MAT1312 Exercises 2

Due date: November 2, 2016

(1) Describe the symplectic quotient $M_{m_1,\ldots,m_n}$ obtained from the action of $U(1)$ on $\mathbb{C}^n$ specified by (nonzero) integers $(m_1,\ldots,m_n)$. If $M_{m_1,\ldots,m_n}$ is a smooth manifold, show that it is equipped with the action of an (n-1)-dimensional torus and describe the moment polytope for this torus action.

(2) Show that the symplectic volume of a toric manifold is equal to the Euclidean volume of its moment polytope.

(3) Show that $\mathbb{C}P^2$ is the toric manifold whose moment polytope is the isosceles right triangle.

(4) Exhibit explicitly the subsets of $\mathbb{C}P^3$ for which the stabilizer under the standard action of $U(1)^3$ is

(a) isomorphic to $U(1)$

(b) isomorphic to $U(1)^2$

(c) isomorphic to $U(1)^3$.

Describe the images of these subsets under the moment map $\mu_{U(1)^3}$: show that they are

(a) facets (i.e. 2-dimensional faces)

(b) edges (i.e. 1-dimensional faces)

(c) vertices.

(5) **(Example of projection of moment polytopes)**

(a) Give an approximate sketch of the image $\mu_H(M)$ in the case when $M = \mathbb{C}P^3$, $T = U(1)^3 \subset U(1)^4$ acting in the usual way (so the moment polytope is a 3-simplex in $\mathbb{R}^4$) and $H = U(1)^2$ acting via some embedding in $U(1)^3$: choose the orthocomplement of Lie(H) in Lie(T) to be some axis $\mathbb{R}\hat{v}$ through the origin in $\mathbb{R}^3$. (The image will depend on the axis $\mathbb{R}\hat{v}$ chosen: draw the axis you have chosen.)

(b) What are the *regular values* for the moment map of H?

(6) Suppose $\mathbb{C}^{p+q}$ is acted on by $U(1)$ by the map

$$u \in U(1) : (z_1,\ldots,z_p,w_1,\ldots,w_q) \mapsto (uz_1,\ldots,uz_p,u^{-1}w_1,\ldots,u^{-1}w_q).$$

How does the reduced space $\mu^{-1}(\epsilon)/U(1)$ change as we pass from $\epsilon > 0$ to $\epsilon < 0$? (Consider in particular the case $p = 1$.)

(7) **(Pushforward of the moment map under a collection of weights)**

Let $T = \{(z_1,z_2,z_3) \in U(1)^3 : z_1z_2z_3 = 1\}$ act on $\mathbb{C}^3$ via the weights β_1,β_2,β_3 given by

$$\beta_1(X_1,X_2,X_3) = X_1 - X_2,$$
$$\beta_2(X_1,X_2,X_3) = X_2 - X_3,$$
$$\beta_3(X_1,X_2,X_3) = X_1 - X_3.$$

(Thus $\beta_3 = \beta_1 + \beta_2$ and β_1 and β_2 form an angle of $2\pi/3$ in $Lie(T) \cong \mathbb{R}^2$. The β_j are the positive roots of $SU(3)$.)

Compute the pushforward of the Liouville measure under the moment map for this action of T, or equivalently the pushforward of Lebesgue measure from $(\mathbb{R}^+)^3$ to $Lie(T)$ under the map

$$(y_1,y_2,y_3) \mapsto \sum_j y_j\beta_j.$$

(You will recover a piecewise linear function which is supported on the cone $C(\beta_1, \beta_2, \beta_3)$ and is equal to 0 on the boundary of this cone.)