Lecture 2: Review of Schemes

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1 Complex Varieties

Before reviewing schemes, let us briefly recall the way people used to do algebraic geometry; that is the language of varieties. The domain for this is always complex affine n-space $\mathbb{C}^n$, on which we have the ring of polynomials $\mathbb{C}[x_1, x_2, \ldots, x_n]$.

Definition. *A* Complex Subvariety V *of* $\mathbb{C}^n$ *is a subset of* $\mathbb{C}^n$*, such that there are finitely many polynomial equations* $f_1, \ldots, f_m$ *of which* V *is precisely the zero-set. That is,*

$$V = \{\vec{z} \mid f_1(\vec{z}) = \cdots = f_n(\vec{z}) = 0\}.$$

In fact, it is a bit clumsy to speak of defining polynomial equations since many equations can define the same variety. Thus, it is more natural to look the ideal I of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$, as every element in I will still vanish in V. Thus, to every variety I we can define a variety $V(I) \subset \mathbb{C}^n$ by letting $V(I)$ be the intersection of the zero sets of all the elements in I. Likewise, given a variety V we can define the ideal $I(V)$ of all polynomials vanishing on V. Now, this correspondence is not quite as nice as we'd like. For instance, while $V(I(V)) = V$, it is not quite true that $I(V(I)) = I$. However, this can be clarified:

Definition. *An ideal* I *in a ring* R *is* radical *if* $r^2 \in I \Leftrightarrow r \in I$.

It is easy to see that $I(V)$ is always radical. And in fact, there is a bijective correspondence

$$\{\text{Complex Subvarieties of } \mathbb{C}^n\} \longleftrightarrow \{\text{Radical Ideals of } \mathbb{C}[x_1, \ldots, x_n]\}.$$

This correspondence moreover takes irreducible varieties to prime ideals, and points to maximal ideals(this is the famous Hilbert Nullstelensatz). However, this notion of varieties has several drawbacks:

- It does not work very well in families. Namely, consider the variety given by $V(x^2 - y^2)$ in $\mathbb{C}^2$. This has a projection map to $\mathbb{C}$ given by taking the x co-ordinate. The pre-image under this projection of any given point is 2 points almost everywhere, but 1 point above $x = 0$. It would be better if we could somehow remember that the point above $x = 0$ comes from two distinct components.

- It does not work well over non-algebraically closed fields. Namely, if we consider the equation $x^2 + y^2 + 1 = 0$ over $\mathbb{R}$, we have no solutions. However, ideal generated by $x^2 + y^2 + 1$ is radical and non-trivial, so this situation is not great. Obviously, over arithmetic fields like $\mathbb{Q}$ the situation is even worse, and its not even clear how to begin dealing with varieties over *rings* like $\mathbb{Z}$.

- Even if we just work over $\mathbb{C}$, our notion of variety is not intrinsic, in that it relies on an embedding into an affine space $\mathbb{C}^n$. Much like abstract manifolds are often more natural to work with then subsets of $\mathbb{R}^n$, we would like a notion of an 'abstract variety'.

2 Locally Ringed Spaces

To begin our quest for a definition of an abstract variety, we make the following definition:

Definition. *Let X be a topological space, with $op(X)$ the underlying category of open subsets of X, with the morphisms being inclusions. A pre-sheaf of sets $\mathcal{F}$ on X is a contravariant functor $\mathcal{F} : op(X) \to \{sets\}$. Concretely, for every open set $U \subset X$ we get a set $\mathcal{F}(U)$, and for every inclusion of opens sets $U \subset V$ we get a restriction map*

$$r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that if $U \subset V \subset W$ we have

$$r_{W,U} = r_{V,U} \circ r_{W,V}.$$

Moreover, if each $\mathcal{F}$ has the additional structure of a group(resp. ring, field) which is preserved by all the restriction maps, then we say that $\mathcal{F}$ is a sheaf of groups(resp. rings, fields).

We say that $\mathcal{F}$ is a sheaf if for every open cover $(U_\alpha)_\alpha$ of an open set V, for every sequence $(s_\alpha \in U_\alpha)_\alpha$ such that

$$(*) \forall \alpha, \beta,\ r_{U_\alpha, U_\alpha \cap U_\beta} s_\alpha = r_{U_\beta, U_\alpha \cap U_\beta} s_\beta$$

there is a unique element $s \in \mathcal{F}(V)$ such that

$$\forall \alpha, r_{V,U_\alpha} s = s_\alpha.$$

We pause to rewrite (*) in more systematic language. Note that there are two natural maps from $\prod_\alpha \mathcal{F}(U_\alpha)$ to $\prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$. One takes $(s_\alpha)_\alpha$ to $(t_{\alpha,\beta})_{\alpha,\beta}$ where $t_{\alpha,\beta} = r_{U_\alpha, U_\alpha \cap U_\beta} s_\alpha$ and the other where $t_{\alpha,\beta} = r_{U_\beta, U_\alpha \cap U_\beta} s_\beta$, and the set of sequences on which these two maps agree are precisely the ones for which (*) holds. Thus, we can rewrite (*) as saying that

$$\mathcal{F}(V) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact in the middle, and the first map is an inclusion. If $\mathcal{F}$ has the additional structure of a group, then we can shorten this by saying that

$$0 \to \mathcal{F}(V) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact.

Definition. *A* ringed space *is a pair $(X, \mathcal{O})$ where X is a topological space, and $\mathcal{O}$ is a sheaf of rings on X.*

Examples:

- $\mathcal{O}(U)$ is the ring of complex continuous functions $\phi : U \to \mathbb{C}$
- X is a complex manifold, and $\mathcal{O}(U)$ is the ring of holomorphic functions $\phi : U \to \mathbb{C}$
- $X = \mathbb{C}$, where the topology is the *Zariski* topology; that is, open sets are the cofinite subsets of X; and $\mathcal{O}(U)$ is the ring of rational functions on $\mathbb{C}$ which are finite on U.

Definition. *Let $(X, \mathcal{O})$ be a ringed space, and $x \in X$. Then we define the* stalk *of $\mathcal{O}$ at x, written $\mathcal{O}_x$, as the direct limit of the rings $\mathcal{O}(U)$, where U ranges over open sets containing x and the maps are given by the restriction morphisms. Written formally,*

$$\mathcal{O}_x := \varinjlim \mathcal{O}(U).$$

In each of the three examples given above, note that the stalk at a point x is a local ring, where the maximal ideal is given by the (resp. continuous, holomorphic, rational) functions vanishing at x.

Definition. *A* locally ringed space *is a ringed space* $(X, \mathcal{O})$ *where all the stalks are local rings. If* $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ *are two locally ringed spaces, then a morphism between them is defined to be the following data:*

- *A continuous map* $\phi : X \to Y$
- *For each open set* $U \subset Y$ *, a map* $\phi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ *such that these maps commute with the restriction homomorphisms.*
- *For each point* $x \in X$*, the induced map* $\phi_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ *of local rings is a* local map*, which is to say it maps the maximal ideal to the maximal ideal.*

3 Schemes

We are now ready to define a scheme. Given a ring R, we define the affine scheme $X = \text{Spec } R$ to be the following locally ringed space:

1. As a set,Spec R is the set of all prime ideals of R.
2. As a topological space, a basis for the open sets is given by the sets $U(f)$, where $f \in R$, consisting of all prime ideals NOT containing f.
3. As a ringed space, the sheaf $\mathcal{O}_X$ is given by localization, so that $\mathcal{O}_X(U(f)) = R_f := R[x]/(xf - 1)$. On other open sets the sheaf is defined by gluing.
4. To see that Spec R is a locally ringed space, it suffices to notice that for a prime ideal P, $\mathcal{O}_{X,P} \cong R_P$.

Definition. *A* scheme *is a locally ringed space* $(X, \mathcal{O}_X)$ *which locally around every point is isomorphic to an affine scheme. I.e.*

$$\forall x \in X, \exists U \in op(X) \mid x \in U, (U, \mathcal{O}_X \mid U) \cong \textit{Spec } \mathcal{O}_X(U).$$

Morphisms between schemes are simply morphisms in the category of locally ringed spaces.

Theorem 3.1. *Hom(Spec* R*, Spec* S*)* $\cong$ *Hom*(S, R).

Proof. There are natural maps in both directions, one way by pulling back prime ideals, and the other by looking at the induced map on global sections. The content of the theorem is that if $\phi : \text{Spec } R \to \text{Spec } S$ is a morphism of locally ringed spaces, with $\phi^{\#} : S \to R$ the induced map on rings, then we must have $\phi(P) = \phi^{\#,-1}(P)$. To prove this, suppose $\phi(P) = Q$. Then we get an induced map on stalks $\phi_Q : S_Q \to R_P$, which by assumption is local. Moreover, this must be compatible with $\phi^{\#}$. Since elements of $S\backslash Q$ are units in S_Q, it follows that $\phi^{\#}(S\backslash Q) \subset (R\backslash P)$, and since the map ϕ_Q is local it follows that $\phi^{\#}(Q) \subset P$. It then follows that $\phi^{\#,-1}(P) = Q$, as desired.

□

We list some examples of schemes:

- As $\mathbb{Z}$ is an initial object in the category of rings, Spec $\mathbb{Z}$ is a final object in the category of schemes. Note that the primes ideals are (p), where p is prime, and (0). Thus as a set, Spec Z has countably many points. One should check that the open sets of Spec $\mathbb{Z}$ are precisely the cofinite subsets which include (0).

- Spec $\mathbb{C}[x]$ consists of the points $(x - c), c \in \mathbb{C}$ and (0). The open sets are again precisely the cofinite subsets that include (0). Thus, we get a refinement of the Zariski topology on $\mathbb{C}$.

- Our old friend Spec $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, which has no real points, has lots of complex points. That is, for every complex solution $x = a + ib, y = c + id$ of $x^2 + y^2 + 1$ we get a different prime ideal $P \subset \mathbb{R}[x, y]/(x^2 + y^2 + 1)$ by pulling back $Q = (x - a - ib, y - c - id)$ from $\mathbb{C}[x, y]/(x^2 + y^2 + 1)$, except that $(a + ib, c + id)$ gives the same prime ideal as $(a - ib, c - id)$, Also, we have the prime ideal (0). Thus, the scheme Spec $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, as a topological space, looks like the complex solutions of $x^2 + y^2 + 1 = 0$, quotiented out by conjugation, together with the everywhere-dense point (0).

- We have many interesting one point schemes by looking at rings with nilpotents. For instance, Spec $\mathbb{F}[t]/t^m$ has only the one point (t), for each positive integer m and for each field $\mathbb{F}$. Thus, the sheaf of rings on a scheme is a truly important piece of structure.

- Given a subvariety $V \subset \mathbb{C}^n$ as we have defined it in the beginning of this lecture, we can now give it an intrinsic structure by considering the scheme Spec $\mathbb{C}[x_1, \ldots, x_n]/I(V)$. Moreover, there is no longer a need to restrict ourselves to radical ideals. For any ideal I, we have a scheme Spec $\mathbb{C}[x_1, \ldots, x_n]/I$. If I is not radical, this means our scheme will have nilpotent global sections. This is sometimes very useful; indeed, the systematic use of nilpotents is one of the major tools Grothendieck pioneered.

- We end by giving an example of a non-affine scheme. Consider the (isomorphic) schemes Spec $\mathbb{C}[x]$ and Spec $\mathbb{C}[y]$. If we remove the points $(x), (y)$ we get the two schemes $U(x) \cong$ Spec $\mathbb{C}[x, 1/x], U(y) \cong$ Spec $\mathbb{C}[y, 1/y]$. We construct a scheme by gluing Spec $\mathbb{C}[x]$ and Spec $\mathbb{C}[y]$ along the two open sets $U(x), U(y)$ using the isomorphism $x \to 1/y$. This gives us the scheme $\mathbb{P}^1_{\mathbb{C}}$; the projective complex line.