**MAT327 MIDTERM EXAM SOLUTIONS**

(1) (10 points) Prove that the map $f : [-1, 1] \to [0, 1]$ given by $f(t) = t^2$ is a quotient map.

**Solution:** First, $f(t)$ is continuous, since $t \to t^2$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ and restrictions of continuous functions to subspaces are also still continuous (this claim did not require proof). Moreover,$f(0) = 0, f(1) = 1$ and so by the intermediate value theorem $f$ is surjective.

Finally, we show that $f$ is open. This will finish the proof since an open, continuous, surjective map is a quotient map. To show $f$ is open it is sufficient to show that the image of every basis element $B$ is an open set.

- $B = [-1, a)$. If $a \geq 0$ then $f(B) = [0, 1]$. Else $f(B) = (a^2, 1]$.
- $B = [a, 1]$. If $a \leq 0$ then $f(B) = [0, 1]$. Else $f(B) = (a^2, 1]$.
- $B = (a, b), -1 \leq a < b \leq 1$. If $a > 0$ then $f(B) = (a^2, b^2)$. If $b < 0$ then $f(B) = (b^2, a^2)$. Else, $f(B) = [0, \max(a^2, b^2))$.

**Alternative Solution:** Notice that the domain and range of $f$ are compact. Let $V \subset [-1, 1]$ be a closed set. Then $V$ is compact since closed subsets of compact sets are compact. So $f(V)$ is also compact, since images of compact sets under continuous functions are compact. Finally, we conclude $f(V)$ is closed since compact subspaces of Hausdorff spaces are closed. Thus $f$ is a closed, continuous, surjective map, which means its a quotient map.

(2) (10 points) Let $O = (0, 0) \in \mathbb{R}^2$ be the origin. Prove that $\mathbb{R}^2\backslash\{O\}$ is connected. *You may use without proof that $\mathbb{R}$ is connected.*

**Solution:** Consider the four subsets $A = \mathbb{R}_{>0} \times \mathbb{R}, B = \mathbb{R} \times \mathbb{R}_{>0}, C = \mathbb{R}_{<0} \times \mathbb{R}, D = \mathbb{R} \times \mathbb{R}_{<0}$. Since $\mathbb{R}_{>0}, \mathbb{R}_{<0}$ are both isomorphic to $\mathbb{R}$, $\mathbb{R}$ is connected, and the product of two connected spaces is connected, it follows that $A, B, C, D$ are all connected. Since $A, B$ have a point in common it follows that $A \cup B$ is connected. Similarly for $C \cup D$. Now since $A \cup B$ and $C \cup D$ have a point in common (for example, $(-1, 1)$) it follows that $A \cup B \cup C \cup D = \mathbb{R}^2\backslash\{0\}$ is connected.

**Alternative Solution:** Since path connected spaces are connected, it is enough to show that $X = \mathbb{R}^2\backslash\{0\}$ is path- connected. Recall that the relation $a \sim b$ if $a$ has a path to $b$ is an equivalence relation. Now $(0,1)$ has a path to $(a,b)$ given by a straight line whenever $(a,b)$ is not of the form $a<0, b=0$. Formally, the path is $f(t)=(ta, tb+(1-t))$. Thus $(0,1)$ is equivalent to all points except perhaps the negative x-axis. However, each point in the negative x-axis has a path to $(1,1)$ via a straight line. Thus, they are all equivalent to $(1,1)$ and thus to $(0,1)$ since $(1,1)\sim(0,1)$. We have shown that all the points lie in a single equivalence class, and thus the space is path-connected.

(3) (10 points) Let $(X,d)$ be a compact metric space.

(a) Prove that there is a positive real number $M$ such that

$$d(x,y)<M$$

holds for all $x,y\in X$.

(b) Define $D_X := \sup\{d(x,y) : x,y\in X\}$. Prove there exist points $a,b\in X$ such that $d(a,b)=D_X$.

**Solution:** Since $X$ is compact and finite products of compact spaces are compact, it follows that $X\times X$ is also compact. Now the distance function $d : X\times X\to\mathbb{R}_{\geq 0}$ is continuous. We showed this in class, but it is possible to show directly: If $d(x,y)=t$, then $d^{-1}(t-\epsilon, t+\epsilon)$ contains $B_{\frac{\epsilon}{2}}(x)\times B_{\frac{\epsilon}{2}}(y)$ by the triangle inequality.

Now by the extreme value theorem, continuous functions on compact spaces have bounded image, and achieve their maximum.This proves both parts of the question.

(4) (10 points) Let $X$ be a normal topological space. Let $A,B,C$ be closed subsets of $X$ which are pairwise disjoint. In other words, $A\cap B=B\cap C=A\cap C=\emptyset$. Prove that there exists a continuous function $f : X\to\mathbb{R}$ such that $f(A)=0, f(B)=1, f(C)=2$.

**Solution:** The key is to apply Uryshons lemma, but to the disjoint closed sets $A\cup B$ and $C$. This gives us a continuous function $f_{AB} : X\to\mathbb{R}$ such that $f_{AB}$ is 0 on $A\cup B$ and 1 on $C$. Similarly, we get a continuous function $f_{BC}$ which is 0 on $A$ and 1 on $B\cup C$. Now $f=f_{AB}+f_{BC}$ is the desired function.

Finally, to show that $f$ is continuous, note that it is the composition of the continuous function $(f_{AB}, f_{BC})$ to $\mathbb{R}\times\mathbb{R}$ with the addition function $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

**Alternative Solution:** Let $Y=A\cup B\cup C$. Since $A,B,C$ are closed and disjoint, it follows that they are each clopen subsets of $Y$. Thus the function $g : Y\to\mathbb{R}$ defined by $g(A)=0, g(B)=1, g(C)=2$ is continuous. By the Tietze extension theorem, this can be extended to a continuous function $f : X\to\mathbb{R}$.

(5) (10 points) Let $X$ be the topological space, which as a set is $[0,1]^{\mathbb{N}}$ endowed with the product topology. Prove that $X$ has a countable dense subset. In other words, there is a countable subset $A \subset X$ such that $\bar{A} = X$. *You may use without proof that a countable union of countable sets is countable, and that the product of finitely many countable sets is countable.*

**Solution:** Let $P = \mathbb{Q} \cap [0,1]$. Note that since $\mathbb{Q}$ is countable, $P$ is countable as well, and since the rationals are dense in the reals, $P$ is dense in $[0,1]$. Now, we cannot simply take $A = P^{\mathbb{N}}$ since that is uncountable. So we modify the idea slightly. We set $A_n \subset [0,1]^{\mathbb{N}}$ to consist of all sequences $(a_i)_{i\in\mathbb{N}}$ satisfying

$$\begin{cases} a_i \in P & \forall i \in \mathbb{N} \\ a_i = 0 & i > n \end{cases}$$

Then $A_n$ is bijective to $P^n$ and therefore countable. Finally, we set $A = \cup_{n\in\mathbb{N}} A_n$. Note that $A$ can be described as the set of all sequences all of whose co-ordinates are rational, and only finitely many of which are non-zero. Since countable unions of countable sets is countable, it follows that $A$ is countable as well. It remains to show that $A$ is dense.

Let $U = \prod_{i\in\mathbb{N}} U_i$ be a basis element for $[0,1]^{\mathbb{N}}$. For each $i \in \mathbb{N}$ we pick an element $a_i \in P \cap U_i$. Since all but finitely many of the $U_i$ are the entire interval $[0,1]$, it follows that there is some natural number $n$ such that we may pick $a_i = 0$ for all $i > n$. We take $\vec{a} = (a_i)_{i\in\mathbb{N}}$. Then $\vec{a} \in U \cap A_n$, which proves that $A$ is dense as desired.