Lecture 7: Etale Fundamental Group - Examples

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In this lecture our only goal is to give lots of examples of etale fundamental groups so that the reader gets some feel for them. Some of the examples will involve scheme-theoretic concepts that we have not covered such as normality, smoothness, dimension etc. We pause to say a few words about these when appropriate, but we do not stop to explain all the details, rather hoping that giving the reader a presentation of the general picture is useful even if some of the technical pieces are missing.

As a matter of notation, when we do not care about the based-point we merely write $\pi_1(X)$ to denote the fundamental group of X.

1 Fields

We begin with the most basic example, to demonstrate how the etale fundamental group provides a generalization of Galois theory[1].

Let $X = \text{Spec } k$ where k is some field. Then picking an algebraic closure $\overline{k}$, and an algebraically closed field K containing $\overline{k}$ we write $\overline{x} : \text{Spec } K \to \text{Spec } k$ to be the geometric point of X. Then finite etale schemes over X[2] correspond to finite unions $\cup_i \text{Spec } L_i$, where L_i is a separable field extension of k.

Now, consider an increasing sequence of Galois covers $t_i^\# : L_i \subset K$ of k whose union is k^{sep}, the separable closure of k. Then $(\tilde{X}, \tilde{\overline{x}} := \varprojlim(\text{Spec } L_i, t_i)$ is a universal cover of X. We thus see that $\pi_1(X, \overline{x}) \cong \text{Gal}(k^{sep}/k)$.

Now suppose $\phi : \text{Spec } L \to \text{Spec } k$ is a morphism of fields, taking a geometric point $\overline{y} : \text{Spec } \overline{L} \to \text{Spec } L$ to $\overline{x} : \text{Spec } \overline{L} \to \text{Spec } k$.

Then the induced map on fundamental groups is the natural map from $\text{Gal}(L^{sep}/L)$ to $\text{Gal}(k^{sep} \cap /k)$. Note that if L is a finite extension of k then

[1] this should not be surprising, as most of our proofs essentially reduced studying etale maps to studying separable field extensions

[2] in fact, every etale morphism to X is finite!

the map will be an injection, whereas in general it does not have to be! As an exercise, give an example where the map is not injective, but IS surjective.

2 Complete Discrete Valuation Rings

Say that A is a discrete valuation ring, with maximal ideal M, such that A is complete, in that $A \cong \widehat{A}$. Let k be the residue field A/M. By Hensel's lemma[3], the base change functor gives an equivalence of categories $FET/\text{Spec } A \cong FET/\text{Spec } k$.

Another way of stating that is as follows: if $\overline{x} : \text{Spec } \overline{k} \to \text{Spec } k$ is a geometric point with image $\overline{y}$ in Spec A, then

$$\pi_1(\text{Spec } k, \overline{x} \to \pi_1(\text{Spec } A, \overline{y})$$

is an isomorphism. Thus, the etale fundamental group of the spectrum of a complete discrete valuation ring is isomorphic to the Galois group of its residue field.

3 Normal Schemes

Definition. *We say that a scheme X is normal if all local rings $\mathcal{O}_{X,x}$ are domains, and are integrally closed in their field of fractinos.*

Let X be a (noetherian) normal scheme. Then X must be irreducible, as the local ring at an intersection point of two irreducible components is necessarily not a domain. Thus X has a generic point $j : \text{Spec } K(X) \to X$. Moreover, if Y is a connected, finite etale cover of X, then one can show – See [Milne, Etale Cohomology, I,3.17] that Y is also normal. Hence Y must be the normalization of X in $K(Y)$.

In particular, a connected finite etale cover of X remains connected when base-changed to Spec $K(X)$. It is easy to see that a homomorphism of groups $r : G \to H$ is surjective iff every H-set with a single orbit also has a single orbit as a G-set. It follows that $\pi_1(\text{Spec } K(X))$ surjects onto $\pi_1(X)$.

In fact, if we let $K(X)^{un}$ be the compositum of all finite extensions of $K(X)$ in which the normalization of X is unramified over X, we hae $\pi_1(X) \cong \text{Gal}(K(X)^{un}/K(X))$.

[3] Which we will cover in a future lecture, or see Atiya-Macdonald

4 Examples from number theory

4.1 Spec $\mathbb{Z}$

Let $X = \text{Spec } \mathbb{Z}$, so that $K(X) = \mathbb{Q}$. Since $\mathbb{Z}$ is normal, it follows from the previous section that the connected, finite etale schemes over X are of the form Spec $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in some number field K. However, every extension of K is known to be ramified about at least 1 prime, it thus follows that $\pi_1(\text{Spec } \mathbb{Z}) = 1$.

4.2 Spec $\mathbb{Z}[\frac{1}{n}]$

Let $X = \text{Spec } \mathbb{Z}[\frac{1}{n}]$. As before, finite etale schemes over X are of the form Spec $\mathcal{O}_K[1/n]$, where $\mathcal{O}_K$ is unramified above Spec $\mathbb{Z}$ away from primes dividing n (or just, 'away from n' for short). Thus $\pi_1(\text{Spec } \mathbb{Z}[\frac{1}{n}]) \cong \text{Gal}(\mathbb{Q}^{(n)}/\mathbb{Q})$ where $\mathbb{Q}^{(n)}$ denotes the maximal extension of $\mathbb{Q}$ unramified away from n.

4.3 Spec $\mathbb{Z}_{(p)}$

Let $X = \text{Spec } \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at (p). then X has two points, corresponding to (p) and (0), with maps $i : \text{Spec } \mathbb{F}_p \to \text{Spec } X$ and $j : \text{Spec } \mathbb{Q} \to \text{Spec } X$. As we saw from the last section, since X is normal the induced map j_* on fundamental groups is surjective. On the other hand, i_* is injective, as there are Galois fields K above $\mathbb{Q}$ unramified above p with prime ideals $\mathcal{P}$ above p such that $\mathbb{F}_{\mathcal{P}}$ is of arbitrary degree over $\mathbb{F}_p$.

The following is a good exercise in getting practise at switching back and forth between the fundamental group and finite etale covers using the main theorem:

- For a subgroup H of a group G, let N be the normal subgroup of G generated by H. Prove that G/N is the largest quotient of G on which the action of H by left multiplication is trivial.
- Let K be the compositum of all finite extensions L of $\mathbb{Q}$ which in which the prime p totally splits; i.e. $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p^{[L:\mathbb{Q}]}$. Prove that the $\text{Gal}(K/\mathbb{Q})$ is the quotient of $\pi_1(\text{Spec } X)$ by the normal subgroup generated by $i_*\pi_1(\text{Spec } \mathbb{F}_p)$.

5 Examples from complex geometry

5.1 $X = \mathbb{A}^1_{\mathbb{C}}$

Let $X = \mathbb{A}^1_{\mathbb{C}}$. We claim that $\pi_1(X) = 1$. To see this, note suppose that $\phi : Y \to X$ is a finite etale cover of X. Then Y must be a normal curve, and therefore smooth. Let Y' be the unique smooth proper curve over $\mathbb{C}$ with an open immersion $Y \to Y'$ [4] and ϕ extends to a map $\phi' : Y' \to \mathbb{P}^1_{\mathbb{C}}$. Now, consider the differential form $\omega = \phi'^*(\frac{dt}{t})$ on Y'. Since ϕ is etale and $\frac{dt}{t}$ has a pole at 'infinity', ω has no zeroes anywhere, and has at least simple poles at each pre-image of 0 and ∞. Thus, if $\deg \phi \geq 2$, ω would have degree at most -3, which never happens. Hence the degree of ϕ must be 1, and so ϕ is an isomorphism.

Note that this proof does not work in finite characteristic because the pullback under a wildly ramified map can turn a differential form with a pole into one with zeroes. For instance, $y \to y^p - y + x$ defines a non-trivial finite etale cover of $\mathbb{A}^1_k$ for k an algebraically closed field of char. p.

Next, we do two examples of singular curves:

5.2 X is a cuspidal cubic curve.

Let $X = \text{Spec } \mathbb{C}[x, y]/(y^2 - x^3)$, so that X is what's usually known as a 'cuspidal cubic curve'. Consider a connected etale cover $Y \to X$. We claim that Y must be trivial.

To see this, first consider the normalization of X by $\psi : \text{Spec } \mathbb{C}[t] \to X$, given by $x \to t^2, y \to t^3$. Then ψ is bijective. We claim that $\psi_Y : Y \times_X \text{Spec } \mathbb{C}[t] \to Y$ is also bijective. To prove this, first note that it suffices to check first note that as ϕ is an isomorphism away from the point (x, y), ϕ_Y is an isomorphism above points not mapping to (x, y). Hence, we are free to base change to the closed point $i : \text{Spec } \mathbb{C} \to X$, $i^\# : P(x, y) \to P(0, 0)$. Letting Y' be the correspondig fiber product $= Y \times_X \text{Spec } \mathbb{C}$, and noting that

$$\mathbb{C}[t] \otimes_{\mathbb{C}[t^2,t^3]} \mathbb{C} \cong \mathbb{C}[t]/(t^2)$$

we get the map $\psi' : Y' \times_{\text{Spec } \mathbb{C}} \mathbb{C}[t]/(t^2) \to Y'$. But as t is nilpotent, the two schemes have the same reduced subscheme, and so ψ' is clearly a bijection[5].

[4] see [Hartshorne, AG, I.8] for details

[5] Note that this part of the argument used nothing about Y being etale. Thus we have proven that any base change of ϕ is a bijection, and in fact if we are a bit careful, a homeomorphism. Thus ϕ is what's called a universal homeomorphism, and it is a general—highly non-trivial—fact that these induce an equivalence of categories on etale sites.

It follows that Y' is a connected finite etale cover of $\mathbb{A}^1_{\mathbb{C}}$, hence must trivial by the previous subsection. Hence Y is also trivial, and so $\pi_1(X) = 1$.

5.3 X is a nodal cubic

Let $X = \text{Spec } \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$ so that X is what's usually called a 'nodal cubic curve'. Consider a connected etale cover $Y \to X$.

Now, to understand Y, we record the normalization of X by $\psi : \text{Spec } \mathbb{C}[t] \to X$ given by $x \to t^2-1, y \to t(t^2-1)$[6]. By two subsections ago, $Y \times_X \text{Spec } \mathbb{C}[t]$ is a disjoint union of n different Spec $\mathbb{C}[t]$'s, each mapping isomorphically to Spec $\mathbb{C}[t]$. We refer to the i'th one as Spec $\mathbb{C}[t]_i$. Moreover, the map $Y \times_X \text{Spec } \mathbb{C}[t] \to Y$ is an isomorphism on the complement of the point (x, y).

Next, above x, y, Y must have n distinct points $y_1, \ldots, y_n$, as $\mathbb{C}$ is algebraically closed and thus the closed point of X is also a geometric point. Moreover, as Spec $\mathbb{C}[t] \to X$ has 2 closed points above (x, y), the same must be true for each lift to Y. That is, on the pre-image of (x, y), the $2n$ points $(t-1)_i, (t+1)_i, 1 \leq i \leq n$ map to the n-points $y_1, \ldots, y_n$ in a two-to-one fashion. Sine Y is connected, these must form a 'chain', and so we can arrange things so that $(t-1)_i$ maps to the same point of Y as $(t+1)_{i+1}$, where we interpert the subscripts cyclically modulo n.

We thus understand the topological structure of Y, and it remains to understand its scheme structure. Since $\mathbb{C}[x,y]/(y^2 - x^3 - x^2) \to \mathbb{C}[t]$ is injective, and Y is affine and flat over X, we see that $Y = \text{Spec } B$ where B injects into $\oplus_i \mathbb{C}[t]_i$. Moreover, It must map onto the subring R_n of elements $(P_i(t))_{1 \leq i \leq n}$ such that $P_i(1) = P_{i+1}(-1)$

We claim that Spec R is etale over Spec X. Clearly, this need only be checked above the point (x, y). Since at a point y_i above (x, y), Spec R_n looks locally like Spec R_2 - where $R_2 \subset \mathbb{C}[t_1] \oplus \mathbb{C}[t_2]$ is the subring of polynomials $(P(t_1), Q(t_2))$ where $P(1) = Q(-1), P(-1) = Q(1)$- it suffices to prove that Spec R_2 is etale above X at $((t_1 - 1, 0), (0, t_2 + 1))$. Convince yourself now that ψ identifies $\mathbb{C}[x,y]/(y^2 - x^3 - x^2)$ with the ring T of polynomials $A(t)$ such that $A(1) = A(-1)$ (hint: subtract a constant to get $A(1) = 0$ and then subtract a constant multiples of x^n or yx^n. Now iterate on degree).

We claim that $((1,1), (t, -t))$ is a basis for R_2 over T away from (t). To prove this, note that we can write

$$(P(t_1), Q(t_2)) = (1,1) \cdot \frac{P+Q}{2} + (t, -t) \cdot \frac{P-Q}{2t},$$

[6]Morally, we plug in $t = y/x$ into the defining equation.

which proves that our set is a basis. Thus R_2 is finite flat over T. As the pre-image of $(0,0)$ in Spec R_2 has two closed points and is of degree 2, both closed points must be simple. Hence Spec R_2 is etale over X as desired.

Thus, Spec R_n and Y are both etale over X, and thus Spec R_n is etale over Y of generic degree 1, and thus Spec $R_n = Y$.

It follows that the only finite etale covers of X are the Galois coversSpec R_n with automorphism group $\mathbb{Z}/n\mathbb{Z}$, and thus $\pi_1(X) = \widehat{\mathbb{Z}}$.

6 The interaction of geometry and arithmetic

Consider now a scheme X_k of finite type over a non-algebraically closed field k, and let $X_{\overline{k}}$ denote the base change to Spec $\overline{k}$. We break up the fundamental group of X_k into so-called geometric and arithmetic parts. For the rest of this section, we pick a geometric point of $X_{\overline{k}}$ which we omit from the notation from here on out. We have a sequence of maps

$$\pi_1(X_{\overline{k}}) \xrightarrow{i} \pi_1(X_k) \xrightarrow{j} \pi_1(\text{Spec } k) .$$

We study this sequence;

1. $j \circ i$ **is trivial:** This is because the map $X_{\overline{k}} \to \text{Spec } k$ factors through Spec $\overline{k}$, which has trivial fundamental group.

2. i **is injective:** Suppose not, so that $1 \neq g \in \pi_1(X_{\overline{k}})$ is in the kernel of j. Since g is non-trivial, we can find a finite etale cover Y on whose geometric points g acts non-trivially. If we can find a cover Z_k of X_k whose base change $Z_{\overline{k}}$ contains Y as a connected component, then $j(g)$ acts non-trivially on the geometric points of Z_k, which is a contradiction.

   To construct such a Z, first note that since Y is finite over $X_{\overline{k}}$ and is of finite type over $\overline{k}$ we can find a finite etale cover Y_L over X_L for some finite Galois extension L of k, whose base change via Spec $\overline{k} \to \text{Spec } L$ is Y. It suffices to make the finite descent to a cover of X.

   Consider, for each element $\sigma \in \text{Gal}(L/k)$ the base change Y_L^σ of Y along the map $\sigma : \text{Spec } L \to \text{Spec } L$. Then $\text{Gal}(L/k)$ acts on

$$Z_L := \bigcup_\sigma Y^\sigma$$

in a natural way, compatible with its action on X_L. Thus, taking the quotient by this group, we get a finite etale cover Z of X_k which base changes to Z_L, which completes the proof[7].

3. j **is surjective if** $X_{\overline{k}}$ **is connected**[8] **:** In this case, X_L is a Galois cover of X_k whose Automorphism group is naturally isomorphic to $\mathrm{Gal}(L/k)$. The claim easily follows.

The now-known-to-be-normal subgroup $j(\pi_1(X_{\overline{k}}))$ is called the *geometric fundamental group* of X.

7 Comparison with usual fundamental group

Suppose X is a finite type scheme over $\mathbb{C}$. There is a naturally associated complex analytic space $X(\mathbb{C})$ to X, and likewise to finite etale covers of Y. Moreover, it is not hard to see using our standard form for etale maps together with the inverse function theorem that if $Y \to X$ is finite etale, then $Y(\mathbb{C}) \to X(\mathbb{C})$ is a covering space. Thus there is a functor $F : FET/X \to FCOV/X(\mathbb{C})$, where $FCOV/X$ is the category of finite-degree covering spaces.

Theorem 7.1. *The functor X is an equivalence of categories.*

The hardest part by far of this theorem is showing that covering spaces of $X(\mathbb{C})$, which are a-priori only complex analytic spaces, in fact come from schemes. This is usually called the Riemann existence theorem, and even for curves it is quite an achievement. For a point $x \in X(\mathbb{C})$ let $\overline{x}$ be the corresponding geometric point. Then, denoting by $\pi_1(X(\mathbb{C}), x)$ the topological fundamental group, we have the following:

Corollary 7.2. *There is a natural map $\pi_1(X, x) \to \pi_1(X(\mathbb{C}), \overline{x})$ whose image is dense, and thus identifies the etale fundamental group with the profinite completion of the usual fundamental group.*

Proof. The finite quotients of the first group and the finite,continuous quotients of the second group, which implies the lemma.

□

As an exercise, the reader can go back over our list of examples and verify the compatibility of the above corollary with our computations.

[7]Can you fill in the details? We know the quotient exists by the previous lecture, but why is it finite etale over X_k, and why is the base change Z_L?

[8]By definition, this is equivalent to X_k being geometrically connected.

7.1 Don't get greedy!

One cannot hope for an algebraic definition of $\pi_1(X(\mathbb{C}))$. The reason is that Serre found an example of a variety X over Spec $\mathbb{C}$, such that if we set X^σ to be the base change along $\sigma : \text{Spec } \mathbb{C} \to \text{Spec } \mathbb{C}$ then $\pi_1(X(\mathbb{C}), x)$ is NOT isomorphic to $\pi_1(X^\sigma(\mathbb{C}), x)$[9] But of course, algebraicaly the covers $X \to \text{Spec } \mathbb{C}$ and $X^\sigma \to \text{Spec } \mathbb{C}$ are isomorphic.

7.2 The most interesting group in mathematics

Is arguably $\pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\})$. By theorem 7.2, $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}) = \widehat{F_2}$, the profinite completion of the free group on two elements[10]. Thus, as we saw in the previous section there is a sequence

$$1 \to \widehat{F_2} \to \pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Understanding anything about this sequence would be pretty sweet.

[9] Of course, they have the same profinite completion.

[10] There is still no purely 'algebraic' proof of this fact!