Toric Manifolds

Introduction

1 The Jurkiewicz-Danilov Theorem

Suppose M is a toric manifold of dimension $2n$ equipped with the (effective) Hamiltonian action of $T = U(1)^n$.

Theorem 1.1 *(Jurkiewicz-Danilov)*

The cohomology of M (with $\mathbb{Z}$ coefficients is

$$H^*(M; \mathbb{Z}) \cong \mathbb{C}[x_1, \ldots, x_d]/(\mathcal{I}, \mathcal{J}$$

where the Newton polyhedron of M has d facets and x_j are generators of degree 2, and the ideal $\mathcal{I}$ is generated by

$$\prod_{j \in I} x_j$$

where v_j are a collection of facets with no common intersection point in the polytope. Also the ideal $\mathcal{J}$ is defined by

$$\mathcal{J} = \sum_i \alpha_i x_i$$

for $\alpha \in \pi^*((\mathbf{R^n})^*)$.

Recall the short exact sequence

$$0 \to n \to \mathbf{R^d} \to \mathbf{R^n} \to \mathbf{0}$$

where we take the symplectic quotient of $\mathbf{R^d}$ *by* N *and inherit a manifold with a Hamiltonian action of* $\mathbf{R^n}$*.*

(Proof: see Theorem 7, S. Tolman, J. Weitsman, The cohomology ring of symplectic quotients, **CAG 11** *(2003) no. 4, 751–773.*

Example:

$$H^*(\mathbb{C}P^n) = \mathbb{C}[x]/ < x^{n+1} >$$

The moment polytope of $\mathbb{C}P^n$ *is an* n*-simplex in* $\mathbf{R^{n+1}}$*. It has* $n+1$ *facets* $D_0, \ldots, D_n$*, so we assign* $n+1$ *degree* 2 *generators* $x_0, \ldots, x_n$*. This gives* N *has dimension* 1 *and* $d = n+1$*. The only collection of facets that do not have a common intersection point is* $\{D_0, D_1, \ldots, D_n\}$*. So this gives a relation* $x_0 x_1 \ldots x_n$*. The linear relations are* $x_0 - x_j$*,* $1 \leq j \leq n$*.*

So this yields the familiar characterization of the cohomology ring of $\mathbb{C}P^n$*.*

Example 2:

$$H^*(\mathbb{C}P^1)$$

This is a special case of Example 1. There are two facets D_+, D_- *(which are the points* $+1$ *and* -1*). They do not intersect, so we choose two degree* 2 *generators*

x_+ and x_- and assign one relation $x_+x_- = 0$ (the ideal $\mathcal{I}$ in this case is generated by x_+x_-).

The linear relation is $x_+ - x_- = 0$, so the ideal $\mathcal{I}$ is generated by $x_+ - x_-$.

This means that the cohomology ring of $\mathbb{C}P^1$ is

$$\mathbb{C}[x_+, x_-]/ < x_+ - x_-, x_+x_- > \cong \mathbb{C}[x_+]/ < (x_+)^2 > .$$

Example 3: $H^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$

The moment polytope is a square with vertices $F_{++} = (+1, 1)$, $F_{+-)=(1,-1),}$ $F_{-+} = (-1, 1)$, $F_{--} = (-1, -1)$. There are four faces, each determined by setting one of the coordinates (either the first or the second) to a constant value of $+1$ or -1. Write these as $D_{+\cdot}, D_{-\cdot}.D_{\cdot +}, D_{\cdot -)}$ (where $\cdot$ denotes the coordinate that is allowed to vary). Put corresponding degree 2 generators $x_{+\cdot}, x_{-\cdot}, x_{\cdot +}, x_{\cdot -}$.

$D_{+\cdot} \cap D_{\cdot} = \emptyset$, and likewise $D_{\cdot +} \cap D_{\cdot -} = \emptyset$. This gives two relations $x_{+\cdot}x_{-\cdot} = 0$ and $x_{\cdot +}x_{\cdot -)=0}$. These generate the ideal $\mathcal{I}$.

The linear relations are as follows. The image of π^* is generated by $x_{\cdot +} - x_{\cdot -}$ and $x_{+\cdot} - x_{-\cdot}$...

This means

$$H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) = \mathbb{C}[x_{+\cdot}, x_{-\cdot}, x_{\cdot+}, x_{\cdot-}]/ < x_{+\cdot} - x_{-\cdot}, x_{\cdot+} - x_{\cdot-}, x_{+\cdot}x_{-\cdot}, x_{+\cdot}x_{-\cdot} >$$

$$\cong \mathbb{C}[x_{\cdot+}, x_{+\cdot}]/ < (x_{\cdot+})^2, (x_{+\cdot})^2 > .$$