Sketch of wave trace formula for Riem. manifolds.

Let (M, g) be Riemannian mfold cpt closed (no boundary)

$g(p, X, Y) = g(p, Y, X)$ bi-linear in X, Y

$p \in M$, $X \in T_pM$, $Y \in T_pM$

smooth in p.

$g(p, X, X) > 0$ — g defines an inner product on TM

which allows to compute lengths, angles, volumes

$\rightsquigarrow$ Riemannian Volume associated to g

if U is a chart of M, locally

$$g^{ij}(x) = g\left(x, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

Metric tensor in coordinate

Volume form $\sqrt{\det g}\, dx_1 \wedge \dots \wedge dx_n$

- This allows to define integration (and $L^2(M, Vol)$)
- A Riemannian metric allows to construct a second-order differential operator $-\Delta_g$ on $C^\infty(M)$

in charts:

$$-\Delta_g f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x_j}\right)$$

Example let $(M, g) = (\mathbb{T}_\Lambda, \text{Euclidean})$

$g^{ij} = $ Identity, and $-\Delta_g f = -\Delta f$

Def $-\Delta_g$ is Laplace Beltrami operator

Facts: $-\Delta_g$ is a self-adjoint positive operator in $L^2(M, \mathrm{Vol})$

- $-\Delta_g$ admits an orthonormal basis of eigenfunctions of $-\Delta_g$ (recall for $M = \mathbb{T}_\Lambda$

$\forall \xi \in \Lambda^* \quad x \longmapsto e^{-2\pi i \langle \xi, x \rangle}$ has

e-value $4\pi^2 \|\xi\|^2$)

$\rightarrow$ spectrum of $-\Delta_g$ is discrete, accumulate only at ∞.

Fact: $\# \{ Sp_{\Delta_g}(M, g) \cap [0, L] \} \sim L^{n/2}$

Weyl's Law: $\# \{ \quad \} = \underbrace{C_n}_{= \frac{\Omega_n}{(2\pi)^n}} \cdot \mathrm{Vol}(M) L^{n/2} + O(L^{n/2})$

Ω_n volume of B_n

Wave trace of $-\Delta_g$:

Claim $\sum_{\lambda \in Sp_\Delta} e^{i\sqrt{\lambda} t}$ makes sense as a tempered distribution

Pf: Recall $S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) \mid \|x^n D^m f\|_\infty < \infty \ \forall n, m \in \mathbb{Z}_{\geq 0} \}$

$S'(\mathbb{R}) = \{$ continuous linear functionals on $S \}$

is the set of tempered distributions. WTP $\forall f \in S$

$$\int \sum_{\lambda \in Sp\Delta} e^{i\sqrt{\lambda}t} f(t)\, dt < \infty.$$

Recall $$\sum_{\lambda \in Sp\Delta} e^{i\sqrt{\lambda}t} = \mathcal{F}\left(\sum_{\lambda \in Sp\Delta} \delta(\omega - \sqrt{\lambda})\right)$$

Since $\#\{Sp\Delta \cap [0,L]\}$ is at most polynomial

then $\forall f \in S$ $$\left[\sum_{\lambda \in Sp\Delta} \delta(\omega - \sqrt{\lambda})\right] f =$$

$$\lim_{L\to\infty} \left[\sum_{\lambda \in Sp\Delta \cap [0,L]} \delta(\omega - \sqrt{\lambda})\right] f =$$

Notice $$\sum_{\lambda \in Sp\Delta \cap [L, L+1]} \delta(\omega - \sqrt{\lambda})$$ by Weyl's law

$$\leq \#\{Sp\Delta \cap [0, L+1]\} \cdot \|f\|_{\{|x| \geq L\}}$$

$$\leq (L+1)^{-n/2} \cdot L^{-10n} < L^{-2n}$$

by Weyl's law — Schwartz — summable, $\square$

$$\sum_{\lambda \in Sp_\Delta(M)} e^{i\sqrt{\lambda}t} = \text{.............}$$

geometric object

Theorem (Chazarain 1976, Duistermaat-Guillemin 1978, Andersson-Melrose 1978 (for bounded domains $\Omega \subset \mathbb{R}^n$))

1) $S(t) = \sum_{\lambda \in Sp_\Delta(M)} e^{i\sqrt{\lambda}t}$ is C^∞ on $\mathbb{R} \setminus LS(M,g)$

$LS = \mathbb{Z}\{\text{length of periodic geodesics on } (M,g)\} \cup \{0\}$

$(\text{sing supp } S(t) \subset LS(M,g))$

2) in fact each periodic geodesic contributes to the distribution:

$$S(t) = \sum_{\gamma \text{ periodic geodesics}} E_\gamma(t)$$

where $E_\gamma(t)$ is singular at $t = \text{length}(\gamma)$ and $\exists$ singularity expansion for E_γ

i.e. $E_\gamma(t) = c_\gamma t^{-\alpha_\gamma} + \text{.....}$

$\alpha_\gamma > 0$ $\quad \alpha_\gamma = 1/2$ if γ is non-deg

$= 1/2 + d$ for a d-parameter family of geodesics

C_γ has an explicit expression in terms of dyn quantities

Maslov index (e.g. $\det(P - \mathbb{1})$)

Poincaré map

$= \sigma$

Note: It is known (due to V. Ivrii) that

$$N(L) = \#\{Sp_\Delta \cap [0,L]\} = c_n \operatorname{Vol} M \cdot L^{n/2} + c'_{n-1} \operatorname{Vol} \partial M \cdot L^{n-1/2} + o(L^{n-1/2})$$

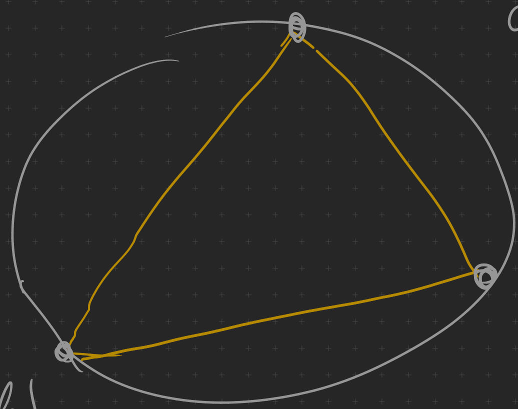
if the set of periodic geodesics has "zero measure" (i.e. $P = \{(x,\xi) \in T^*M$ s.t. $\exists t\ \varphi_t(x,\xi) = (x,\xi)\}$)

Fun Remark: for billiards one can classify periodic geodesics depending on # collisions w/ boundary. $\rightsquigarrow q$

Periodic geodesics $= \bigcup_{q \geq 2}$ Periodic geo. w/ q bounces

Proved that $\{\ldots\}$ has 0 (Liouville) measure for $q = 2, 3, 4$

$q = 3$



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• Weyl's Law for Bounded Domains in $\mathbb{R}^n$.

Example: let $B_\ell = [0,\ell]^n$ be a cube of side ℓ in $\mathbb{R}^n$

$$\boxed{\circledast \quad \begin{aligned} -\Delta u &= \lambda u \\ u(\partial B) &= 0 \end{aligned}} \; ?$$

Recall; $\forall\, k \in \mathbb{Z}^n$ where $u \in L^2(B_\ell)$

$\forall\, x \in B$ $\quad u_k(x) = e^{2\pi i \frac{1}{\ell}\langle k, x\rangle}$ satisfies $-\Delta(u_k) = 4\pi^2 \frac{1}{\ell^2}\|k\|^2 u_k$

↗ periodic conditions

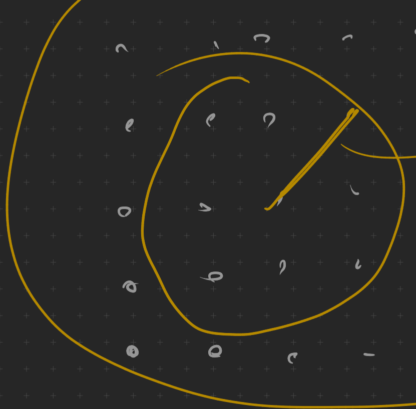
$$\boxed{u_k(x) = \sin\left(\pi \tfrac{1}{\ell} k_1 x_1\right) \cdot \sin\left(\pi \tfrac{1}{\ell} k_2 x_2\right) \cdot \ldots \sin\left(\pi \tfrac{1}{\ell} k_n x_n\right)}$$

satisfy Dirichlet conditions

$$Sp_\Delta(B_\ell) = \left\{ \frac{\pi^2}{\ell^2}\|k\|^2 \right\}_{k \in \mathbb{Z}^n}$$

$$\# \left\{ \frac{\pi^2}{\ell^2}\|k\|^2 \right\}_{k \in \mathbb{Z}^n} \cap [0, L]$$

$$= \# \left\{ k \in \mathbb{Z}^n \;\middle|\; \|k\|^2 \frac{\pi^2}{\ell^2} < L \right\}$$



$L^{1/2} \cdot \frac{\ell}{\pi}$

$$\|k\| < L^{1/2} \cdot \frac{\ell}{\pi}$$

$$\simeq \mathrm{Vol}\left(B\left(L^{1/2} \cdot \frac{\ell}{\pi}\right)\right)$$

$$\frac{\omega_n}{\pi^n} \cdot L^{n/2} \cdot \underbrace{\ell^n}_{Vol(B_\ell)} = C_n L^{n/2} \cdot Vol(B_\ell) + o(L^{n/2})$$

so Weyl's law holds on B_ℓ.

How to consider $\Omega \subset \mathbb{R}^n$ which are not Boxes?

Functional detour.

Sobolev space $H^1(\Omega) \subset L^2(\Omega)$

Consider $u, v \in C^1(\Omega)$, define inner product.

$$\langle u, v\rangle_{H^1} := \int_\Omega u \cdot v \, dV + \int_\Omega \langle \nabla u, \nabla v\rangle dV$$

$$\| u \|_{H^1} = \sqrt{\langle u, u\rangle}.$$

$H^1 =$ closure of C^1 w.r.t. $\| \cdot \|_{H^1}$ Sobolev space

$H^1 \subset L^2$ since $\| \ \|_{H^1} \geq \| \ \|_{L^2}$

Notice: $C_0^\infty(\Omega) \subset \cancel{H^1(\Omega)}$

let $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ wrt $\| \ \|_{H^1}$

$H_0^1 \subset H^1 \subset L^2$

let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then $\forall v \in C_0^\infty(\Omega)$

$$\int_\Omega \langle \nabla u, \nabla v \rangle = \underset{\text{by parts}}{=} \cancelto{0}{\int_{\partial\Omega}} \int (-\Delta) u\, v$$

since $v \in C_0^\infty$

so if $-\Delta u = \lambda u$ we have

$$\int_\Omega \langle \nabla u, \nabla v \rangle = \lambda \int u v \qquad (**)$$

$\forall\, v \in C_0^\infty(\Omega)$

Def we say $u \in H_0^1(\Omega)$ is a weak solution of Dirichlet problem $(*)$ if $(**)$ holds for any $v \in C_0^\infty$

Interpret $(*)$ as a variational problem for functions in $H_0^1(\Omega)$.

Let $H_0^1(\Omega) \ni u \neq 0$. Define the Rayleigh quotient

$$\rho(u) = \frac{\int \langle \nabla u, \nabla u \rangle \, dV}{\|u\|_{L^2}^2}$$

Note: 1) Recall $-\Delta$ on Ω admits a orthonormal basis of e-functions for L^2, then

$$\forall\, i,j \qquad \langle \nabla u_i, \nabla u_j \rangle_{L^2} = \lambda_i \langle u_i, u_j \rangle$$

$$\langle \qquad \rangle = \delta_{ij} \lambda_i$$

$$\langle u_i, u_j \rangle_{H^1} = \delta_{ij} (1+\lambda_i)$$

$\tilde{u}_i = \frac{1}{\sqrt{1+\lambda_i}} u_i$ form a H^1-orthonormal basis of weak solutions of (*)

Lemma (Variational Principle I):

Let $H^{(m)} = \operatorname{span}(u_1, \dots, u_m)$

$$H_{(m)} = H^{(m)\perp} \cap H_0^1$$

then $\lambda_n = \inf_{u \in H_{(n-1)}} \rho(u) = \sup_{u \in H^{(n)}} \rho(u)$

Pf $\rho(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$

$$\|u\|^2 = \langle u, u \rangle_{L^2} = \sum_n \langle \langle u, u_n \rangle u_n, u \rangle$$

$$= \sum_n \langle u, u_n \rangle^2$$ Parseval's Identity.

$$\|\nabla u\|^2 = \langle \nabla u, \nabla u \rangle_L = \sum_n \langle u, u_n \rangle \langle \nabla u_n, \nabla u \rangle$$

$$\overset{(*)}{=} \sum_n \langle u, u_n \rangle \lambda_n \langle u_n, u \rangle$$

$$= \sum_n \lambda_n \langle u, u_n \rangle^2$$

if $u \in H_{(n-1)}$, then $\langle u, u_n \rangle = 0$

for $n = 1, \ldots, n-1$

$$\|\nabla u\|^2 = \sum_{k \geq n} \lambda_k \langle u, u_k \rangle^2$$

$$\geq \lambda_n \sum_{k \geq n} \langle u, u_k \rangle^2$$

$$\geq \lambda_n \sum_{k \geq 0} \langle u, u_k \rangle^2 = \lambda_n \cdot \|u\|^2$$

on $H_{(n-1)}$ $\quad \rho(u) = \dfrac{\|\nabla u\|^2}{\|u\|} \geq \lambda_n$

$$\inf_{u \in H_{(n-1)}} \geq \lambda_n$$

(prove $= \lambda_n$ by letting $u \to u_n$ in H_0^1

$\square$