MAT 475 WEEK 3: EXTREMAL PRINCIPLE

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Extremal Principle

The Idea behind the extremal principle is to pick an object which **maximizes**. or **minimizes** an object. You can then say a lot about this object, because any perturbation of it must yield something less extreme!

Here are some problems to get you started:

(1) If n points in the plane do not all lie in a single line, then there is a line passing through exactly 2 of them. *Hint: Consider shortest height in any non-degenerate triangle formed by the 3 points.*
(2) There are n points given in the plane. Any three of the points form a triangle of area 1. Prove that all the points lie in a triangle of area 4.
(3) S is a finite set of (distinct) positive real numbers. For every pair of elements $a, b \in S$ the element $\sqrt{a^2 + b^2}$ is also contained in S. Prove that S contains at most 2 elements.
(4) Prove that $\sqrt{5}$ is not a rational number.

2.1. More Problems.

(1) Say you have finitely many red and blue points on a plane with the interesting property: every line segment that joins two points of the same color contains a point of the other color. Prove that all the points lie on a single straight line.
(2) Rooks are placed on an $n \times n$ chessboard such that whenever a square does not have a rook on it, the row and column of that square, combined, have at least n rooks. Prove at least half the squares contain rooks.
(3) Fifteen sheets of paper of various sizes and shapes lie on a desktop covering it completely. The sheets may overlap and may even hang over the edge. Show that five of the sheets may be removed so that the remaining ten sheets cover at least $\frac{2}{3}$ of the desktop.
(4) Let $p(x)$ be a real polynomial such that for all x we have $p(x) + p'(x) \geq 0$. Prove that for all $x, p(x) \geq 0$.
(5) There are n identical cars on a circular track. Together they have enough gas for one car to complete a single lap. Show that there is a car which can complete a lap by collecting gas from every car it passes (Assume there is no inefficiency in starting/stopping/refilling).

(6) Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

2.2. **hints.**

(1) Consider the monochromatic (all vertices have the same color) triangle of the smallest area.
(2) Consider the row or column with the fewest rooks.
(3) Consider the sheet of paper such that removing it would uncover the least area.
(4) Consider the smallest value of the polynomial $p(x)$. Note, you have to show this exists (so that $p(x)$ does not tend to $-\infty$ as $x \to \infty$ or $x \to -\infty$.
(5) Let a car with tons of gas complete a circular lap, collecting gas along the way. Consider the point at which it had the least gas.
(6) Consider the matching with the shortest total length of line segments.