# 12 Geometric quantization

## 12.1 Remarks on quantization and representation theory

**Definition 12.1** *Let $M$ be a symplectic manifold. A* prequantum line bundle with connection *on $M$ is a line bundle $\mathcal{L} \to M$ equipped with a connection $\nabla$ for which the curvature $F_\nabla$ is equal to the symplectic form $\omega$.*

Suppose a symplectic manifold $M$ is equipped with a complex structure compatible with the symplectic structure (i.e. $M$ is a Kähler manifold). Then if $(\mathcal{L}, \nabla)$ is a prequantum line bundle with connection, $\mathcal{L}$ naturally acquires a structure of holomorphic line bundle (we define the $\bar{\partial}$ operator on sections of $\mathcal{L}$ as the antiholomorphic part $\nabla''$ of the prequantum connection, in other words a section $s$ is holomorphic if and only if $\nabla'' s = 0$).

**Definition 12.2** *Suppose $M$ is a symplectic manifold equipped with a prequantum line bundle with connection $(\mathcal{L}, \nabla)$. The* quantization *of $M$ is the virtual Hilbert space*

$$\mathcal{H}(M, \mathcal{L}) = \bigoplus_{i \text{ even}} H^i(M, \mathcal{L}) \ominus \bigoplus_{i \text{ odd}} H^i(M, \mathcal{L}).$$

**Remark 12.3** *In many natural situations, only one of the vector spaces $H^i(M, \mathcal{L})$ is nonzero.*

**Remark 12.4** *If $M$ is compact, all the vector spaces $H^i(M, \mathcal{L})$ are finite-dimensional, and the dimension of the quantization is given by the Riemann-Roch theorem.*

Suppose $M$ is equipped with a prequantum line bundle with connection, and suppose a group $G$ acts in a Hamiltonian fashion on $M$, and that the group action lifts to the total space of $\mathcal{L}$ in a way that is compatible with the connection. (The choice of such a lift is in fact equivalent to the choice of a moment map for the group action: cf. Remark 8.44.) Then each of the vector spaces $H^i(M, \mathcal{L})$ is acted on by the group $G$, in other words the quantization of $M$ is a (virtual) representation of $G$.

If $M$ is acted on by a torus $T$, the multiplicities with which the weights for the action appear in the representation $\mathcal{H}$ of $T$ are related to the moment polytope: all weights that have nonzero multiplicity lie within the moment polytope, and the asymptotics of the multiplicities of weights are in a natural sense given by the Duistermaat-Heckman polynomial $f$ from Theorem 10.23. (For a precise statement, see Section 3.4 of [19].)

## 12.2 Integral closed 2-forms and line bundles

$\omega$ is integral iff for any cover $\{U_i\}$ of $M$ there exists $\alpha_i \in \Omega^1(U_i)$ with $\omega|_{U_i} = d\alpha_i$ and $f_{ij} \in C^\infty(U_i \cap U_j)$ with $(\alpha_j - \alpha_i)|_{U_i \cap U_j} = df_{ij}$. Then on $U_i \cap U_j \cap U_k$,

$$f_{ij} + f_{jk} - f_{ik} = a_{ijk} \in \mathbb{R}$$

is a constant. $[\omega]$ is integral iff $a_{ijk} \in \mathbb{Z}$. We define transition functions

$$g_{jk} : U_j \cap U_k \to \mathbb{C}^*$$

by

$$g_{jk} = \exp i f_{jk}.$$

In order that these should satisfy

$$g_{ij} g_{jk} = g_{ik}$$

it is necessary that

$$f_{ij} + f_{jk} - f_{ik} \in 2\pi\mathbb{Z}.$$

On $M = S^2$, the usual symplectic form is $\omega = d\phi \wedge dz$. Take a different closed 2-form $\omega'$ on $S^2$ defined by $\omega' = d\phi \wedge df$ where $f : S^2 \to \mathbb{R}$ is a smooth function such that

(a) $f(z, \phi) = z$ for $z > 2\epsilon$ (b) $f(z, \phi) = -1$ for $z < \epsilon$. Then $\int_{S^2} \omega' = 2\pi \int_{-1}^{1} df = 4\pi$. In fact $\omega'$ is in the same class as $\omega$ in de Rham cohomology. so one is integral iff the other is. Take $U_0 = \{z < \epsilon\}$, $U_1 = \{z > -\epsilon\}$. On $U_0$, $\omega'|_{U_0} = 0$ so $\omega' = d\alpha_0$ where $\alpha_0 = 0$. On $U_1$, $\omega' = -d(f d\phi)$ so $\omega' = d\alpha_1$ where $\alpha_1 = -f d\phi$. On $U_0 \cap U_1$, $(\alpha_1 - \alpha_0)|_{U_0 \cap U_1} = df_{01} = -f d\phi = -d(f\phi) = d\phi$ since $f = -1$ on $U_0 \cap U_1$. Hence

$$f_{01}(z, \phi) = \phi$$

The effect on this calculation of replacing $\omega$ by $\lambda\omega$ (where $\lambda$ is a constant) is that $\alpha_0, \alpha_1$ and $f_{01}$ are multiplied by $\lambda$. So

$$g_{01}(z, \phi) = \exp i f_{01} = \exp i\lambda\phi.$$

This is single valued and defines a transition function iff $\lambda \in \mathbb{Z}$.

**Definition 12.5** *A symplectic form $\omega$ on a manifold $M$ is* integral *if $[\omega] \in H^2(M, \mathbb{Z})$, or equivalently if for any oriented 2-dimensional submanifold $S$ of $M$, $\int_S \omega \in \mathbb{Z}$.*

**Example 12.6** *$\omega = dz$ on $S^2$ (where $z$ is the height function): $\int_{S^2} \omega = 4\pi$ so*

$$\frac{n\omega}{4\pi}$$

*is integral for $n \in \mathbb{Z}$.*

**Lemma 12.7** *$\omega$ is integral iff $[\omega] = c_1(L)$ for a line bundle $L$ over $M$.*

**Remark 12.8** *Chern classes $c_j(E)$ always take values in $H^{2j}(M, \mathbb{Z})$.*

## 12.3 Prequantum line bundles with connection

**Definition 12.9** *A prequantum line bundle with connection over $M$ is a complex line bundle $L$ equipped with a connection $\nabla$ whose curvature $iF_\nabla/(2\pi) = \omega$.*

**Definition 12.10** *A connection $\nabla$ on a line bundle $L$ is a linear operator $\nabla : \Gamma(L) \to \Omega^1(M, L) = \Gamma(T^*M \otimes L)$ with the property that for a smooth function $f$ and a section $s$ on $L$,*

$$\nabla(fs) = (df)s + f\nabla s.$$

*One may define $\nabla : \Omega^1(M, L) \to \Omega^2(M, L)$ where $\Omega^2(M, L) = \Gamma(\Lambda^2 T^*M \otimes L)$. Then*

$$\nabla(\nabla s) = F_\nabla s$$

*where $F_\nabla \in \Omega^2(M, \mathbb{C})$ is a 2-form (the curvature form).*

## 12.4 Quantization and polarizations

In quantum mechanics, the standard phase space is $\mathbb{R}^{2n} = \{(q, p)\}$, the space of positions $q$ and momenta $p$. One wants to pass from the phase space to the space of wave functions

$$\mathcal{H} = \{\Psi(q)\} = L^2(\mathbb{R}^n).$$

These are "functions of half the variables" (the Heisenberg uncertainty principle says that the more precisely one knows $q$, the less precisely one knows $p$, so the wave function $\Psi$, where $|\Psi|^2(q)$ is the probability of the particle being at position $q$, is a function only of $q$. One could equivalently write the wave function as $\hat{\Psi}(p)$, a function only of $p$. $\hat{\Psi}$ is the Fourier transform of $\Psi$.

Another way of defining "half the variables": take $z_j = q_j + ip_j$ so $\mathbb{R}^{2n} = \mathbb{C}^n$ and define

$$\mathcal{H} = \{f : \mathbb{C}^n \to \mathbb{C} | f \text{is holomorphic in the } z_j \text{ and } \int_{\mathbb{C}^n} e^{-|z|^2} f(z) dz_1 \dots dz_n < \infty.$$

In general, given a manifold $M$ equipped with a prequantum line bundle with connection, in order to define a quantization one needs a polarization.

**Definition 12.11** Real polarization *(analogue of choice of $\{q\}$ or $\{p\}$ on $\mathbb{R}^{2n}$): choice of a foliation of $M$ by Lagrangian submanifolds, in the case of $\mathbb{R}^{2n}$ these are $\{p = \text{const}\}$ or $\{q = \text{const}\}$.*

**Definition 12.12** *Complex polarization: a choice of an almost complex structure $J$ on $M$ which is compatible with $\omega$. We assume $J$ is integrable i.s. comes from a structure of Kähler manifold on $M$.*

## 12.5 Holomorphic line bundles

Holomorphic line bundle over a complex manifold:

**Definition 12.13** *A complex line bundle is specified by an open cover* $\{U_\alpha\}$ *on* $M$ *and transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ *and* $L = \cup_\alpha U_\alpha \times \mathbb{C}/\sim$ *where* $(x, z_\alpha) \cong (x, z_\beta)$ *if* $z_\alpha = g_{\alpha\beta}(x) z_\beta$.

**Definition 12.14** *The line bundle is holomorphic if the transition functions* $g_{\alpha\beta}$ *are holomorphic.*

**Definition 12.15** *A section* $s$ *of* $L$ *is a collection of maps* $s_\alpha : U_\alpha \to \mathbb{C}$ *satisfying* $s_\alpha(z) = g_{\alpha\beta}(z) s_\beta(z)$ *for* $z \in U_\alpha \cap U_\beta$. *(This makes sense since* $\frac{\partial}{\partial \bar{z}_j} g_{\alpha\beta} = 0$ *so on* $U_\alpha \cap U_\beta$ $\frac{\partial}{\partial \bar{z}_j} s_\alpha = 0$ *iff* $\frac{\partial}{\partial \bar{z}_j} s_\beta = 0$.

**Definition 12.16** *Complex (co) tangent space:*

$$T_{\mathbb{C}} M = TM \otimes \mathbb{C}$$

$$T^*_{\mathbb{C}} M = T^* M \otimes \mathbb{C}$$

*In local complex coordinates* $z_j$*, a basis for* $T^*_{\mathbb{C}} M$ *is* $\{dz_j, d\bar{z}_j, j = 1, \ldots, n$.

**Definition 12.17** *Holomorphic and antiholomorphic cotangent spaces*

$$T^*_{\mathbb{C}} M = (T^*)^{(1,0)} M \oplus (T^*)'' M$$

*where in local complex coordinates* $(T^*)'' M$ *is spanned by* $\{d\bar{z}_j\}$ *and* $(T^*)' M$ *is spanned by* $\{dz_j\}$.

**Definition 12.18** $\bar{\partial}$ *-operator on functions on* $M$

*Choose local complex coordinates* $z_1, \ldots, z_n$ *on the* $U_\alpha$ *and define*

$$\bar{\partial} : C^\infty(U_\alpha) \to \Omega^{0,1}(U_\alpha)$$

$$\bar{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

**Definition 12.19** *(*$\bar{\partial}$ *operator on sections of* $L$ *on* $M$*) Given a section* $s : M \to L$, $s = \{s_\alpha\}$*, define* $\bar{\partial} s \in \Gamma(T^*)'' M \otimes L)$ *by*

$$\bar{\partial} s = \bar{\partial} s_\alpha$$

*on* $U_\alpha$.

This is well defined since $\bar{\partial} g_{\alpha\beta} = 0$.

**Proposition 12.20** *Specifying a structure of holomorphic line bundle on a complex line bundle $L$ is equivalent to specifying an operator $\bar{\partial} : \Gamma(L) \to \Omega^{0,1}(M, L)$ satisfying $\bar{\partial} \circ \bar{\partial} = 0$.*

Proof: We have seen that a holomorphic line bundle determines a $\bar{\partial}$ operator. Conversely, given a complex line bundle $L$ with $\bar{\partial}$, we can choose an open cover $\{U_\alpha\}$ with locally defined solutions $s_\alpha \in \Gamma(L|_{U_\alpha})$ to $\bar{\partial} s_\alpha = 0$, and $s_\alpha(x) \neq 0 \ \forall x \in U_\alpha$. Define transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ by

$$g_{\alpha\beta} = s_\alpha s_\beta^{-1}.$$

It follows that $\bar{\partial} g_{\alpha\beta} = 0$ so $g_{\alpha\beta}$ gives $L$ the structure of a holomorphic bundle. □

**Proposition 12.21** *Let $(L, \nabla)$ be a prequantum line bundle over $M$. Suppose $M$ is equipped with a complex structure $J$ compatible with $\omega$. (in other words, on $M$ there are locally defined complex coordinates $\{z_j\}$). Then $\nabla : \Gamma(L) \to \Gamma((T^*)M \otimes L)$ decomposes as*

$$\nabla = \nabla'' \oplus \nabla'$$

*where*

$$\nabla'' : \Gamma(L) \to \Gamma((T^*)'' M \otimes L)$$

*and*

$$\nabla' : \Gamma(L) \to \Gamma((T^*)' M \otimes L).$$

*Note that $\nabla'', \nabla'$ depend on the almost complex structure $J$ on $M$.*

**Proposition 12.22** *We may define a structure of holomorphic line bundle on $L$ by defining $\nabla''$ as a $\bar{\partial}$ operator: a section $s$ of $L$ is defined to be holomorphic if*

$$\nabla'' s = 0.$$

**Definition 12.23** *The quantization of the symplectic manifold $(M, \omega)$ equipped with the prequantum line bundle $L$ with connection $\nabla$ and the complex structure $J$ is*

$$\mathcal{H} = H^0(M, L),$$

*in other words the global holomorphic sections of $L$.*

**Remark 12.24** *If $M$ is compact, $\mathcal{H}$ is a finite-dimensional complex vector space.*

## 12.6 Quantization of $\mathbb{C}P^1 \cong S^2$

$$\mathbb{C}P^1 = \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\}/ \sim$$
$$= \{[z_0 : z_1]\}$$

The hyperplane line bundle over $\mathbb{C}P^1$ is $L_{[z_0:z_1]} = \{f : \{\lambda(z_0, z_1) \to \mathbb{C}\}$

$$f(z) = f_0 z_0 + f_1 z_1$$

Its dual is the tautological line bundle

$$L^*_{[z_0:z_1]} = \{(\lambda z_0, \lambda z_1) : \lambda \in \mathbb{C}\}$$

This is the line through the point $(z_0, z_1)$. The $k$-th power of the tautological bundle is

$$L^k_{[z_0,z_1]} = \{f : \{(\lambda z_0, \lambda z_1) \to \mathbb{C} : f(\lambda z_0, \lambda z_1) = \lambda^k f(z_0, z_1)\}$$

in other words $f$ is a polynomial of degree $k$ on the line through $(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$. Its zero-th power is the trivial bundle $L^0 = \mathbb{C}P^1 \times \mathbb{C}$.

*Global holomorphic sections:*

$H^0(L)$ is spanned by the restrictions to $\mathbb{C}^2 \setminus \{0\}$ of the linear functions on $\mathbb{C}^2$. This is a space of dimension 2. $H^0(L^k)$ is spanned by the restrictions to $\mathbb{C}^2 \setminus \{0\}$ of the polynomials of degree $k$ on $\mathbb{C}^2$:

$$f(z_0, z_1) = \sum_{j=0}^{k} a_j z_0^j z_1^{k-j}.$$

This is a space of dimension $k + 1$.

## 12.7 Link to representation theory

Suppose a (compact) group $G$ acts on $M$ (from the left), preserving the complex structure $J$ as well as the symplectic structure (in other words, for each $g \in G$, $L_g : M \to M$ is a holomorphic diffeomorphism).

Suppose the $G$ action lifts to an action on the total space $L$ of a prequantum line bundle which preserves the connection $\nabla$, and that this action is linear in the fibres: in other words

$$L_g : \pi^{-1}(m) \to \pi^{-1}(gm)$$

is a linear map.

**Proposition 12.25** *In this situation, the $G$ action defines an action of $G$ on $\mathcal{H}$ (from the right).*

*Define $(s \cdot g)(m) = s(g(M))$, in other words $s \cdot g = s \circ L_g$. Thus since $L_g$ is a holomorphic diffeomorphism, the composition $s \circ L_g$ is a holomorphic section.*

**Proposition 12.26** *The action of $G$ on the space of holomorphic sections is linear. Thus $\mathcal{H}$ is a linear representation of $G$*

Proof: $(s_1 + s_2) \cdot g = s_1 \cdot g + s_2 \cdot g$. □

**Proposition 12.27** *Let $M$ be a symplectic manifold acted on by $T$, and suppose $\omega$ is an integral symplectic form. Then the weights $\beta \in \mathbf{g}^*$ of the representation of $T$ on $\mathcal{H}$ lie in the moment polytope $\Phi_T(M) \subset \mathbf{t}^*$. These will in general appear with some multiplicities $m_\beta$, in other words $\mathcal{H} = \oplus_{\beta \in \Lambda^W} m_\beta \mathbb{C}_\beta$, $m_\beta \in \mathbb{Z}^+$. (This is given by the Kostant multiplicity formula, and its generalizations due to Guillemin.)*

**Remark 12.28**

1. *For toric manifolds, a weight appears with multiplicity* $1$ *iff it is in* $\Phi(M)$ *(and* $0$ *otherwise).*
2. *The multiplicity function $m : \Lambda^W \to \mathbb{Z}^{\geq 0}$ is related to the pushforward $\frac{\Phi_* \omega^n}{n!}$. The pushforward is obtained from the asymptotics of the multiplicity function under replacing $\omega$ by $k\omega, k \in \mathbb{Z}^+$ (this operation dilates the moment polytope by $k$).*

## 12.8 Holomorphic bundles over $G/T$: the Borel-Weil theorem

**Theorem 12.29 (Kostant)** *Suppose $\lambda \in\subset \mathbf{t}^*$. The symplectic form $\omega$ on the coadjoint orbit $\mathcal{O}_\lambda$ is integral iff $\lambda \in \Lambda^W \subset \mathbf{t}^*$.*

Let $\lambda \in \Lambda^W$, $\text{Stab}(\lambda) = T$. We may define a complex line bundle $L_\lambda$ over $G/T \cong \mathcal{O}_\lambda$ as follows.

$$\rho_\lambda = \exp \lambda \in \text{Hom}(T, U(1))$$

so define

$$L_\lambda = G \times_{T, \rho_\lambda} \mathbb{C}$$

$= (G \times \mathbb{C})/ \sim$ where

$$(g, z) \sim (gt^{-1}, \rho_\lambda(t) z).$$

Sections of $L_\lambda$ are given by equivariant maps $G \to \mathbb{C}$

$$= \{ f : G \to \mathbb{C} | f(gt^{-1}) = \rho_\lambda(t) f(g) \}$$

The action of $G$ on the space of sections is

$$g \cdot f(hT) = f(ghT).$$

**Proposition 12.30** *$G/T = G^{\mathbb{C}}/B$ where $G^{\mathbb{C}}$ is the complexification of $G$ and $B$ (Borel subgroup) is a complex Lie group defined by*

$$\mathrm{Lie}(B) = \big(\mathrm{Lie}(T) \otimes \mathbb{C}\big) \oplus \bigoplus_{\gamma>0} \mathbb{C}\gamma.$$

*Recall that* $\mathrm{Lie}(G) \otimes \mathbb{C}$ *decomposes under the adjoint action of $T$ as*

$$(\mathrm{Lie}(T) \otimes \mathbb{C}) \oplus \bigoplus_{\gamma>0} \mathbb{C}_\gamma \oplus \bigoplus_{\gamma>0} \mathbb{C}_{-\gamma}.$$

Examples of complexifications of Lie groups:

$$SU(n)^{\mathbb{C}} = SL(n, \mathbb{C})$$

$$U(1)^{\mathbb{C}} = \mathbb{C}^*$$

$$U(n)^{\mathbb{C}} = GL(n, \mathbb{C})$$

Examples of Borel subgroups:

$$G = U(n)$$

$$G^{\mathbb{C}} = GL(n, \mathbb{C})$$

$B$ is the set of upper triangular matrices in $GL(n, \mathbb{C})$ (in other words $z_{ij} = 0$ if $i > j$).

The groups $G^{\mathbb{C}}$ and $B$ have obvious complex structures: so, therefore, does $G^{\mathbb{C}}/B$. This holomorphic structure is compatible with $\omega_\lambda$ (it comes from the complex structure $J$ on $\mathrm{Lie}(G) \otimes \mathbb{C}$).

$$\omega_\lambda([\lambda, X], [\lambda, Y]) =< \lambda, [X, Y] > .$$

gives $\omega_\lambda(JZ_1, JZ_2) = \omega_\lambda(Z_1, Z_2)$. Here, the almost complex structure $J$ is defined on $T_\lambda(G/T)$ and is defined at $T_{g\cdot\lambda}(G/T)$ by identifying this with $T_\lambda(G/T) \cong \oplus_{\gamma>0}\mathbb{C}_\gamma$. It is integrable.

Thus $L_\lambda$ acquires the structure of a holomorphic line bundle.

**Lemma 12.31** *There is a homomorphism $p : B \to T_{\mathbb{C}}$.*

Proof: $B$ has a normal subgroup $N_{\mathbb{C}}$ for which $T_{\mathbb{C}} = B/N_{\mathbb{C}}$. □

**Example 12.32** $GL(n, \mathbb{C})$

*$T_{\mathbb{C}}$ is the invertible diagonal matrices*
*$B$ is the upper triangular matrices*
*$p$ is projection on the diagonal*

Hence $\rho_\lambda = \exp(\lambda) : T \to U(1)$ extends to $\rho_\lambda : T_{\mathbb{C}} \to \mathbb{C}^*$ and to $\bar{\rho_\lambda} : B \to \mathbb{C}^*$ via $\bar{\rho_\lambda} = \rho_\lambda \circ p$. Thus we can define

$$\begin{aligned} L_\lambda &= G_{\mathbb{C}} \times_{B,\rho} \mathbb{C} \\ &= \{(g,z)\}/\sim \end{aligned}$$

where $(g,z) \sim (gb^{-1}, \rho_\lambda(b)z)$ for all $b \in B$.

The space of holomorphic sections of $L_\lambda$ is

$$H^0(\mathcal{O}_\lambda, L_\lambda) = \{f : G^{\mathbb{C}} \to \mathbb{C} : f \text{ holo.}, \ f(gb^{-1}) = \rho_\lambda(b) f(g)\}$$

for all $g \in G^{\mathbb{C}}$ and $b \in B$.

**Theorem 12.33 (Borel-Weil-Bott)** *: If $\lambda \in \Lambda^W$ is in the positive Weyl chamber, then $H^0(\mathcal{O}_\lambda, L_\lambda)$ is the irreducible representation of $G$ with highest weight $\lambda$.*

**Representations of $SU(2)$:**

The representations of $SU(2)$ arise by quantizing $S^2$.

$$H^0(M,L) = \{a_0 z_0 + a_1 z_1\}$$

$$H^0(M,L^k) = \{\sum_j a_j z_0^j z_1^{k-j}\}$$

$$\tau := \mathrm{diag}(t, t^{-1}) \in SU(2)$$

acts on $\mathbb{C}^2$ by sending

$$\tau : \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} t z_0 \\ t^{-1} z_1 \end{pmatrix}$$

So $z_0^{k-j} z_1^j \mapsto t^{k-2j} z_0^{k-j} z_1^j$

There are $k+1$ weights in total, each appearing with multiplicity 1.

**Roots:**

1. Decompose $\mathrm{Lie}(G) \otimes \mathbb{C}$ under the adjoint action of the maximal torus $T$. The roots are the weights of this action of $T$. They appear in pairs (if $\beta$ is a root, so is $-\beta$).

2. Choose a polarization to enable us to designate some roots $\beta$ positive, while $-\beta$ is designated as negative.

3. *Simple roots* are a collection of roots which form a basis of $\mathrm{Lie}(T)$.

**Example 12.34**

$$SU(n)$$

$$\mathrm{Lie}(T) = \{\mathrm{diag}(X_1, \ldots, X_n) | \sum_j X_j = 0\}$$

*The roots are $\gamma_{ij}(X) = X_i - X_j$, and the positive roots are $\gamma_{ij}$ with $i < j$. The simple roots are $\gamma_{12}, \ldots, \gamma_{(n-1)n}$. The positive Weyl chamber consists of the subset of* $\mathbf{t}$ *for which the inner product with all simple roots is $> 0$.*