold associated to the fan* Σ *is* $M_\Sigma = U_\Sigma / N_{\mathbb{C}}$ *where* U_Σ *is an open set in* $\mathbb{C}^d$*:*

$$U_\Sigma = \mathbb{C}^d \setminus \cup_{I: C_I \notin \Sigma} e_I.$$

Alternative definition:

$$U_\Sigma = \cup_{I, C_I \in \Sigma} U_I$$

where

$$U_I = \{z \in \mathbb{C}^d : z_j = 0 \implies j \in I\}$$
$$= (\mathbb{C}^*)^{\bar{I}} \times \mathbb{C}^I$$

Conditions for a fan to correspond to a compact smooth toric variety:

1. Fan is complete
2. $C_I \in \Sigma$ implies $e_I \cap \mathbf{n} \otimes \mathbb{C} = \emptyset$. The preceding item is a consequence of
3. Each cone of Σ is generated by $\{u_i, i \in I\}$, which forms part of an integer basis of the integer lattice Λ^I
4. All n-dimensional cones of Σ (which correspond to vertices of the Newton polytope) are generated by part of a $\mathbb{Z}$-basis of Λ^I.

Example 6.14 *1.* $n = 2, d = 2$ $\{I\} = \emptyset, \{1\}, \{2\}, \{1, 2\}$

We have all possible indexing sets so $U_\Sigma = \mathbb{C}^2$

2. $n = 2, d = 2$ $\{I\} = \emptyset, \{1\}, \{2\}$

$$C_{\bar{I}} \notin \Sigma \to \bar{I} = \{12\}, I = \emptyset$$

$$e_{I=\emptyset} = \{0\}$$

so

$$U_\Sigma = \mathbb{C}^2 \setminus \{0\}$$

3. $n = 2, d = 3$

$$I = \emptyset, \{1\}, \{2\}, \{3\}, \{12\}, \{23\}, \{13\}$$

$C_{\bar{I}} \notin \Sigma$ *implies* $\bar{I} = \{123\}$*, which implies* $I = \emptyset$*, which implies* $e_I = \{0\}$.

$$U_\Sigma = \mathbb{C}^3 \setminus \{0\}$$

Since $\mathbf{n} = \mathbb{R}(1, 1, 1) \subset \mathbb{R}^3$ *and* $N = \{(\lambda, \lambda, \lambda) | \lambda \in U(1)\} \subset U(1)^3$ *we have*

$$N_\mathbb{C} = \{(\lambda, \lambda, \lambda) | \lambda \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3$$

We have recovered the more usual description of $\mathbb{C}P^2$*:*

$$\mathbb{C}P^2 = (\mathbb{C}^*)^3 \setminus \{0\})/\mathbb{C}^*.$$

4. Recovering a symplectic structure on a toric manifold specified via a fan

As before we have

$$\mathbb{C}^d \xrightarrow{J} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathbf{n}^*$$

$$J(z_1, \ldots, z_d) = -\frac{1}{2}(|z_1|^2, \ldots, |z_d|^2)$$

(the inclusion $0 \to \mathbf{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0$ specifies i^*.)

For any regular value $\xi \in \mathbf{n}^*$ ($\xi = i^*(\lambda_1, \ldots, \lambda_d)$ in our previous notation) we saw that a toric manifold was specified as

$$M_{(\lambda_1, \ldots, \lambda_d)} = (i^* \circ J)^{-1}(\xi)/N.$$

For any regular value ξ, the manifolds $M_{(\lambda_1, \ldots, \lambda_d)}$ have the diffeomorphism type of the manifold

$$M_\Sigma = U_\Sigma / N_\mathbb{C}$$

.

(a) M_Σ inherits an action of $(\mathbb{C}^*)^n = (\mathbb{C}^*)^d / N_\mathbb{C}$ (the complexification of $U(1)^n$).

(b) This action preserves the complex structure but not the symplectic structure. The action of $U(1)^n$ preserves both complex and symplectic structures (Kähler structure).

Remarks:

(a) Two constructions of M_Σ:

(i) as a complex manifold, as quotient of an open set in $\mathbb{C}^d$ by the action of the complex group $N_\mathbb{C}$

(ii) As a symplectic manifold, as symplectic quotient of $\mathbb{C}^d$ by the compact group N.

Construction (i) is an example of a general geometric constrution ("geometric invariant theory quotient"): Delete " unstable points" from $\mathbb{C}^d$ (points which would cause quotient by $N_\mathbb{C}$ to be non-Hausdorff): get

$$M_\Sigma = \left(\mathbb{C}^d \setminus \text{set of complex codimension} \geq 2\right)/N_\mathbb{C}.$$

General principle: Symplectic quotient of a Kähler manifold by a compact group N is same thing as geometric invariant theory quotient by complexified group $N_\mathbb{C}$ (Atiyah-Bott; Kirwan).