SOLUTIONS: INVARIANTS

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(1) Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

Proof. Consider all the sides of all the doors. Since every door has 2 sides, there is an even number of such. Now, every side of a door is either facing a room or the outside. By assumption, every room has an even number of doors, so summing over all rooms there must an even number of sides of doors which face rooms. Since there is an even number of door sides total, there must also be an even number of sides of doors facing the outside. □

(2) At first a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{10} minutes, could the room contain exactly 1001 people?

Proof. Let x be the number of people in a room on turn n. Then on turn $n+1$, we have either $x+1$ or $x-2$ people. These are the same modulo 3. Thus, whatever happens, modulo 3 the number of people on turn n is n. Now 3^{10} is 0 modulo 3, whereas 1001 is not, so this cannot happen. □

(3) A quadrominoe is a 4×1 tile, which can be oriented horizontally or vertically. Can a 10×10 square be tiled with 25 4×1 Quadrominoes?

Proof. Label the individual squares (i, j) for $1 \leq i, j \leq 10$. Then inside every square write the remainder of $i + j$ modulo 4. Note that inside every 1×4 tile the numbers $0, 1, 2, 3$ appear once each. However, there are 26 tiles with 3 written inside of them. Since any tiling would involve 25 colors, the answer is no. □

(4) If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

Proof. The total number of pairs (G, P) where G was a game played and P was a participant in G is even, since for every game there are 2 people playing. We can count this in 2 ways, by first summing over G or first summing over P. Formally,

$$\sum_G \#\text{players playing in G} = \sum_P \#\text{games that P played in.}$$

Thus, if we sum the total number of games played by each player we get an even number. Thus there must be an even number of odd numbers in the sum, which implies the statement. □

(5) Let $P_1, \ldots, P_{2015}$ be distinct points in the plane, with no three points lying on a line. Connect the points with the line segments

$$P_1P_2, P_2P_3, \ldots, P_{2014}P_{2015}, P_{2015}P_1$$

(some of these line segments may intersect each other). Can one draw a line that passes through the interior of each of these 2015 line segments?

Proof. We argue by contradiction. Assume such a line L exists. Note that L divides the plane into 2 sides. Since P_1P_2 intersects L, it follows that P_1 and P_2 are on opposite sides. Likewise, P_2 and P_3 are on opposite sides. Thus, P_1 is on the same side as P_3. Continuing on in this way, we see that P_1 is on the same side as P_{2015}. But this contradicts that L intersects P_1P_{2015}.

$\square$

(6) The n cards of a deck (where n is an arbitrary positive integer) are labeled $1, 2, \ldots, n$. Starting with the deck in any order, repeat the following operation: if the card on top is labeled k, reverse the order of the first k cards. Prove that eventually the first card will be 1 (so no further changes occur). **Hint: This is not quite an invariants question, but more of a mix between an induction question and an invariants question**

Proof. We claim that every number $m > 1$ appears in the top of the deck at most 2^{n-m} times, which obviously implies the statement. We argue by reverse induction on m. First, suppose $m = n$. Note that if n appears at the top of the deck, it will immediately after appear in the n'th position. Now for the number in the n'th position to change, the top of the deck must be n. Thus, from then on n will always be in the n'th position. This establishes the base case.

Now for the induction step, suppose m appears at the top of the deck. Then immediately afterwards, it appears in the m'th position. Now, for the position of m to change from that point, a number larger then m must appear at the top of the deck. Thus, between any 2 instances of m appearing at the top of the deck, there must be a larger number appearing at the top of the deck. This happens $1 + 2 + 4 + \cdots + 2^{n-m-1} = 2^{n-m} - 1$ times by our induction. Thus, m can appear at the top of the deck at most 2^{n-m} times, as desired.

$\square$

(7) You have a stack of $2n + 1$ cards, which you can shuffle using the two following operations:

- Cut: Remove any number of cards from the top of the pile, and put them in the bottom (in the same order)
- Riffle: Remove the top n cards, and put them in order in the spaces between the remaining $n + 1$ cards.

Prove that, no matter how many operations you perform, you can reorder the cards in at most $2n(2n + 1)$ different ways.

Proof. Label the positions $1, 2, \ldots, 2n + 1$ and work modulo $2n + 1$. Then observe that a cut takes the card in position x to the card in position $x + k$ for some k, and a riffle takes x to $2x$. Thus, by a straightforward induction any combination of these moves takes the card in position x to the card in position $ax + b$ for some a, b modulo $2n + 1$, and $a \neq 0$. There are only $2n(2n + 1)$ such pairs, completing the proof.

$\square$