The wave trace on $\overline{\Gamma\backslash}$.

Lecture 5

- PSF: $\forall f \in S(\mathbb{R}^n)$, $\forall \Lambda$ full-rank lattice $\subset \mathbb{R}^n$

$$\sum_{\xi \in \Lambda^*} f(\xi) = \operatorname{covol} \Lambda \sum_{s \in \Lambda} \check{f}(s)$$

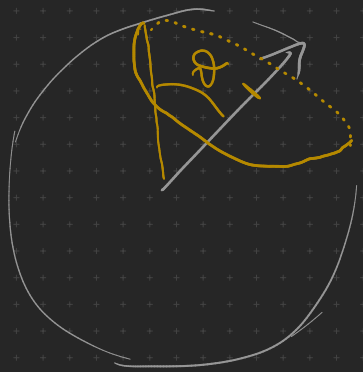
Assume $f(\xi) = \varphi(\|\xi\|)$ is radial function

$$\sum_{\xi \in \Lambda^*} \varphi(\|\xi\|) = \operatorname{covol} \Lambda \sum_{s \in \Lambda} \check{\varphi}(\|s\|)$$

where $$\check{\varphi}(\ell) = \int_0^\infty \varphi(\omega)\, \omega^{n-1} K_{n-2}(\omega \ell)\, d\omega$$

and $$K_m(x) = \omega_m \int_0^\pi e^{2\pi i x \cos\vartheta} \sin^m \vartheta \, d\vartheta$$

ω_m = volume of S^m



Observe: Bessel function

$$K_{n-2}(x) = x^\nu J_\nu(x)$$

where $\nu = \frac{n-2}{2}$

Lemma: Compute Fourier transform of $K_m(x)$

$$\hat{K}_m(\xi) = \omega_m \mathbb{1}_{[-1,1]}(\xi) \cdot \sqrt{1-\xi^2}^{m-1}$$

Pf Notice $\frac{K_m(x)}{\omega_m} = \int_0^{\pi} e^{2\pi i x \cos\vartheta} \sin^m\vartheta \, d\vartheta$

$\vartheta = \pi/2 + \vartheta'$

$$= \int_{-\pi/2}^{\pi/2} e^{-2\pi i x \sin\vartheta'} \cos^m\vartheta' \, d\vartheta'$$

This shows $K_m(x) = K_m(-x)$

$$= \int_{-\pi/2}^{\pi/2} \cos(2\pi x \sin\vartheta) \cos^m\vartheta \, d\vartheta$$

$$\hat{K}_m(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi x} \int_{-\pi/2}^{\pi/2} e^{-2\pi i x \sin\vartheta} \cos^m\vartheta \, d\vartheta \, dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos^m\vartheta \int_{\mathbb{R}} e^{2\pi i x(\xi - \sin\vartheta)} \, dx \, d\vartheta$$

$$\int_{-\pi/2}^{\pi/2} \cos^m\vartheta \, \delta_{\sin\vartheta}(\xi) \, d\vartheta$$

let $\tau = \sin\vartheta$

$d\tau = \cos\vartheta \, d\vartheta$

$$\int_{-1}^{1} \sqrt{1-\tau^2}^{\,m-1} \, \delta_{\tau}(\xi) \, d\tau$$

$$= \mathbb{1}_{|\xi|\le 1}(\xi)\sqrt{1-\xi^2}^{\,m-1} \qquad \square$$

Recall: PSF can be written in distro. form as:

$$\left[\sum_{\xi\in\Lambda^*}\delta_{\|\xi\|}\right]\varphi = \text{covol.}\int_0^\infty \sum_{e\in\Lambda} K_{n-2}(\eta\|e\|)\,\xi^{n-1}\,\varphi(\eta)\,d\eta$$

$$\sum_{\xi\in\Lambda^*}\delta_{\|\xi\|} = \text{covol}\,\Lambda \sum_{e\in\Lambda} K_{n-2}(\eta\|e\|)\,\eta^{n-1}$$

Take Fourier transform:

$$\sum_{\xi\in\Lambda^*} e^{2\pi\|\xi\|t} = \text{covol}\,\Lambda\,\mathcal{F}\left(K_{n-2}(\eta\|e\|)\,\eta^{n-1}\right)$$

$$= \frac{\text{covol}\,\Lambda}{(2\pi)^{n-1}}\frac{\partial^{n-1}}{\partial t^{n-1}}\sum_{e\in\Lambda}\mathcal{F}\left(K_{n-2}(\eta\|e\|)\right)$$

since $\mathcal{F}(x\varphi) = [\mathcal{F}\varphi]$

$$= \frac{\text{covol}\,\Lambda}{(2\pi)^{n-1}}\frac{\partial^{n-1}}{\partial t^{n-1}}\sum\frac{1}{\|e\|}\hat{K}_{n-2}\left(\frac{t}{\|e\|}\right)$$

$$\mathcal{F}(\varphi(\alpha x)) = \frac{1}{|\alpha|}\hat{\varphi}\left(\frac{\omega}{\alpha}\right)$$

$$= \frac{\operatorname{covol}\Lambda}{(2\pi)^{n-1}} \sum_{e\in\Lambda} \frac{1}{\|e\|} \mathbb{1}_{[-1,1]}\left(\frac{t}{\|e\|}\right) \cdot \frac{\partial^{n-1}}{\partial t^{n-1}} \left(1-\left(\frac{t}{\|e\|}\right)^2\right)^{\frac{n-3}{2}}$$

$$= \frac{\operatorname{covol}\Lambda}{(2\pi)^{n-1}} C_n \sum_{e\in\Lambda} \frac{1}{\|e\|} \mathbb{1}_{[-1,1]}\left(\frac{t}{\|e\|}\right) \times$$

universal numerical constant

$$\times \left(1-\left(\frac{t}{\|e\|}\right)^2\right)^{\frac{n-3}{2}} \cdot (-n-1)\cdot \frac{(2t)^{n-1}}{\|e\|^{2(n-1)}}$$

$$+ \text{lower order of singularity} \dots$$

So RHS is manifestly singular at $t = \|e\|$ for $e\in\Lambda \iff e \in LS(\Pi_\Lambda)$

singularity of order $\frac{n-3}{2} - \frac{2n-2}{2} = \frac{-n-1}{2}$

Hence: wave trace is singular at $t\in LS(\Pi_\Lambda)$

What happens if $t \notin LS(\Pi_\Lambda)$?

$$\sum_{e\in\Lambda} \mathcal{O}\left(\|e\|^{-1} \cdot \|e\|^{-2(n-1)}\right) = \mathcal{O}\left(\|e\|^{-2n+1}\right)$$

is it smooth? $\sum_{e\in\Lambda,\ \|e\|\geq t>0} \|e\|^{-2n+1}$ smooth if Λ is a rank-n lattice

$\Rightarrow$ if $t \notin LS(\overline{\Pi}_\Lambda)$ RHS converges.

Core can even show convergence to C^∞ function)

$$\Rightarrow \sum_{s \in \Lambda^*} e^{2\pi i \|s\| t}$$

singular at $t \in LS(\overline{\Pi}_\Lambda)$ $\sim t^{\frac{-n-1}{2}}$

smooth at $t \notin LS(\overline{\Pi}_\Lambda)$

$$\Rightarrow \operatorname{sing\,supp}\left(\sum e^{-i\sqrt{\Delta}t}\right) = LS(\overline{\Pi}_\Lambda)$$

(!!!)