MAT 475 WEEK 2: COLORING

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Colouring Proofs

The Idea behind colouring is very simple: We partition a set into subsets by assigning each element a 'color'. Then, when faced with a tiling question, or a walking process, one can reason about how they interact with the coloring to draw conclusions. Examples of common techniques include:

- Keep track of the parity of the number of cells of each color.
- More generally, consider congruences for the number of cells of each color.
- You can often consider many different colorings and draw different conclusions from each one.
- The points of a plane are colored red or blue. Prove there is an equilateral triangle

Here are some problems to get you started:

(1) Eight 1×3 rectangles and one 1×1 square cover a 5×5 board. Prove the 1×1 piece is in the center.
(2) Let m, n, k be positive integers. When can you tile an $m \times n$ board with $1 \times k$ rectangles?
(3) Every point of the plane is coloured red or blue. Prove that there is a rectangle with all vertices of the same colour.
(4) What is the smallest number of squares one has to delete from an 8×8 chessboard so that no L-shaped trimino fits? What about a 9×9 chessboard?

2.1. **Coloring Problems.**

(1) Every point of the plane is coloured red or blue or green . Prove that there are 2 points of the same colour of distance 1.
(2) Can a $10 \times 10 \times 10$ cube be tiled with 250 $1 \times 1 \times 4$ tiles?
(3) Can an 8×8 board be tiled with fifteen 1×4 rectangles and one 2×2 square?
(4) A 6×6 board is tile by 1×2 dominoes. Prove that among the ten horizontal and vertical lines going through the board, one of these lines does not cut a single a domino in half.
(5) Every point of the plane is coloured red or blue. Prove that there is an equilateral triangle all of whose vertices are the same colour.
(6) Prove that on a $4 \times n$ board, a chess knight cannot visit every square once and return to the same square it started on.

(7) Every element of a 25×25 matrix is either 1 or -1. For each row and column, compute the product of all the elements in that row or column. Prove that these 50 numbers do not sum to 0.

(8) Every point in space is coloured red, blue, or green. Prove that for at least one of the 3 coloors, the set of distances between points of that colour contains all positive real numbers.