University of Toronto Faculty of Arts & Science

MAT327

October 17, 2019

Duration: 1 Hour 50 minutes

You are expected to write complete proofs to the following problems - leaps of logic or unproven assumptions will be penalized. You may use any statement we've proven in class freely, so long as its not identical to the statement you are asked to prove. The exam is graded out of 40 points with a maximum score of 50. **Please ask me if any question is unclear!**

(1) (10 points) Prove that the map $f : [-1, 1] \to [0, 1]$ given by $f(t) = t^2$ is a quotient map.

(2) (10 points) Let $O = (0, 0) \in \mathbb{R}^2$ be the origin. Prove that $\mathbb{R}^2 \setminus \{O\}$ is connected. *You may use without proof that $\mathbb{R}$ is connected.*

(3) (10 points) Let (X, d) be a compact metric space.
  (a) Prove that there is a positive real number M such that
  $$d(x, y) < M$$
  holds for all $x, y \in X$.
  (b) Define $D_X := \sup\{d(x, y) : x, y \in X\}$. Prove there exist points $a, b \in X$ such that $d(a, b) = D_X$.

(4) (10 points) Let X be a normal topological space. Let A, B, C be closed subsets of X which are pairwise disjoint. In other words, $A \cap B = B \cap C = A \cap C = \emptyset$. Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(A) = 0, f(B) = 1, f(C) = 2$.

(5) (10 points) Let X be the topological space, which as a set is $[0, 1]^{\mathbb{N}}$ endowed with the product topology. Prove that X has a countable dense subset. In other words, there is a countable subset $A \subset X$ such that $\bar{A} = X$. *You may use without proof that a countable union of countable sets is countable, and that the product of finitely many countable sets is countable.*