Wave trace formula

Weyl's law

$$S(t) = \sum_{\lambda \in Sp_\Delta} e^{-i\sqrt{\lambda} t} \quad \text{is a tempered distribution} \in S'(\mathbb{R})$$

$$\Re$$

$$\sum_{\lambda \in Sp_\Delta} \cos(\sqrt{\lambda} t)$$

Recall

$-\Delta$ admits a orthonormal basis $\{u_n\}$ of $L^2(\mathbb{T}_1) / L^2(M, vol_g)$

$$\underset{\substack{\Uparrow \\ L^2}}{(-\Delta) u} = -\Delta \sum_{n=1}^{\infty} u_n \langle u_n, u \rangle = \sum_{n=1}^{\infty} \lambda_n u_n \langle u_n, u \rangle$$

$\forall f \in C^\infty(\underset{t}{\mathbb{R}} \times \underset{x}{\mathbb{R}}; \mathbb{R})$ $\quad f(t, -\Delta) = \sum_{n=1}^{\infty} f(t, \lambda_n) u_n \langle u_n, u \rangle$ $\quad (*)$

is defined on subspace of L^2

Examples:

- $f(x) = e^{-\pi x t} \Rightarrow f(t, -\Delta) = e^{-\pi \Delta t}$

Heat operator

- $f(x) = e^{-i\sqrt{x} t} \Rightarrow e^{-i\sqrt{-\Delta} t} = U(t)$

wave operator.

By $(*)$

$$[U(t)v](t,x) = \sum_{n=1}^{\infty} e^{-i\sqrt{\lambda_n}\, t} \langle u_k, v\rangle \cdot u_n(x)$$

Lemma $[U(t)v](t,x)$ satisfies the following problem:

wave equation.

$$\begin{cases} \left[\dfrac{\partial^2}{\partial t^2} + [-\Delta]\right] u(t,x) = 0 \\[2mm] u(t,x)\big|_{t=0} = v(x) \\[2mm] \dfrac{\partial}{\partial t} u(t,x)\big|_{t=0} = 0 \end{cases}$$

Pf:

- $$\left(\frac{\partial^2}{\partial t^2} + [-\Delta]\right) \sum_{n=1}^{\infty} e^{-i\sqrt{\lambda_n}\, t} u_n(x) \langle u_n, v\rangle$$
  $$= \sum_{n=1}^{\infty} \cancel{\left(-\lambda_n + \lambda_n\right)} e^{-i\sqrt{\lambda_n}\, t} u_n(x) \langle u_n, v\rangle$$

- $$[U(0)v](x) = \sum_{n=1}^{\infty} u_n(x) \langle u_n, v\rangle = v$$

- $$\frac{\partial}{\partial t} u(t,x) = \sum_{n=1}^{\infty} \cancel{(-i)\sqrt{\lambda_n}\, e^{-i\sqrt{\lambda_n}\, t} u_n \langle u_n, v\rangle} \to 0$$

$$[\mathcal{U}(t)\,\cdot\,](x) = \sum_{n=1}^{\infty} e^{-i\sqrt{\lambda}t}\, u_n(x) \langle u_n, \cdot \rangle_{L^2}$$

$$= \int \underbrace{\sum_{n=1}^{\infty} e^{-i\sqrt{\lambda}t}\, u_n(x)\, \overline{u_n(y)}}\, \cdot(y)\, dy$$

$$[\mathcal{U}(t)\,v](x) = \int \ulcorner K(t,x,y) \lrcorner\, v(y)\, dy$$

Distribution kernel of $\mathcal{U}(t)$

$$\ulcorner \int K(t,x,x)\, dx \lrcorner$$

$$= \sum_{n=1}^{\infty} e^{-i\sqrt{\lambda}t} \int \underbrace{u_n(x)\, \overline{u_n(x)}\, dx}_{1}$$

$$= \sum_{n=1}^{\infty} e^{-i\sqrt{\lambda_n}t} = S(t)$$

What are the singularity properties of $K(t,x,y)$?

Answer: K will be smooth in R^{c} where

$$\mathcal{M} \supset R = \{(t,x,y) \mid \exists \text{ geodesic of length } t \text{ connecting } x \text{ to } y\}^{c}$$

* means $\displaystyle\int k(t,x,y)\,\chi^n\, v(y)\,dy = \int \varphi(t,x,y)\,\chi^n\, v(y)\,dy$ $\quad \forall v \in C^\infty$

let $\chi^n_R \to 1\!\!1_R$ where $\varphi \in C^\infty((\mathbb{R}\times M)\times M)$
$\hat{C^\infty}$ bump functions ✓

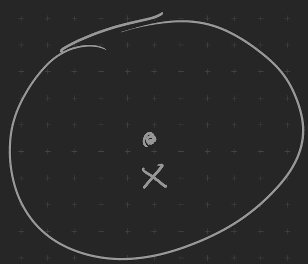
Recall $\displaystyle S(t) = \int_M k(t,x,x)\,dx$, so

"it makes sense" to believe that $S(t)$ will be smooth for $t \neq$ length of a geodesic loop

Chazarain
Guillemin + Duistermaat

Pretend we are in $\mathbb{T}^n$, (t small)

so really we are in $B \subset \mathbb{R}^n$



How does $k(t,x,y)$ looks like?

$u(x)$ supported in B

$[\check{\mathcal{F}}\, u(\xi)](x)$ $\quad = \langle u_\xi , u\rangle$

$$u(x) = \int e^{2\pi i\langle \xi, x\rangle}\,\hat{u}(\xi)\,d\xi$$

$$U(t) = \int e^{-i2\pi\|\xi\| t}\, e^{i2\pi\langle \xi, x\rangle}\, e^{-2\pi i\langle \xi, y\rangle}\,\hat{u}(\xi)\,d\xi$$

$$[U(t)u](x) = \int e^{-2\pi i(\|\xi\| t + \langle \xi, x-y\rangle)}\,\hat{u}(\xi)\,d\xi\,dy$$

$$\hookrightarrow \nabla_\xi(\ ) = \left(\frac{\xi}{\|\xi\|}t + (x-y)\right)$$

WARM UP — Stationary phase.

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau\phi(\xi)}\, a(\xi)\, d\xi$$

- ϕ smooth function of $\mathbb{R}^n$, $\mathbb{R}$-valued. phase
- $a \in C_0^\infty(\mathbb{R}^n)$. amplitude

First notice: $$|I(\tau)| = \left| \int_{\mathbb{R}^n} e^{i\tau\phi(\xi)} a(\xi)\, d\xi \right|$$

$$\le \int_{\mathbb{R}^n} |\cancelto{1}{e^{i\tau\phi}}|\, |a(\xi)|\, d\xi$$

$$\le \|a\|_{L^1} < \infty.$$

- Example - $\phi(x) = c$

$$I(\tau) = \int e^{2\pi i\tau\phi}\, a(\xi)\, d\xi$$

$$= e^{2\pi i\tau c} \int a(\xi)\, d\xi$$

$$= A \cdot e^{2\pi i\tau c}$$

. shows no decay at ∞.

- let $\phi(x) = x_\ell$ $\quad \ell = 1, \dots, n$.

$$I(\tau) = \int e^{2\pi i\tau\xi_\ell}\, a(\xi)\, d\xi$$

$$= \hat{a}(0, \dots, \tau, \dots 0)$$

$\uparrow$ i-th component.

if τ is large $\hat{a}(x)$ decays as $x \to \infty$

$I(\tau)$ will also decay as $\tau \to \infty$. In particular I is rapidly decaying.

Def $I(\tau)$ is rapidly decaying if $\forall N > 0$,

$$|\tau^N I(\tau)| < \infty$$

- Non-stationary phase regime.

Lemma Assume $\nabla\phi$ does not vanish on a nbhd A of support a then $I(\tau)$ is rapidly decaying.

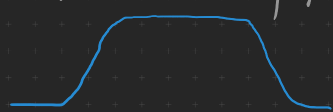
Pf Fact:

$$\nabla\phi \cdot \nabla \left(e^{2\pi i \tau \phi}\right) = 2\pi i \tau \|\nabla\phi\|^2 e^{2\pi i \tau\phi} \quad \checkmark$$

$$\nabla\phi\cdot\nabla = \sum_i \partial_i\phi\,\partial_i,$$

Let χ be C^∞ bump function s.t. $\chi(z) = 1$ if $z \in \operatorname{supp} a$

0 if $z \notin A$



$$L := \frac{\chi}{\|\nabla\phi\|^2}\nabla\phi\cdot\nabla$$

Diff. operator:

$$L e^{i\tau\phi(\xi)} = 2\pi i \tau \chi \, e^{i\tau\phi(\xi)}$$

$$I(t) = \int a \, e^{i\tau\phi(\xi)} = \int a\chi \, e^{i\tau\phi(\xi)} = \int \frac{a}{i\tau} L e^{i\tau\phi(\xi)}$$

$$= \frac{1}{i\tau} \int a(\xi) L e^{i\tau\phi(\xi)} \, d\xi = \frac{i}{\tau} \int e^{i\tau\phi} L^T a$$

$$= \frac{(2\pi i)^n}{\tau^n} \int e^{i\tau\phi} (L^T)^n a$$

$|I(\tau)| \lesssim \tau^{-n} \| (L^T)^n a \| \Rightarrow I(\tau)$ is rapidly decaying. $\square$

<u>Rmk</u> The same arguments show rapid decay for $\phi(\xi, w)$ and w ↰ parameters (e.g. $t, x, y \ldots$)

decay is <u>uniform</u> in w. (a is cptly supported)

- Stationary phase $\nabla\phi = 0$ at $\bar{\xi} \in \operatorname{supp} a$,
  $\det(\text{Hessian } \phi) \neq 0$.

<u>Prototype:</u> $\phi(\xi) = \dfrac{\langle \xi, A\xi \rangle}{2}$ A non-degenerate.

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau \frac{\langle \xi, A\xi \rangle}{2}} a(\xi)\, d\xi \quad \text{as } \tau \to \infty$$

$$\sim \left(\frac{2\pi}{\tau}\right)^{n/2} \frac{e^{i\pi\sigma/4}}{\sqrt{|\det A|}} \sum_{j=0}^{\infty} \alpha_j(a)(0)\,\tau^{-j}$$

where • σ is the signature of A

(i.e $\#\{\lambda > 0\} - \#\{\lambda < 0\}$)

- $\alpha_0(a)(0) = a(0)$
- $\alpha_j(a)$ is a diff. operator of order $2j$