Lectures 12,13 - Derived Functors and Injectives

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1 Introduction to Cohomology

Let C be an abelian category We will explain how, for any left exact functor $F : C \to D$ to another abelian category, to define its *right derive functors* $R^iF : C \to D$, at least under sufficiently nice circumstances. To motivate their definition, we list the two properties we would like to have:

1. $R^0F = F$.
2. For any short exact sequence $0 \to A \to B \to C \to 0$ we get, functorially, a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \longrightarrow R^2F(A) \cdots$$

   so that any morphism between two short exact sequences yields a morphism between the associated long exact sequences in a functorial way.

We shall define R^iF to be universal with respect to the above properties. It turns out that it is very helpful to identify the following class of objects:

Definition. *An object $I \in ob(C)$ is* injective *if the contravariant functor $X \to Hom(X, I)$ is exact. In other words, if $A \hookrightarrow B$ is an injection, then any map from A to I can be lifted to a map from B to I. We say that an abelian category C has* enough injectives *if every object has a monomorphism into an injective object.*

Thus, whenever we have a sequence $0 \to I \to B \to C$ with I injective, we have a map $B \to I$ extending the identity map $I \to I$ and so we have a non-canonical splitting $B \cong I \oplus C$. From this it follows that the identity map $C \to C$ factors through the surjection $B \to C$, and thus the identity map on $F(C)$ factors through the map $F(B) \to F(C)$. It follows that this latter map is surjective, and thus $0 \to F(I) \to F(B) \to F(C) \to 0$ is exact. This suggests the property $R^iF(I) = 0$ for all $i > 0$. We shall see that for categories with enough injectives, this determines the functors R^iF! Thus, let us suppose first that we have functors R^iF satisfying 1.

Suppose A is any object. Embedding A into an injective object I and letting B be the cokernel, we get a short exact sequence. $0 \to A \to I \to B \to 0$. Taking the associated long exact sequence gives $R^1F(A) = \text{coker}{:}F(I) \to F(B)$ and for $i > 1$, $R^iF(A) \cong R^{i-1}F(B)$. Thus, to get the higher cohomology groups of A we should repeat the procedure for B by embedding it into an injective object as well, and so on. This proves that the functors R^iF are determined by conditions 1 and their vanishing on injectives.

With this discussion in mind, we make the following definition:

Definition. *An* injective resolution *of A is an exact complex $0 \to A \to I^0 \to I^1 \to \cdots$ such that all the I^n are injective. We often right $A \to I^\cdot$ for short.*

Given an injective resolution of A, we can break it up into short exact sequences $0 \to K^i \to I^i \to K^{i+1} \to 0$ where $K^0 = A$. For any n we deduce by forming the associated long exact sequence that

$$R^nF(A) = R^nF(K^0) = R^1F(K^{n-1}) = \text{coker}{:}F(I^{n-1}) \to F(K^n).$$

By left exactness, $F(K^n) \cong \ker{:}F(I^n) \to F(I^{n+1})$ and it follows that $R^nF(A)$ is the n'th cohomology group of the resolution $F(I^0) \to F(I^1) \to \cdots$ written shorthand as $F(I^\cdot)$. We thus make this our definition of right derived functors.

Definition. *The right derived functors $R^iF(A)$ are defined to be the cohomology groups of $F(I^\cdot)$ for any injective resolution $A \to I^\cdot$.*

The following theorem proves that the above definition is well defined and does indeed yield a functor; it is the main technical input in the theory of injective resolutions:

Theorem 1.1. *If $A \to I^\cdot$ and $A \to J^\cdot$ are two injective resolutions of A, then there is a natural isomorphism $H(F(I^\cdot)) \cong H(F(J^\cdot))$. As a consequence, the functors $R^iF(A)$ are well defined.*

Proof. Suppose $M^\cdot, N^\cdot$ are 2 complexes. We call $f : M^\cdot \to N^\cdot$ a map of complexes if $f : M^i \to N^i$ and $d_N f = f d_M$. It is easy to see that f induces maps $f : H^i(M^\cdot) \to H^i(N^\cdot)$. Given maps $h_i : M^i \to N^{i-1}$ it is easy to see that $d_N h + h d_M$ is a map of complexes inducing the 0 maps on homology. We call 2 maps $f, g : M^\cdot \to N^\cdot$ *homotopically equivalent* if there exists a map h such that $f - g = d_N h + h d_M$.

Lemma 1.2. *For two objects A, B, and injective resolutions $A \to I^\cdot$ and $B \to J^\cdot$, and a morphism $\bar{f} : A \to B$, there exists a morphism of complexes $f : I^\cdot \to J^\cdot$ inducing $\bar{f}$ on $A \cong H^0(I^\cdot) \to H^0(J^\cdot) \cong B$. Moreover, any two such lifts f_1, f_2 are homotopically equivalent.*

Proof. First we construct such an f. Since we have a map $\bar{f} : A \to B$,composing with the inclusion of B into J^0 we get a map $A \to J^0$ and as J^0 is injective, this lifts to a map $f^0 : I^0 \to J^0$. Now suppose by induction we have defined $f_n : I^n \to J^n$ in a way such that $d_J f_k = f_{k+1} d_I$ for $k < n$. Then

$$d_J \circ f_n(ker : I^n \to I^{n+1}) = d_J \circ f_n \circ d_I(I^{n-1}) = d_J \circ d_J \circ f_{n-1}(I^{n-1}) = 0.$$

It follows that the map $I^n \xrightarrow{d_J \circ f_n}$ induces a map $\overline{f_{n+1}} : d_I(I^n) \to J^{n+1}$, and by injectivity of J^{n+1}, this extends to a map $f_{n+1} : I^{n+1} \to J^{n+1}$. Moreover, $d_J f_n = f_{n+1} d_I$ by construction. Continuing by induction, we get a morphism $f : I^\cdot \to J^\cdot$ as desired.

Next, suppose that we have two such lifts f_1, f_2. Then $f_1 - f_2$ induces the zero map $A \to B$ and so it suffices to prove that any lift of the 0 map is of the form $d_N h + h d_M$. We do so by induction. First, as $f_0(A) = 0$, we see that $f_0 : I^0 \to J^0$ extends to a map $d(I_0) \to J^0$, which then lifts to a map $h_1 : I^1 \to J^0$ by injectivity of J^0. Now suppose we have define $h_n : I^n \to J^{n-1}$ by induction so that $f_k = d_J h_k + h_{k+1} d_I$ for $k < n$. Now, as

$$\begin{aligned}(f_n - d_J h_n)(ker : I^n\ xrightarrow d_I I^{n+1}) &= (f_n - d_J h_n) \circ d_I(I^{n-1}) \\ &= (d_J f_{n-1} - d_J \circ (f_{n-1} - d_J h_{n-1}))(I^{n-1}) \\ &= 0\end{aligned}$$

we see that $f_n - d_J h_n$ descends to a map on $d_I(I^n)$, which can by injectivity of J_n be extended to a map $h_{n+1} : I^{n+1} \to J^n$. Continuing by induction, we construct our homotopy h. □

To finish the proof of the theorem, construct a morphism $f : I^\cdot \to J^\cdot$ lifting the identity map on A. Applying the functor f gives the desired morphism $RFf : H^i(F(I^\cdot)) \to H^i(F(J^\cdot))$. If f' were another lift, then $f' - f = d_J h + h d_I$ by the lemma, and thus $Ff' - Ff = d_J Fh + Fhd_I$, and so it easily follows that $RFf = RFf'$. Finally, to see that the map RFf is an isomorphism construct a lifting $g : J^\cdot \to I^\cdot$ of the identity and compose.

□

1.1 Acyclic Resolutions

It is often the case that obtaining injective resolutions is very inconvenient. As such, we present a much more accessible way of computing right derived functors.

Definition. *An object A is said to be F-acyclic if $R^iFA = 0$ for $i \geq 1$.*

Lemma 1.3. *Suppose $0 \to A \to J^\cdot$ is an F-acyclic resolution of A. That is, it is an exact complex and J^i are $F-$acyclic for all $i \geq 0$. Then R^iFA is naturally isomorphic to the i'th cohomology of $0 \to FJ^\cdot$. In other words, cohomology can be computed using acyclic objects.*

Proof. As always, let us split up the long exact sequence into short exact sequences:

$$0 \to A \to J^0 \to K^0$$

and

$$\forall i \geq 1, 0 \to K^i \to J^i \to K^{i+1} \to 0.$$

By writing out the corresponding long exact sequences of right derived functors, and using the F-acyclicity of the J^i, we learn that

$$R^nFA \cong R^{n-1}K^0 \cong \cdots \cong R^1K^{n-2} \cong \text{coker}(FJ^{n-2} \to FK^{n-1}).$$

As F is left exact, FK^{n-1} is the kernel of $FJ^{n-1} \to FJ^n$, and the claim follows.

□

2 The category of Abelian Groups

Let us now apply the preceding theory to the category *Ab* of Abelian groups. In this case one may give a very concrete interpretation of injective objects:

Definition. *We say that an abelian group I is* divisible *if for every positive integer n, the multplication by n map $\times n : I \to I$ is surjective.*

Lemma 2.1. *An abelian group I is injective iff it is divisible/*

Proof. Suppose that I is injective, and let $x \in I$. Consider the map $f_x : \mathbb{Z} \to I$ which is defined by $f_x(1) = x$. Since I is injective this lifts to a map $g_x : \mathbb{Q} \to I$. Now $ng_x(\frac{1}{n}) = g_x(1) = f_x(1) = x$. Since x was arbitrary, this shows that I is divisible.

Now suppose I is divisible, $A \hookrightarrow B$ is an injection and $f : A \to I$ is a morphism. We wish to extend f to B. It suffices by transfinite induction to assume that B is generated over A by a single element x. Now either $B/A \cong \mathbb{Z}$ or $B/A \cong \mathbb{Z}/n\mathbb{Z}$. In the former case, simply extend f by sending x to 0. In the latter case,

$$B \cong (A \oplus \mathbb{Z} \cdot x)/(a - nx)$$

for some $a \in A$. Now as I is divisible, we may pick $y \in I$ so that $ny = \phi(a)$. Then f can be extended by sending x to y. Thus I is injective.

□

Corollary 2.2. *Ab has enough injectives.*

Proof. Let A be an Abelian group. For each non-zero $x \in A$, let C_x be the group generated in A by x (so it is either $\mathbb{Z}$ or cyclic).C_x imbeds into an injective object I_x (either $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}$) and by injectivity, we can extend this to a map $\phi_x : A \to I_x$. Now consider the product map $\phi : A \to \prod_{x \in A} I_x$. Clearly ϕ is an embedding, and fro the universal property of products a product of injectives is injective. Thus ϕ is an embedding of A into an injective object. Since A was arbitrary, the claim follows.

□

As such we can talk about cohomology freely in *Ab*. Moreover, since every quotient of a divisible group is divisible, every object A has an injective resolution of length 2. That is, if we imbed A into an injective I, then

$$0 \to A \to I \to I/A \to 0^{\cdot}$$

is an injective resolution of A. We thus have the following corollary:

Corollary 2.3. *For every left exact functor $F : Ab \to C$,the functors R^iF vanish for $i > 1$.*

3 (Pre)sheaves have enough injectives

We warn the reader that this section gets a little more category-theoretically and set-theoretically hairy, and for the purpose of these lecture notes (and, indeed, much of ones existence) one may comfortable accept that the categories of sheaves and presheaves on any site have enough injectives without worrying about the proofs.

The following exercise is extremely useful in proving that various categories have enough injectives, by 'bootstrapping' of other categories:

Excercise 3.1. *Suppose $F : C \to D$ is a functor with a left adjoint $G : D \to C$, so that $Hom(A, FB) \cong Hom(GA, B)$, and G is exact. Then F preserves injectives.*

We being by showing that on any category C, the corresponding category of presheaves (of abelian groups) has enough injectives.

Lemma 3.2. *For any category C, the category of presheaves $P(C)$ has enough injectives.*

Proof. Denote C to be a category, $Ob(C)$ to be the category with the same objects as C but NO maps except for the identity maps. The proof proceeds in stages:

1. $P(ob(C)$ has enough injectives. In fact, $P(ob(C))$ is just $\prod_{x\in Ob(C)} Ab$ as a category. Since by theorem 2.2 we know Ab has enough injectives, the same follows for the product category (as one may easily verify that a product of injectives is injective).

2. Consider the forgetful functor $f : P(C) \to P(ob(C))$ and the functor $u : P(ob(C) \to P(C)$ which to a presheaf $A \in P(ob(C))$ assigns the presheaf
$$uA(X) := \prod_{\phi:Y\to X} A(Y),$$
where the product is over ALL maps $\phi : Y \to X$ and the transition map associated to $\psi : X' \to X$ is given by
$$(a_\phi)_\phi \to (a_{\psi\circ\phi'})_{\phi'}.$$
We claim that u is the right adjoint to f. To prove this, we must show that $\mathrm{Hom}(fP_1, P_2) \cong \mathrm{Hom}(P_1, uP_2)$. Given an element $\phi \in \mathrm{Hom}(fP_1, P_2)$ we get an element $F(\phi) : P_1 \to uP_2$ as follows: for

$t \in P(X)$, $F(\phi)_X(t) = (\phi_Y(t_Y))_{\phi:Y\to X}$. Conversely, given an element $\psi \in \mathrm{Hom}(P_1, uP_2)$ we define an element $G(\psi) : fP_1 \to P_2$ as follows: For $t \in P(X)$, $G(\psi)(t) = G(t)_{Id_X}$. We leave it to the reader to show that F, G define inverse maps.

3. A naturally imbeds into ufA via the canonical map

$$m \in A(X) \to (m_Y)_{\phi:Y\to X},$$

which can be gotten via the adjointness above.

4. Since u has an exact left adjoint, it follows by excercise 3.1 that u preserves injectives. Now, take $A \in P(C)$. Then since we saw that $P(ob(C))$ has enough injectives, we can write $fA \hookrightarrow I$ where I is injective. Then $A \hookrightarrow ufA \hookrightarrow uI$, which completes the proof.

□

Our next goal is to see that $S(C')$ has enough injectives where C' is an arbitrary cite with underlying category C, and $S(C')$ is the site of sheaves on it. For a presheaf P, let $P \to J(P)$ be an imbedding of P into an injective presheaf $J(P)$. Moreover, recall the forgetful functor $i : S(C') \to P(C)$ and the sheafification functor $a : P(C) \to S(C')$. a is exact and left adjoint to i. Now, given a sheaf $\mathfrak{F}$, we define $J_1(\mathfrak{F}) = aJ(i\mathfrak{F})$. Clearly $\mathfrak{F} \to J_1(\mathfrak{F})$ and this map is injective a a is exact. Now define $J_{i+1}(\mathfrak{F}) = J_1(J_i(\mathfrak{F}))$ and for a limit ordinal α define $J_\alpha(\mathfrak{F})$ to be the direct limit $\varinjlim_{\beta<\alpha} J_\beta(\mathfrak{F})$. As direct limit are exact, we have an injective map $J_\alpha(\mathfrak{F}) \to J_{\alpha'}(\mathfrak{F})$ whenever $\alpha < \alpha'$. Now, given an injective map of sheaves $\mathfrak{F}_1 \hookrightarrow \mathfrak{F}_2$ and a map $\mathfrak{F}_1 \to J_\alpha(\mathfrak{F})$, we get maps

$$i\mathfrak{F}_1 \to iJ_\alpha(\mathfrak{F}) \to J(iJ_\alpha).$$

Thus, by injectivity we can extend this to a map $i\mathfrak{F}_2 \to J(iJ_\alpha(\mathfrak{F}))$ and by adjointness we get a map

$$\mathfrak{F}_2 \to J_{\alpha+1}(\mathfrak{F}).$$

Thus, this somehow says that the direct system $J_\alpha(\mathfrak{F})$ gives an injective embedding of $\mathfrak{F}$. Of course we cannot take the limit over all ordinals, as they do not form a set[1]. We will now show that in fact we can stop at some point and take a fixed $J_\beta(\mathfrak{F})$.

[1]This may seem like we're being stubborn at this point, to those who dislike set-theoretic issues. But in fact using ordinals in the first place was sort of a cheat to try and make things work. Now that we've chosen to dip our toe in the water, we should make sure that there aren't any sharks!

Lemma 3.3. *For each $X \in ob(C)$ define $\mathbb{Z}_X$ to be the presheaf given by $\mathbb{Z}_X(U) := \oplus_{\phi:U\to X}\mathbb{Z}$. Let I be a sheaf such that for every X and every subsheaf S of $a\mathbb{Z}_X$ and morphism from S to I, there is an extension to a morphism from $a\mathbb{Z}_X$ to I. Then I is injective.*

Proof. Suppose $\mathfrak{F}_1 \hookrightarrow \mathfrak{F}_2$ and let $\phi : \mathfrak{F}_1 \to I$ be a morphism. Now consider an $X \in ob(C)$ and a section $s \in \mathfrak{F}_2(X)$ that does not occur in $\mathfrak{F}_1(X)$. Now, we have by adjointness that

$$\hom(a\mathbb{Z}_X, \mathfrak{F}_2) \cong \hom(\mathbb{Z}_X, i\mathfrak{F}_2) \cong \mathfrak{F}_2(X).$$

Thus, we have a morphism $\psi : a\mathbb{Z}_X \to \mathfrak{F}_2$ sending $1 \in \mathbb{Z}_X(X)$ to s. Now, let $S = \psi^{-1}(\mathfrak{F}_1)$. By assumption, the induces morphism $\phi \circ \psi : S \to I$ extends to a morphism $\phi' : a\mathbb{Z}_X \to I$.Since ϕ' kills the kernel of ψ, we get a map from $\Im(\phi')$ to I which agrees on $\mathfrak{F}_1 \cap \Im(\phi')$ by construction. Thus we can glue and get an extension of ϕ to the strictly bigger subsheaf $\mathfrak{F}' = \mathfrak{F} + \Im\phi'$. Continuing by transfinite induction completes the proof.

□

Theorem 3.4. *$S(C')$ has enough injectives.*

Proof. Let α be the cardinality of $ob(C)$ multiplied by the cardinality of $\bigcup_{X,U} |a\mathbb{Z}_X(U)|$. Now choose an ordinal β with cofinality larger than α. We claim that for any sheaf $\mathfrak{F}, J_\beta(\mathfrak{F})$ is injective. Since we have already seen that $\mathfrak{F}$ embeds into $J_\beta(\mathfrak{F})$, this completes the proof.

To prove this, we apply lemma test. So let S be a subsheaf of $a\mathbb{Z}_X$ for some $X \in ob(C)$. Consider a map from $\phi : S \to J_\beta(\mathfrak{F})$. Since the cofinality of β was chosen to be larger than the cardinality of the set of all sections of S, it follows that ϕ factors through $J_\alpha(\mathfrak{F})$. Thus $a\mathbb{Z}_X$ maps into $J_{\alpha+1}(\mathfrak{F})$ which then maps into $J_\beta(\mathfrak{F})$, as desired. This completes the proof.

□

3.1 Neat fact

Because its cute and much easier, we give an independent proof that the category of etale sheaves has enough injectives:

Corollary 3.5. *The category $S(X_{et})$ has enough injectives.*

Proof. First, if X is the spectrum of a separably closed field then $S(X_{et}) \cong$ *Ab*, and the corollary follows from lemma **??** .Next, for any X, consider the scheme $X^0 := \bigcup_{\bar{x}} \bar{x}$ and the natural map $f : X^0 \to X$. Now f^* is exact since taking stalks is exact by lemma **??**. Moreover, f^* is left adjoint to f_* and thus it follows from exercise 3.1 that f_* preserves injectives. Hence, for any sheaf F on X_{et}, imbed $f^*F \hookrightarrow I$ where I is injective. Then $f_*f^*F \hookrightarrow f_*I$ is a monomorphism since f_* is exact. Finally, since $\mathrm{Hom}(F, f_*f^*F) \cong \mathrm{Hom}(f^*F, f^*F)$ by adjointness we have a natural map $F \to f_*f^*F$ corresponding to the identity morphism. By lemma **??** this is an imbedding, so we get an imbedding $F \hookrightarrow f_*I$ as desired. □

3.2 Cohomology of sheaves

We can now define right derived functors of any left exact functor on the category of Sheaves, $S(X_{et})$. We list some examples of functors that we shall use:

1. Most importantly, the global sections functor $F \to \Gamma(X, F)$ is left exact, and we write $R^i\Gamma(X, F)$ or more frequently just $H^i(X, F)$ to denote it's right derived functors. This group is called *the i'th cohomology group of X with values in F.*

2. For any etale cover $U \subset X$ we write $H^i(U, F)$ for the right derived functors of $F \to \Gamma(U, F)$. Note that these are not a-priori the same as the groups $H^i(U, F \mid U)$. In fact, they are isomorphic, as we shall later show.

3. The inclusion functor $i : S(X_{et}) \to P(X_{et})$ is left exact. We denote its right derived functors by $\underline{H}^i(F)$.

4. For any fixed F_0, the functor $F \to \mathrm{Hom}(F_0, F)$ is left exact. We denote its right derived functors by $\mathrm{Ext}^i(F_0, F)$.

5. Likewise, the functor $S(X) \to S(X)$ given by $U \to \mathrm{Hom}(F_0 \mid U, F \mid U)$ is left exact. We denote its right derived functors by $\underline{Ext}^i(F_0, F)$.

6. Fially, for any continuous morphism $\pi : X_{et} \to X'_{et}$ we denote the right derived functors of $F \to \pi_*F$ by $R^i\pi F$. These are called the *higher direct image* sheaves of F.