MAT1312F Exercises 2

Due date February 17, 2020

(1) **(Equivariant cohomology of subgroups)**

(a) Let G be a compact Lie group. Suppose $H \subset G$ is a Lie subgroup of G. Show that there is a map

$$H^*_G(M) \to H^*_H(M).$$

(Either use the Cartan model, or else use the existence of a fibration $M \times_H EH \to M \times_G EG$. This fibration exists since, as EG is a contractible space on which G acts freely, it is also a contractible space on which H acts freely, and hence may be identified with EH.)

(b) Show that in particular there is a map $H^*_G(M) \to H^*_T(M)$ where T is the maximal torus of G.

(c) In (b), let M be a point. Using the Cartan model to identify $H^*_G(pt) \cong S(\mathfrak{g}^*)^G$ and $H^*_T(pt) \cong S(\mathfrak{t}^*)$, show that the image of the map $H^*_G(pt) \to H^*_T(pt)$ (which is in fact isomorphic to $H^*_G(pt)$) is $S(\mathfrak{t}^*)^W \subset S(\mathfrak{t}^*)$.

(d) Show that if $G = SU(2)$ and $T = U(1)$ the image of $H^*_G(pt)$ in $H^*_T(pt) \cong \mathbb{R}[X]$ is the polynomials of even degree in X.

(2) **(Functorial properties of equivariant cohomology)** Show that if M_1 and M_2 are two manifolds equipped with actions of G and $f : M_1 \to M_2$ is a G-equivariant map then it induces a map $f^* : H^*_G(M_2) \to H^*_G(M_1)$. (This correspondence has the property that if M_3 is another manifold with a G action and $g : M_2 \to M_3$ is G equivariant then $(g \circ f)^* = f^* \circ g^*$.)

(3) (a) Show that if G is a compact Lie group and H is a Lie subgroup of G then

$$H^*_G(G/H) \cong H^*_H(pt).$$

(Hint: rewrite $EG \times_G G/H$ as a quotient by H.)

(b) Let $G = SO(3)$ and $H = U(1)$. What is the ring $H^*_G(S^2)$ where $SO(3)$ acts on S^2 by rotation? (Hint: write $S^2 = SO(3)/U(1)$.)

(4) **(Functorial properties of pushforward)** Let M be a G-manifold. Show that if $\alpha \in S(\mathfrak{g}^*)^G$ and $\pi^*\alpha$ is its inclusion as an element of $\Omega^*_G(M)$, and $\beta \in \Omega^*_G(M)$, then

$$\int_M (\beta \pi^* \alpha) = (\int_M \beta)\alpha :$$

It follows that the map $\int_M : H^*_G(M) \to H^*_G(pt)$ is a homomorphism of $H^*_G(pt)$-modules.