# MAT 475 WEEK 1: INVARIANTS

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

## 1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

## 2. Invariants

An invariant is a quantity which always remains the same. When dealing with a problem, look for some invariant lurking in the background! Often when you have a very complicated system with lots going on, if you can find at least one quantity which doesn't change, you can often go a long way. Some types of invariants to look for:

- If you're dealing with a set of numbers $a_1, \ldots, a_n$, consider the sum $a_1 + \cdots + a_n$ or the product $a_1 a_2 \cdots a_n$.
- Consider the parity of a number.
- Consider numbers modulo other numbers.
- Use algebra to your advantage!
- (Trickier) Look for *monovariants*: quantities which can get bigger, but not smaller, or which can get smaller, but not bigger!

Here are some problems to get you started:

- We initially have the integer numbers between 1 and 100 written on the board. You choose two numbers $a$ and $b$, you erase them, and you write the number $|b - a|$. You repeat this operation until there is only one number left. Prove that the last number left will always be even.
- The opposite corners of a chessboard are removed. Is it possible to cover the remaining 62 squares using 2-by-1 dominoes?
- Vancouver Island has a small population of chameleons. In this island, chameleons come in three colours: green, white, and black. Initially, there are 15 green chameleons, 7 white chameleons, and 11 black chameleons. If two chameleons of different colours meet, both of them change to the third colour. No other colour changes happen. Is it possible that after a few such transitions all of the chameleons have the same colour?

### 2.1. Invariants Problems.

(1) Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.
(2) At first a room is empty. Each minute, either one person enters or two people leave. After exactly $3^{10}$ minutes, could the room contain exactly 1001 people?

(3) A quadrominoe is a $4 \times 1$ tile, which can be oriented horizontally or vertically. Can a $10 \times 10$ square be tiled with 25 $4 \times 1$ Quadrominoes?

(4) If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

(5) Start with the integer $7^{2015}$. At each step, delete the leading digit and add it to the remaining number (for example, 1367 would become 367+1=368). This is repeated until a number with exactly 10 digits remains. Prove that this number has two equal digits.

(6) Let $P_1, \ldots, P_{2015}$ be distinct points in the plane, with no three points lying on a line. Connect the points with the line segments

$$P_1P_2, P_3P_4, \ldots, P_{2014}P_{2015}, P_{2015}P_1$$

(some of these line segments may intersect each other). Can one draw a line that passes through the interior of each of these 2015 line segments?

(7) (The 'Lights out' game) Suppose $n \geq 2$ light bulbs are arranged in a row, numbered 1 through $n$. Under each bulb is a button. Pressing the button will change the state of the bulb above it (from on to off or vice versa), and will also change the neighbors states. (Most bulbs have two neighbors, but the bulbs on the end have only one.) The bulbs start off randomly (some on and some off). For which $n$ is it guaranteed to be possible that by flipping some switches, you can turn all the bulbs off?

(8) The $n$ cards of a deck (where $n$ is an arbitrary positive integer) are labeled $1, 2, \ldots, n$. Starting with the deck in any order, repeat the following operation: if the card on top is labeled $k$, reverse the order of the first $k$ cards. Prove that eventually the first card will be 1 (so no further changes occur). **Hint: This is not quite an invariants question, but more of a mix between an induction question and an invariants question**

(9) Three bugs are crawling on the coordinate plane. They move one at a time, and each bug will only crawl in a direction parallel to the line between the other two. If the bugs start out at $(0,0), (3,0), (0,3)$, can the bugs end up at $(1,2), (2,5), (2,3)$? Is it possible that after some time the first bug will end up back where it started, while the other two bugs have switched places?

(10) You have a stack of $2n+1$ cards, which you can shuffle using the two following operations:
- Cut: Remove any number of cards from the top of the pile, and put them in the bottom (in the same order)
- Riffle: Remove the top $n$ cards, and put them in order in the spaces between the remaining $n+1$ cards.

Prove that, no matter how many operations you perform, you can reorder the cards in at most $2n(2n+1)$ different ways.

(11) $n \geq 2$ is a positive integer. On an $n \times n$ board, there are $n^2$ squares, of which $n-1$ are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.