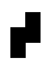
HOMEWORK SET #2: DUE SEPTEMBER 30

(1) A 9×9 grid gets covered by 16 5×1 tiles and a single 1×1 tile. What are the possible positions for the 1×1 tile?

Solution: Place co-ordinates on the grid such that the cells are $\{(i, j) : 1 \leq i \leq 9, 1 \leq j \leq 9\}$. Color the grid with 5 colors C_0, C_1, C_2, C_3, C_4 such that cell (i, j) gets colored C_k if $i + j \equiv 5 \mod 5$. Note that every 5×1 tile contains one cell of each Color. A simple count shows that there is 1 more cell of color C_0 then the other colors, and thus the 1×1 tile must be colored C_0. Thus it must lie on a cell (a, b) with $5 \mid a + b$. Applying a similar coloring with $(i - j) \mod 5$ likewise shows that $5 \mid a - b$. Thus we see that 5 divides both a and b, and so the only option is $a = b = 5$. We conclude that in any tiling, the 1×1 tile must be in the center.

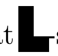
It remains to exhibit a coloring with a 1×1 in the center. To do this, we first group our sixteen 1×5 tiles into four 4×5 tiles. We then place them on the grid in a rotationally symmetric fashion, with the first one having vertices $\{(1, 1), (1, 5), (4, 1), (4, 5)\}$.

(2) Consider the usual tetris piece:  Prove that one cannot tile an 20×20 board with 100 such pieces (rotations and reflections are allowed).

Solution: Note that every tetris pieces covers at most as many tiles from the bottom row as from the second-most bottom row, and strictly fewer if it touches the third-most bottom row at all. Since they both has the same number (20) of cells, it follows that in any tiling the bottom two rows must be completely tiled with tetris pieces. So it suffices to show that a 2×20 board cannot be tiled by tetris pieces. But this is easy. For example, one cannot even tile two adjacent corners.

(3) Consider the tetris piece above. Suppose a 9×9 board is tiled with such pieces together with 1×1 pieces. What is the most number of tetris pieces that can be used?

Solution: Place co-ordinates on the grid such that the cells are $\{(i, j) : 1 \leq i \leq 9, 1 \leq j \leq 9\}$. Notice that every tetris piece contains exactly one cell both of whose co-ordinates are even. Thus there can be at most 16 such pieces. You can fit 16 such pieces by fitting 4 pieces into a 2×9 board 4 times. *Its hard for me to draw stuff in PDF files.*

(4) Prove that -shaped L-tetrominos (4 squares, width 2, height 3) cannot tile a 10×10 board.

Solution: Color the rows alternately black and white. Note that every tetrominoe covers 3 cells of one color and 1 of the other, thus an odd number of each. Any tiling has to use $\frac{100}{4} = 25$ tiles, and would thus cover an odd number of cells of each color. But the total number of cells of each color is even.

(5) An infinite chessboard has a positive integer written in every square. The value in each square is the average of the values in the four squares around it. Prove that all the numbers in all the squares are equal.

Solution: Consider the smallest value M written in any cell (note that this exists since any set of positive integers has a minimum, even if this set is infinite). Notice that if some cell C in the grid has the value M, then all the other 4 cells adjacent to it must also have the value M. Otherwise, their average would be larger and contradict the condition in the question. Now suppose C_0 has the value M. For any other cell C, one may connect C_0 and C via a path of adjacent squares. This means C also have the value M written inside it. Thus all the cells have the same value M and we are done.

(6) Is it possible to find 100 consecutive positive integers with exactly 7 primes among them?

Solution: Let $S_n = \{n, n+1, \ldots, n+99\}$. Let $f(n)$ be the number of primes in S_n. Since S_{n+1} is obtained from S_n by adding a number and then removing a number, it follows that $|f(n+1) - f(n)| = 1$. Thus, $f(n)$ only changes by a most 1 every time n increases to $n+1$. Now note that $f(1) = 25 > 7$. Thus, if $f(n)$ is never equal to 7, it follows that $f(n)$ can never take any value less then 7 either, since it would have had to have 'leapfrogged' over 7[1]. So it is enough for us to exhibit a specific value of n for which $f(n) \leq 7$. To do this, note that $f(100! + 1) = 0$.

(7) On a large flat field, 235 people are positioned so that for each person the distances to all the other people are different. Everyone has a water pistol and at a given time fires the pistol on the person nearest to them. Show that at least 1 person is left dry.

[1] This is intuitive enough that I wouldn't require a proof, but it is easy enough to give a proof by induction

Solution: By induction, we show that the conclusion holds with $2k+1$ people for any non-negative integer k. The base case of $k = 1$: Let A, B, C be three people, and suppose that B and C are the closes pair. Then they shoot each other, and so nobody shoots A and she is left dry.

Now for the inductive step, suppose we established the premise for $k - 1$. Again, let A, B be the closes pair of people. Then they shoot each other. Now, if anyone else shoots A or B then one of them is shot twice, which means that some other person must be left dry (otherwise there would have been more than $2k + 1$ shots!). So we may assume that no-one else shot A and B. Thus, for the other $2k - 1$ people, they all shot the closest person to them who is not A or B. Now we are done by our inductive hypothesis.

(8) n red points and n blue points are drawn in the plane such that no 3 points lie on a line. Prove we can join each red point to a single blue point by a line segment, such that no 2 line segments cross.

Solution: Consider all possible pairings of red points and blue points by line segments, and take the one with the smallest possible sum of all the lengths of the line segments. Now suppose for the sake of contradiction that two line segments cross, so that A, B are red, C, D are blue, and the line sgements AC and BD cross. Now that means that $ABCD$ is a convex quadrilateral whose diagonals are AC and BD. By the triangle inequality, $|AC| + |BD| > |AD| + |BC|$ which would mean that if we instead joined A to D and B to C the sum of the lengths of all the line segments would decrease. This is a contradiction.

(9) In a school there are n kids, such that every kid is friends with exactly 3 others (friendship is mutual, so if A is friends with B then B is friends with A). Prove that we can split the kids up into 2 rooms, such that every kid is friends with at most 2 others in the same room.

Solution: Consider a splitting into rooms such that the number of people in a room with all of their friends is as small as possible. Now suppose for the sake of contradiction that some person A and his 3 friends B, C, D all in the same room. If we move A into the other room, then A will no longer be in a room with all of their friends. Moreover, noone in the other room is friends with A, so the total number of people with all of their friends in the same room will decrease by 1. This is a contradiction.