MAT 475 WEEK 7: BIJECTIONS

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Binomial Coefficients

Binomial coefficients are in some sense the basic combinatorial objects. The definition of $\binom{n}{k}$ is *the number of ways to choose a set of k distinct elements from a set of n distinct elements.* For instance, $\binom{3}{2} = 3$, since if A, B, C are three objects, and we have to pick two of them, we can pick $\{A, B\}, \{A, C\}$ or $\{B, C\}$. The basic facts are the following

Lemma 2.1. *There are $n!$ways[1] to order n elements.*

Proof. We proceed by induction. For $n = 1$ the lemma is obvious. Now suppose it is true for $n - 1$. If we have n elements, we have n ways of choosing the largest. Then by induction, there are $(n-1)!$ ways of ordering the remaining elements. Thus, the number of ways to order n elements is $n \cdot (n-1)! = n!$.

□

Proposition 2.2. *If $k \leq n$ then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Proof. Suppose S is a set of k numbers from the set $\{1, 2, \ldots, n\}$. Let C be the set of orderings of $\{1, 2, \ldots, n\}$. We are going to count the size of C in two different ways. First, to get an element of C we just have to decide what the first k elements are, how to order the first k elements are, how to order them, and how to order the las $n - k$ elements. This gives

[1] By convention, we say $0! = 1$.

$|C| = \binom{n}{k} k!(n-k)!$. On the other hand, an ordering is just a permutation, so $|C| = n!$. The result follows.

$\square$

2.1. Some problems to get you started:

- Let $n \geq k \geq 1$. Show that $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. Give a proof using a bijection! Can you also give a proof using algebra?
- Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
- Johnny lives in New York, where the streets and avenues are numbered by consecutive positive integers. He is at the intersection of street 1 and avenue 1. He wants to get to the intersection of street 5 and avenue 5. At every turn, he is only allowed to go up one street, or right one avenue. How many ways does Johnny have of getting to where he wants to go?
- A 2×2 chessboard has 9 subrectangles including itself. Count the number of subrectangles in a standard, 8×8 chessboard

2.2. Followup Problems.

- Let $n \geq 2, k \geq 1$. Show that $\binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} = \binom{n}{k}$. Give a proof using a bijection! Can you also give a proof using algebra?
- Let $n \geq 1$. Prove that $\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = 0$.
- How many ways are there to write 10 as a sum of 3 distinct positive integers?
- The sum of the areas of the subrectangles in a 2×2 chessboard is 16.Find the sum of the area ins subrectangles in a standard, 8×8 chessboard.

2.3. Bijection Problems.

(1) From the set $\{1, 2, \ldots, n\}$ we select two disjoint subsets, A and B, possibly empty. In how many ways can this be done?

(2) Prove that for $n > 0$, $\displaystyle\sum_{k \text{ is even}} \binom{n}{k} = 2^{n-1}$.

(3) In a special lottery, 6 numbers are chosen from $\{1, 2, \ldots, 49\}$. There are $\binom{49}{6}$ to do this. How many of the ways have at least one pair of consecutive numbers? *Harder: How many of the ways have exactly 1 pair of consecutive numbers?.*

(4) In how many ways can p 1's and q 0's be lined up in a row so that no two 0's are adjacent? Assume $p + 1 >= q$.

(5) Let m, n be positive integers. Compute the number of ordered m-tuples $(x_1, x_2, \ldots, x_m)$ of positive integers, whose sum is n (so that $x_1 + x_2 + \cdots + x_m = n$). Note that $(2, 1)$ is considered different than $(1, 2)$. In other words, the order matters!

(6) n players compete in a Backgammon tournament. Each player plays against every other player exactly once, and there are no draws. Let

w_i and l_i be the number of games that the i'th player won and lost, respectively. Prove that $\sum_i w_i^2 = \sum_i l_i^2$.

(7) Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

(8) Let n, k be positive integers. Find the number of ordered k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets of $\{1, 2, \ldots, n\}$ such that the intersection of all k of these subsets is 0.

(9) Prove that $\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$.

(10) An *awesome* sequence of 0's and 1's is a sequence which doesn't contain $0, 1, 0$ as consecutive terms. A *fantastic* sequence is one which doesn't contain $0, 0, 1, 1$ or $1, 1, 0, 0$ as consecutive terms. Show that there are twice as many fantastic sequences of length 101 as there are awesome sequences of length 100.

(11) Prove that the number of sequences made up of 0's,1's and 2's such that 0 is never followed by a 1 is F_{2n+2}, where F_n is the n'th Fibonacci number.

(12) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1\}$ such that $P(2) = n$.

(13) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.