Lecture 3: Flat Morphisms

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1 A crash course on Properties of Schemes

For more details on these properties, see [Hartshorne, II, §1-5].

1.1 Open and Closed Subschemes

If $(X, \mathcal{O}_X)$ is a scheme, and $U \subset X$ is an open set, then we say that $(U, \mathcal{O}_X \mid U)$ is an open subscheme of X. More generally, we say that $\phi : Y \to X$ is an *open immersion* if the image of ϕ is an open subset U of X, and $\phi : Y \to U$ is an isomorphism of schemes[1].

For closed schemes, one must take care to describe the sheaf of ring, as closed subsets do not have sheaves of rings on them. Thus, we say if $Z \subset X$ is a closed subset, and $\mathcal{O}_Z$ is a sheaf of rings on Z, then we say that $(Z, \mathcal{O}_Z)$ is a closed subscheme of X if there is an (equivalently, for every) affine open cover $U_\alpha \cong \text{Spec } A_\alpha$ of X, the map $(Z \cap U_\alpha, \mathcal{O}_Z \mid Z \cap U_\alpha) \to U_\alpha$ looks like Spec $A_\alpha/I \to$ Spec A_α where I is an ideal of the ring A. More generally, we say that $\phi : Z \to X$ is a *closed immersion* is topologically a homeomorphism of Z with a closed subscheme of X such that the induced map on sheaves is locally like Spec $A_\alpha/I \to$ Spec A_α, or equivalently, the map on sheaves is surjective on stalks.

1.2 Fiber Products

If X, Y are S-schemes, that is schemes with maps to a base scheme S. We say that a morphism between S-schemes is a morphism of schemes that commutes with the morphisms to S. There is a scheme $X \times_S Y$ with morphisms

[1] In this case note that we suppress describing the sheaf of rings on U, with the understanding that its the restriction of the sheaf of rings from X. We frequently suppress the sheaf of rings when its clear from context

to X and Y which is the fiber product of X and Y over S in the category of schemes. That is, for any scheme W, the following sequence is exact:

$$0 \to \mathrm{Hom}(W, X \times_S Y) \to \mathrm{Hom}(W, X) \times \mathrm{Hom}(W, Y) \rightrightarrows \mathrm{Hom}(W, S).$$

Pictorially, we say that

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

is a Cartesian square. For affine schemes, fiber product is dual to the tensor product so that Spec $B \times_{\text{Spec } A}$ Spec $C \cong$ Spec $(B \otimes_A C)$. For more general schemes, the fiber product can be constructed by taking affine open covers and gluing. For details, see [Hartshorne, II, Thm 3.3]. If $\phi : Y \to X$ is a morphism of schemes, and $x \in X$ then we define $Y_x := Y \times_X$ Spec $k(x)$. Y_x i a scheme over Spec $k(x)$ and one can show that its underlying topological space is $\phi^{-1}(y)$. In general, for any morphism $Z \to X$ we say that $\phi_Z : Y \times_X Z \to Z$ is the *base change* of ϕ (resp. Y) to ϕ_Z (resp. $Y \times_X Z$).

Many properties of morphisms are preserved under base change, such as open and closed immersions.

1.3 Reduced Subscheme

We say that a scheme X is *reduced* if each affine open subset of X (equivalently, for some affine open cover) is Spec A where A has no nilpotents. Equivalently, each stalk of X has no nilpotent elements. For any scheme X we may define associated reduced subscheme X^{red} to be the closed subscheme localy defined by Spec $A/N \to$ Spec A where N is the nilpotent ideal of A. Moreover, for each closed subset $Y \subset X$ there is a unique sheaf of ideals $\mathcal{O}_Y$ on Y such that $(Y, \mathcal{O}_Y)$ is a reduced, closed subscheme of X. This is called the *reduced, induced subscheme structure* on Y.

1.4 Various Additional Properties of Morphisms of Schemes

We say that $\phi : Y \to X$ is *finite type* if X may be covered by affine open sets Spec B such that each inverse image ϕ^{-1}(Spec B) can be covered by finitely many affine open sets Spec A such that A is finitely generated as a B-algebra. We say that ϕ is *affine* if each inverse image ϕ^{-1}(Spec B) is an affine open subscheme, and we say that ϕ is *finite* if each inverse image

$\phi^{-1}(\text{Spec } B)$ is an affine open subscheme Spec A where A is a finite B-algebra. All of these properties can be checked on an arbitrary affine open cover of X.

Further, we say that a scheme X is *Noetherian* if X can be covered by finitely many open sets of the form Spec A where A is a Noetherian ring. We shall frequently restrict to Noetherian schemes for simplicity.

1.5 Separated and proper morphisms

The category of schemes carries with it some pathologies. For instance, if we take the two schemes Spec $\mathbb{C}[x]$ and Spec $\mathbb{C}[y]$ and glue them along the open subschemes Spec $\mathbb{C}[x, 1/x]$ and Spec $\mathbb{C}[y, 1/y]$ with the isomorphism $x \to y$ we get a scheme X which looks like the affine complex line $\mathcal{A}^1_{\mathbb{C}}$ except it has two points where 0 should be. I.e. there are two maps from $\mathcal{A}^1_C$ which agree everywhere except at 0. We say that X is *non-separated*, and frequently(though not always) want to avoid this type of pathology.

As with a lot of concepts about schemes, it is better to study separated morphisms rather then schemes. One reason is that you might have two non-separated schemes, and a nice separated morphism between them. Thus, we say that $Y \to X$ is *separated* if $\Delta : Y \to Y \times_X Y$ is a closed immersion, where Δ represents the diagonal map correponding to the two identity maps on Y through the universal property.[2]

Equivalently, in the case that Y is Noetherian, one has a nice check on whether a morphism is separated just using local rings, called the *valuative criterion*: $Y \to X$ is separated if for each discrete valuation ring R with fraction field K, and commutative diagram

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec } R & \longrightarrow & X \end{array} \tag{1}$$

there is at most one morphism from Spec R to Y making the diagram commute.

There is a related notion of a proper morphism of schemes: we say that $\phi : Y \to X$ is *proper* if it is separated, finite type, and *universally closed*. That is, for any morphism $Z \to X$, the base changed map $\phi : Y \times_X Z \to Z$

[2] A separated scheme should be thought of as being analogous to a Hausdorff topological space.

is closed a a map on topological sets; i.e. it maps closed sets to closed sets.[3]

In the case that Y is Noetherian, one has a similar valuative criterion: for each commutative diagram as in (1), there exists a unique morphism from Spec R to Y making the diagram commute.

Both proper and separated are properties that are preserved under base change.

2 Flatness

We begin by studying flatness on the level of rings.

2.1 Flat modules

Let R be a ring and M be a module. For every exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of R modules, the complex

$$M_1 \otimes_R M \to M_2 \otimes_R m \to M_3 \otimes_R M \to 0$$

is exact. Thus, we say that tensoring with M is a *right exact* functor from R-modules to R-modules.

Definition. *We say that M is a* flat *R-module (or M is flat over R) if tensoring with M is exact, so that for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R modules, the complex*

$$0 \to M_1 \otimes_R M \to M_2 \otimes_R m \to M_3 \otimes_R M \to 0$$

is exact. Equivalently, M is flat if tensoring with M preserves injections.

Some examples:

- For every multiplicative set $S \subset R$ the ring R_S is flat as an R-module.
- If M is a flat A-module, and B is an A-algebra, then $M \otimes_A B$ is a flat B module.
- If M is a flat B-module, and B is a flat A-algebra, then M is a flat A-module.

[3] A proper scheme (over, say, Spec $\mathbb{C}$) should be thought of as being analogous to a compact Hausdorff topological space.

- Combining the last two facts, we see that for every multiplicative subset $S \subset R$ and flat module M over R, the module $M_S := M \otimes_R R_S$ is flat over both R_S and R.

Lemma 2.1. *If M is a non-zero R-module, then there exist a maximal ideal $\mathfrak{M}$ of R such that $M_{\mathfrak{M}}$ is non-zero. More generally, $M \to \bigoplus_{\mathfrak{M}} M_{\mathfrak{M}}$ is injective.*

Proof. The kernel of the natural map from M to $M_{\mathfrak{M}}$ is the set of elements $m \in M$ whose Annihilator ideal $Ann(m) := \{r \in R \mid rm = 0\}$ is not contained in $\mathfrak{M}$. Since every proper ideal is contained in some maximal ideal, the claim follows.

□

2.2 Flat morphisms of Schemes

Definition. *A map of schemes $Y \to X$ is a* flat morphism *if $\mathcal{O}_y$ is flat over $\mathcal{O}_{f(y)}$ for all $y \in Y$.*

Lemma 2.2. *A map of rings $f : A \to B$ is flat iff the map of schemes $f^{\#} : Spec\ B \to Spec\ A$ is flat.*

Proof. We must show that f is flat iff for every prime $P \in B$ and prime $Q \in A$ such that $f^{-1}(P) = Q$, the map $A_Q \to B_P$ is flat. Suppose f is flat. Then for all primes Q of A, B_Q is flat over A_Q. Now as B_P is flat over B_Q, it follows that B_P is flat over A_Q. Conversely, suppose that the exact sequence of A-modules $0 \to M \to N$ is no longer exact when tensored with B. Let K be the kernel of $M \otimes_A B \to N \otimes_A B$. Then by lemma 2.1 there exists a prime ideal P of B such that $K_P \neq 0$. The claim now follows from the fact that $M \otimes_A B_P = M \otimes_{A_Q} B_P$ where Q is the prime ideal generate by $f^{-1}P$.

□

2.3 Faithful Flatness

Definition. *We say that M is a* faithfully flat *R-module (or M is faithfully flat over R) if tensoring with M is exact, so that for every complex $0 \to M_1 \to M_2 \to M_3 \to 0$ of R modules, it is exact iff the complex*

$$0 \to M_1 \otimes_R M \to M_2 \otimes_R m \to M_3 \otimes_R M \to 0$$

is exact.

Lemma 2.3. *M is faithfully flat iff it is flat and $M/\mathfrak{M}M$ is non-zero for every maximal ideal $\mathfrak{M}$ in A.*

Proof. Suppose we have a sequence of A-modules

$$K \to M_1 \to M_2 \to M_3 \to C.$$

This sequence remains exact after tensoring by M (since M is flat), so we see that M is faithfully flat iff $N = 0 \leftrightarrow M \otimes_A N = 0$.

Let $n \in N$ be a non-zero element. This then induces an injection $R/I \hookrightarrow N$ by sending 1 to N where I is the annihilator ideal of n in M. Since M is flat, this stays an injection after tensoring with M. Thus, we see that M is faithfully flat iff $M/IM \neq 0$ for all proper ideals I. Since every proper ideal is contained in a maximal ideal, this is equivalent to the statement of the lemma.

□

Since morphisms between schemes induce local maps of local rings on stalks, the following lemma is crucial:

Lemma 2.4. *A flat, local map $f : A \to B$ of local rings is faithfully flat.*

Proof. Let $\mathfrak{M}_A$ and $\mathfrak{M}_B$ denote the maximal ideals. Suppose that M is a non-zero A-module such that $M \otimes_A B = 0$. Pick an injection $A/I \hookrightarrow M$ where I is a proper ideal of A. Since B is flat, it follows also that $B/f(I)B = N \otimes_A B = 0$. However,

$$f(I) \subset f(\mathfrak{M}_A) \subset \mathfrak{M}_B,$$

which is a contradiction.

□

Theorem 2.5. *A map of schemes $f^{\#}$: Spec $B \to$ Spec A is flat and surjective iff $f : A \to B$ is faithfully flat.*

Proof. First suppose that $f^{\#}$ is not surjective. Let Q be a prime ideal of A above B not in the image of $f^{\#}$. Then it is possible to show (excercise) that $f^{\#,-1}(Q)$ as a set is equal to the underlying set of Spec $B \otimes_A k(A/QA)$. Thus $B \otimes_A k(A/QA = 0$, and so f is not faithfully flat.

Conversely, suppose $f^{\#}$ is surjective. Then for every prime ideal P of A, there exist a prime ideal Q of B over P, so that $f^{-1}(Q) = P$. But then $B/f(P)B$ surjects onto $B/QB \neq 0$. Thus B is faithfully flat by lemma 2.3. □

Excercise 2.6. *Prove that Open immersions are flat, compositions of flat morphisms are flat, finite flat maps are open, and flatness is preserved under base extension.*

2.4 Openness of Flat maps

Finally, we wish to give some properties of flat maps. First some definitions:

Definition. *If X is a scheme and $x, y \in X$ then x is called a* specialization *of y if x is in the topological closure of y. Likewise, y is called a* generization *of x.*[4]

Excercises:

- If $X = \text{Spec } A$ is as an affine scheme, then its points are prime ideals. Prove that P is a generization of Q iff $P \subset Q$.
- If X is any scheme, and $x \in X$, show that there is a natural map $\text{Spec } \mathcal{O}_{X,x} \to X$, that on points this map is injective, and its image consists of all generizations of x. **HINT: reduced to the affine case.**

Lemma 2.7. *Let $f : Y \to X$ be a flat map, such that $f(y) = x$. If x' is a generization of x, then there exists $y' \in Y$ s.t. $f(y') = x'$. In other words, the image of a flat map is closed under generizations.*

Proof. Consider the diagram

$$\begin{array}{ccc} \text{Spec } \mathcal{O}_{Y,y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec } \mathcal{O}_{X,x} & \longrightarrow & X \end{array}$$

Since the map $f_y : \text{Spec } \mathcal{O}_{Y,y} \to \text{Spec } \mathcal{O}_{X,x}$ is dual to a local flat map, it is surjective on points by lemma 2.4 and theorem 2.5. The theorem follows by the two exercises above.

□

For our main theorem, we shall need the following input:

[4] The reason for the definition is that if X is a complex variety, then its points correspond to irreducible closed subvarieties, and under this correspondence x is a specialization of y iff the corresponding variety for x is contained in the corresponding variety for y.

Theorem 2.8. *(Chevalley) Let $f : Y \to X$ be a finite-type morphism of schemes, where X, Y are Noetherian. Then $f(Y)$ is a constructible subset of X. In other words, there are finitely many pairs U_i, V_i of open and closed ssubets of X, such that $f(Y) = \bigcup_i U_i \cap V_i$.*

Theorem 2.9. *Let $f : Y \to X$ be a flat, finite-type morphism between Noetherian schemes. Then f is* open. *I.e. f maps open sets to open sets.*

See also [Milne, Etale Cohomology, I, 2.13] for a slightly more general statement.

Proof. Let $W = X - f(Y)$, and $\overline{W}$ the topological closure of W. Let $(Z_i)_{i \in I}$ be the irreducible components of $\bar{W}$. On affine open subsets, these correspond to minimal prime ideals so each Z_i has a generic point z_i. Suppose that $z_i \in f(Y)$. Then by theorem 2.8 there is an open set U and a closed set V such that

$$z_i \in U \cap V \subset f(Y).$$

Since V is closed it follows that V contains Z_i, so wlog $V = Z_i$. But then

$$U_2 := U \cap (X - \bigcup_{j \neq i} Z_j)$$

is an open set within $f(Y)$ containing z_i. But then $W \subset X - U$ so that $\bar{W} \subset X - U$, which is a contradiction since $z_i \in W$. Thus, we conclude that $z_i \in f(Y)$.

Every point in $\overline{W}$ is the specialization of some z_i, hence by lemma 2.7 we see that $W = \overline{W}$ and thus W is closed as desired.

□

Excercises:

- Prove that if $f : Y \to X$ is finite, then the image of f is closed under specialization. As an easy corollary, prove that finite maps between Noetherian schemes are proper. (In fact, one need not assume Noetherianity. See the Stacks Project, Tag 00GU: stacks.math.columbia.edu/tag/00GU).
- Prove that finite flat maps between Noetherian Schemes are surjective.