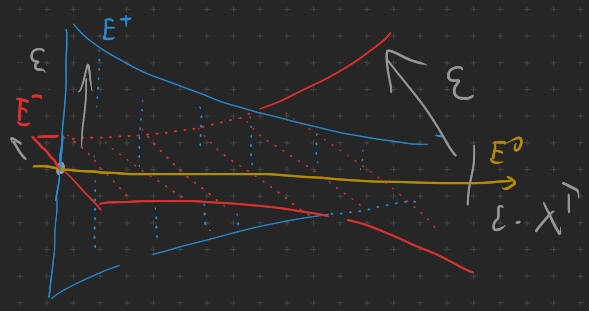
Anosov Closing Lemma Let X be a mfld (smooth),

ϕ_t smooth Anosov flow ($\lambda < 1 < \nu$)

$\forall \alpha \in (\max(\lambda, \nu^{-1}), 1)$

$\exists C, \varepsilon_0$ s.t. if



$\varepsilon = \text{dist}(\phi_T(x), x) < \varepsilon_0$

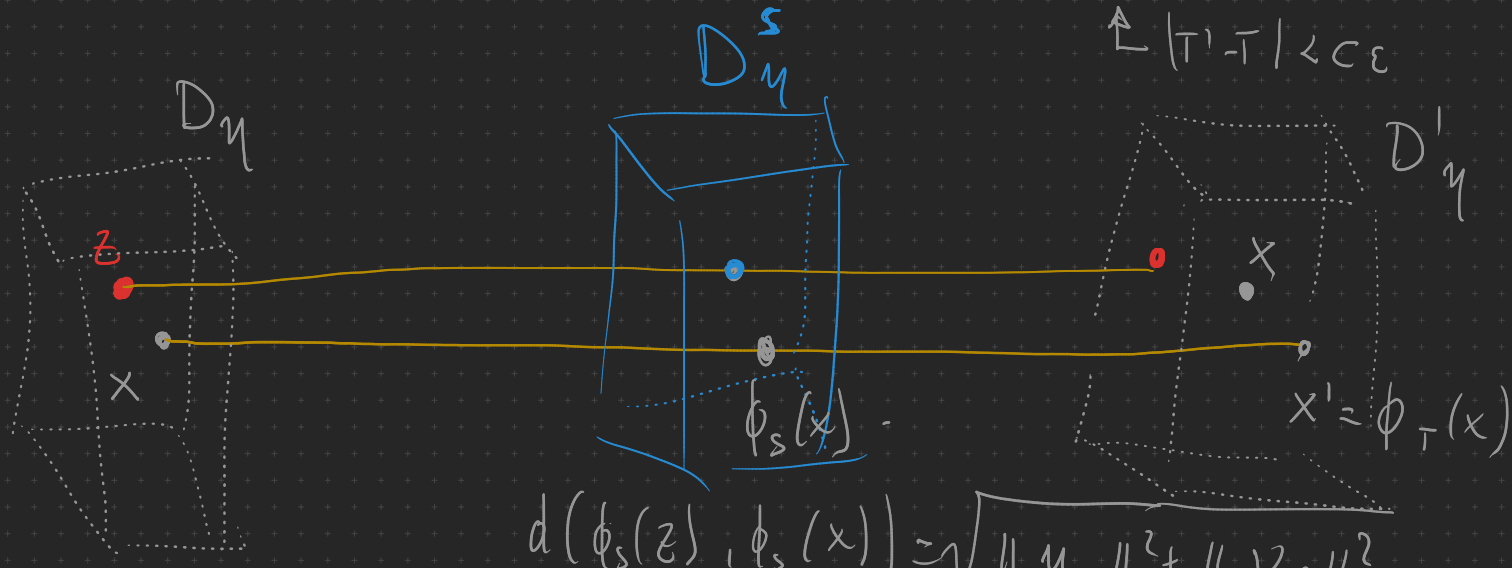
$\exists z \in X$, $z = \phi_{T'}(z)$ $\quad |T - T'| < C\varepsilon$

$\forall s \in (0, T)$ $\text{dist}(\phi_s(x), \phi_s(z)) < C\varepsilon \cdot \alpha^{\min(s, T-s)}$

Pf (Lost line) .....

$\exists z \in X$, $\text{dist}(z, x) < C\varepsilon$ $\quad \phi_{T'}(z) = z$

$\Leftarrow |T' - T| < C\varepsilon$



$d(\phi_s(z), \phi_s(x)) = \sqrt{\|u_s\|^2 + \|v_s\|^2}$

Assume ε to be so small that $x, x', z \in D_\eta^s \cap D_\eta$

In Adapted coords:

$$\phi_s(t, u, v) = \left(t + \tau(t, u, v),\ A^+ u + \alpha^+(t, u, v),\ A^- v + \alpha^-(t, u, v)\right)$$

$A^\pm$ linear $\quad \|A^+\| < \lambda^s \quad \|(A^-)^{-1}\| < \nu^{-s}$

$$\|\tau\|, \|\alpha^{\pm}\|_{C^1} < C\varepsilon$$

Sketch: $z = (0, u, v) \in D_\eta$

$z = (t', u', v') \in D'_\eta$

- $\tilde{D}_\eta$ centered on $z = (0,0,0)$

  $x = (0, \tilde{u}, \tilde{v})$

$\tilde{D}^{T'}_\eta$ centered on $z = (0,0,0)$

$x' = (-\tilde{t}', \tilde{u}', \tilde{v}')$

Def $x'' = \phi_{T'}(x)$ $\quad x'' = (0, \tilde{u}'', \tilde{v}'')$

$\tilde{\tilde{D}}_\eta$

$\|\tilde{u}''\|, \|\tilde{v}\| < C\varepsilon$

$\|\hat{u}\|, \|\hat{v}\| < C\varepsilon$

$$d(\phi_s(z), \phi_s(x)) = \sqrt{\|u_s\|^2 + \|v_s\|^2}$$

for $s \in [0, T'/2)$ $\leq \sqrt{\|\tilde{u}\|^2 \lambda^{2s} + \|\tilde{v}''\|^2 \mu^{-2s}}$

$\leq \alpha^s \sqrt{(C\varepsilon)^2 + (C\varepsilon)^2}$

$\leq \alpha^s C\varepsilon$

$s \in [T'/2, T']$ same estimates .... $\square$

Th'm (Livšic) Let X mfold, ϕ_t be a **transitive** Anosov flow. Let f Hölder on X so that $\int_\gamma f = 0 \quad \forall \gamma$ closed orbit of ϕ_s

Then: $\exists\ \beta$ Hölder on X, diff'ble along ϕ_s s.t.

$$f(x) = \frac{d}{ds} \beta(\phi_s(x))\Big|_{s=0} = \xi \beta \quad (*)$$

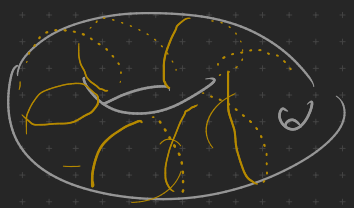
Moreover: β is unique (up to a constant)

Transitive: $\exists$ dense orbit of the flow ϕ_s.

Pf. Let $c \in \mathbb{R}$. Let x_0 be a transitive pt of ϕ_s

$\check{X} := \{\phi_s(x_0)\}_{s \in \mathbb{R}}$ s.t. $\check{X}$ is dense in X

Define β on $\check{X}$ as follows:



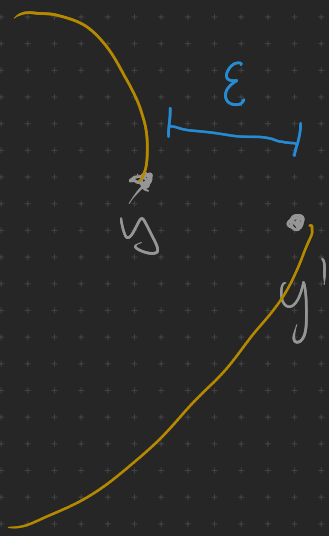
$\forall y \in \check{X}$ $\exists!\ t \in \mathbb{R}$ s.t.

$y = \phi_t(x_0)$.

$$\beta(y) := \int_0^t f(\phi_s(x_0))\, ds + c$$

Of course on $\check{X}$ satisfies (*). I want to extend β to X. It suffices to show (since X is opt) that β (on $\check{X}$) is unif. Hölder continuous.

Let $y, y' \in \check{X}$. Assume $\varepsilon = \text{dist}(y,y') < \varepsilon_0$ (ε_0 given in the Closing Lem.



so: $\exists T \in \mathbb{R}$ s.t.

$$\phi_T(y) = y'$$

by Anosov's Closing Lem

$$\exists z, T' \text{ s.t. } |T' - T| < C\varepsilon$$

$$\text{dist}\left(\phi_s(z), \phi_s(y)\right) < C\varepsilon \cdot \alpha^{\min(s, T-s)}$$

so $\beta(y') - \beta(y) = \int_0^T f(\phi_s(y))\, ds$

$$= \int_0^T f(\phi_s(z)) + \left(-f(\phi_s(z)) + f(\phi_s(y))\right) ds$$

$$= \int_0^{T'} f(\phi_s(z)) + \mathcal{O}(\varepsilon) + \int_0^T f(\phi_s(y)) - f(\phi_s(z))\, ds$$

($|T - T'| < C\varepsilon$)

so $|\beta(y') - \beta(y)| \le C\varepsilon + \int_0^T |f(\phi_s(y)) - f(\phi_s(z))| ds$

Hölder exp: $\Rightarrow$

$\le C\varepsilon + C \int_0^T \text{dist}(\phi_s(y), \phi_s(z))^\beta ds$

$\le C\varepsilon + C\varepsilon^\beta \int_0^T e^{(\log\alpha\beta)\min(s, T-s)} ds$

$O(1)$

$\le C\varepsilon^\beta$

$\Rightarrow \beta$ is Hölder cont. in $\overset{\cup}{X} \Longrightarrow$ can be extended to X

The proof of $\circledast$ for $x \in X \setminus \overset{\cup}{X}$ is an easy exercise ---- □.