Lecture 11 - Decomposition of Sheaves

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1 Behavior of Stalks under Immersions

In this lecture we will develop a way to understand Etale sheaves on a space by cutting it up into 'simpler' pieces. Specifically, we consider the scenario where X is a scheme, Z is a closed subscheme, and $U = X - Z$ is the complementary open subscheme. We denote the maps by

$$Z \xrightarrow{i} X \xleftarrow{j} U.$$

Given a sheaf F on X, we get sheaves $F_1 = i_* F$ and $F_2 = j^* F$ on Z, U resp. Our goal is to 'build' F out of these two sheaves. To do this, we rely heavily on the use of stalks. As such, we have the following useful lemma

Lemma 1.1. *For any morphism $\pi : Y \to X$ and geometric point $\overline{y}$ of Y, and $\mathfrak{F} \in S(X_{et})$, we have $(\pi^* \mathfrak{F})_{\overline{y}} \cong \mathfrak{F}_{\pi(\overline{y})}$. If moreover π is a closed immersion, and $\mathfrak{F}' \in S(Y_{et})$ then $\mathfrak{F}'_{\overline{y}} \cong (\pi_* \mathfrak{F}')_{\pi(\overline{y})}$.*

Proof. The first part of the lemma follows since we know from last lecture that $\overline{y}^* \circ \pi^* \cong (\pi \circ \overline{y})^*$. Assume now that π is a closed immersion. Tracing through the definition of the stalk, we have that

$$\mathfrak{F}'_{\overline{y}} = \varinjlim \mathfrak{F}'(V)$$

where the limit is over based etale covers $(V, \overline{v})$ of $(Y, \overline{y})$ whereas

$$(\pi_* \mathfrak{F}')_{\pi(\overline{y})} = \varinjlim \mathfrak{F}'(U \times_X Y)$$

where the limit is over based etale overs $(U, \overline{u})$ of $(X, \pi(\overline{y}))$. The second limit clearly has a map to the first, and to prove equality we must show that it has cofinal image. Thus, we have to prove that for every $(V, \overline{v})$ there exists a $(U, \overline{u})$ such that $U \times_X Y$ has a Y-morphism to V which carries $\overline{u}$ to $\overline{v}$.

To prove this, it clearly suffices to work locally on X. By our work on standard etale morphisms in lecture 5, we know that by replacing X with an open neighborhood we can assume that $X = \text{Spec } A, Y = \text{Spec } A/I$ and $V = \text{Spec } \overline{B}_{\overline{b}}$ where $\overline{B} = A/I[x]/\overline{P}(x)$ for some monic polynomial $\overline{P}(x)$. Now consider $U = \text{Spec } B_b$ where $B = A[x]/P(x)$ where $P(x)$ is any lift of $\overline{P}(x)$ and $b = P'(x)b_0$ where b_0 is any lift of $\bar{b}$. Then clearly U is etale over X and $U \times_X T \cong V$. This completes the proof. $\square$

1.1 The canonical map

For any morphism $\pi : Y \to X$, and any sheaves $\mathfrak{F} \in S(X_E), \mathfrak{F}' \in S(Y_E)$ we know that $\text{Hom}(\pi^*\mathfrak{F}, \mathfrak{F}') \cong \text{Hom}(\mathfrak{F}, \pi_*\mathfrak{F}')$. Plugging in $\mathfrak{F}' = \pi^*\mathfrak{F}$ we get

$$\text{Hom}(\pi^*\mathfrak{F}, \pi^*\mathfrak{F}) \cong \text{Hom}(\mathfrak{F}, \pi_*\pi^*\mathfrak{F}).$$

The left hand side contains the identity map, and so corresponding to it we get a natural map $\mathfrak{F} \to \pi_*\pi^*\mathfrak{F}$, which we refer to as the canonical map.

If one traces through the definitions, we see that for $U \to X$ an E-morphim,

$$\pi_*\pi^*\mathfrak{F}(U) = \varinjlim_S \mathfrak{F}(V)$$

where the limit is over all squares S given by

$$\begin{array}{ccc} U \times_X Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}\ .$$

There is a natural square S_0 with $V = U$ corresponding to the fiber product, and thus we get a natural map $\mathfrak{F}(U) \to \pi_*\pi^*\mathfrak{F}(U)$. If one traces through the definitions, one can verify that this is the canonical map.

2 Decomposing Sheaves: The Category $T(X)$

It is now easy to confirm using lemma 1.1 that for a sheaf $\mathfrak{F} \in S(X_{et})$ we have the following table. Note that we write $\overline{x} \in Z_{et}$ or $\overline{x} \in U_{et}$ for a geometric point $\overline{x} \in X_{et}$ if the map given by $\overline{x}$ factors through Z or U respectively.

$$(i_*i^*\mathfrak{F})_{\overline{x}} \cong \begin{cases} \mathfrak{F}_{\overline{x}} & \overline{x} \in Z \\ 0 & \text{else.} \end{cases}$$

$$(j_*j^*\mathfrak{F})_{\overline{x}} \cong \begin{cases} \mathfrak{F}_{\overline{x}} & \overline{x} \notin Z \\ ? & \text{else.} \end{cases}$$

What do the stalks at points $\overline{z} \in Z$ look like for j_*j^*F? Tracing through the definitions, one can see that

$$(j_*j^*\mathfrak{F})_{\overline{z}} = \varinjlim \mathfrak{F}(U' \times_X U)$$

where U' runs through the etale neighborhoods of $\overline{z}$ in X. This is a bit of a mysterious object, but nontheless we see that we can recover the stalks of $\mathfrak{F}$ at Z from $i_*\mathfrak{F}_1$ and at U from $j_*\mathfrak{F}_2$. Moreover, the canonical map $\mathfrak{F} \to j_*\mathfrak{F}_2$ induces a map $\phi : \mathfrak{F}_1 \to i^*j_*\mathfrak{F}_2$ by applying the functor i^*. Viewing ϕ as the 'gluing map' between $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we make the following definition:

Definition. *consider the category $T(X)$ whose objects are triples $(\mathfrak{F}_1, \mathfrak{F}_2, \phi)$ with $\mathfrak{F}_1 \in S(Z_{et}), \mathfrak{F}_2 \in S(U_{et})$ and $\phi : \mathfrak{F}_1 \to i^*j_*\mathfrak{F}_2$, and a morphism $(\mathfrak{F}_1, \mathfrak{F}_2, \phi) \to (\mathfrak{F}_1', \mathfrak{F}_2', \phi')$ is a pair of maps $\psi_i : \mathfrak{F}_i \to \mathfrak{F}_i', i = 1, 2$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{F}_1 & \xrightarrow{\phi} & i^*j_*\mathfrak{F}_2 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle i^*j_*\psi_2} \\ \mathfrak{F}_1' & \xrightarrow{\phi'} & i^*j_*\mathfrak{F}_2' \end{array}$$

is commutative.

3 The Main Theorem

Theorem 3.1. *The functor $t : \mathfrak{F} \to (i_*\mathfrak{F}, j^*\mathfrak{F}, \phi)$ induces an equivalence of categories $S(X_{et}) \to T(X)$.*

Remark. *Note that this statement does not depend on the scheme structure given to Z, merely on its being a closed subset of X. This is consistent with our analysis in lecture 5, where we showed that the etale site only depends on the reduced subscheme of a scheme.*

Proof. First, it is easy to see that t is a functor. We now define a functor s in the opposite direction. We do this as follows, given $(\mathfrak{F}_1, \mathfrak{F}_2, \phi)$ we define a sheaf $s(\mathfrak{F}_1, \mathfrak{F}_2, \phi)$ to be the fiber product of $i_*\mathfrak{F}_1$ and $j_*\mathfrak{F}_2$ over $i_*i^*j_*\mathfrak{F}_2$, where the first map is $i_*\phi$ and the second map is the canonical map. From the universal property of fiber products, we see that this indeed induces a functor. For any $\mathfrak{F}$, the canonical maps $\mathfrak{F} \to i_*i^*\mathfrak{F}$ and $\mathfrak{F} \to j_*j^*\mathfrak{F}$ give us a functorial map $\mathfrak{F} \to st(\mathfrak{F})$. To see that this map is an isomorphism, it suffices to prove that the diagram

$$\begin{array}{ccc} \mathfrak{F} & \longrightarrow & i_*i^*\mathfrak{F} \\ \downarrow & & \downarrow \\ j_*j^*\mathfrak{F} & \longrightarrow & i_*i^*j_*j^*\mathfrak{F} \end{array}$$

is Cartesian. To prove this, by our last theorem in the previous lectire it suffices to check that it is Cartesian on each stalk. So let $\overline{x} \in X_{et}$, and we have 2 cases to check. If $\overline{x} \in Z_{et}$ then taking stalks we get

$$\begin{array}{ccc} \mathfrak{F}_{\overline{x}} & \longrightarrow & \mathfrak{F}_{\overline{x}} \\ \downarrow & & \downarrow \\ (j_*j^*\mathfrak{F})_{\overline{x}} & \longrightarrow & (j_*j^*\mathfrak{F})_{\overline{x}} \end{array}$$

which is clearly Cartesian. Likewise,if $\overline{x} \in U_{et}$ then taking stalks gives

$$\begin{array}{ccc} \mathfrak{F}_{\overline{x}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{F}_{\overline{x}} & \longrightarrow & 0 \end{array}$$

which is again clearly Cartesian. Thus $\mathfrak{F} \to st(\mathfrak{F})$ is an isomorphism.

Since s and t can easily be checked (again, by working on stalks) under the identification $\mathfrak{F} \to st(\mathfrak{F})$ to induce inverse maps on morphisms, it suffices to prove that t is essentially surjective. Thus we must prove that $ts(\mathfrak{F}_1, \mathfrak{F}_2, \phi)$ is isomorphism to $(\mathfrak{F}_1, \mathfrak{F}_2, \phi)$. To see this, we first observe that the maps $i^*s(\mathfrak{F}_1, \mathfrak{F}_2, \phi) \to i^*i_*\mathfrak{F}_1 \cong \mathfrak{F}_1$ and $j^*s(\mathfrak{F}_1, \mathfrak{F}_2, \phi) \to j^*j_*\mathfrak{F}_2 \cong \mathfrak{F}_2$ are isomorphisms, again by working on stalks. To check that the induced map is identified with ϕ is now a straightforward check, one more time, by working on stalks. This completes the proof.

□

Now that we have this equivalence, we can define two other functors. The functor $j_! : S(U_{et}) \to S(X_{et})$ is given by $F \to t(0, F, 0)$ and is usually called the 'extension by zero' functor. The functor $i^! : S(X_{et}) \to S(Z_{et})$ is defined by $i^!(t(F_1, F_2, \phi)) = \ker(\phi)$ and is usually called the 'subsheaf with support on Z' functor.

Theorem 3.2. *1. i_* is left adjoint to $i^!$*

2. $j_!$ is left adjoint to j^*.

3. $i_*, i^*, j^*, j_!$ are exact, $i^!, j_*$ are left exact.

Proof. Suppose $\mathfrak{F} \in S(X_{et})$ and $\mathcal{G} \in S(Z_{et})$. Let $t(\mathfrak{F}) = (i^*\mathfrak{F}, j^*\mathfrak{F}, \phi)$. Note that $t(i_*\mathcal{G}) \cong (\mathcal{G}, 0, 0)$. To give a morphism from $i_*\mathcal{G}$ to $\mathfrak{F}$ is, by theorem 3.1 to give a morphism from $\mathcal{G}$ to $i^*\mathfrak{F}$ which lands in the kernel of ϕ, and thus is is simply a morphism from $\mathcal{G}$ to $i^!\mathfrak{F}$, establishing the first claim.

now suppose that $\mathcal{G} \in S(U_{et})$. Then $t(j_!\mathcal{G}) = (0, \mathcal{G}, 0)$ and so to give a morphism from $j_!\mathcal{G}$ to $\mathfrak{F}$ is simply to give a morphism from $\mathcal{G}$ to $j^*\mathfrak{F}$, establishing the second claim.

The exactness of $i_*, i^*, j^*, j_!$ follows from theorem 3.1 and the fact that exactness can be checked on stalks. The left exactness of $i^!, j_*$ follows since they are right adjoints. □

4 Example: Discrete Valuation Ring

To finish, we describe in somewhat more explicit language the above decomposition in the following case: $X = \text{Spec } A, U = \text{Spec } K, Z = \text{Spec } k$ where A is a discrete valuation ring, K is the function field of A and $k = A/\mathfrak{M}$ is the residue field of A. Number theorists can keep in mind the example $X = \text{Spec } Z_{(p)}, U = \text{Spec } \mathbb{Q}, Z = \text{Spec } \mathbb{F}_p$. Now let G_K be the galois group of K, and G_k the Galois group of k. Fix geometric points $\overline{\eta} : \text{Spec } K^{sep} \to X$ and $\overline{s} : \text{Spec } \overline{k} \to X$. Now recall that we proved in lecture 8 that the category of sheaves on Spec K is equivalent to the category of discrete G_K-modules via $\mathfrak{F} \to \mathfrak{F}_{\overline{\eta}}$, via the action of G_K on K^{sep}.

So suppose $\mathfrak{F} \in S(U_{et})$ with $\mathfrak{F}_{\overline{\eta}} \cong M$. To apply theorem 3.1 we must compute $i^*j_*\mathfrak{F}$, or equivalently $(j_*\mathfrak{F})_{\overline{s}}$ together with the action of G_k. Since X is normal, as observed in lecture 8 given a connected based etale cover $(Y, \overline{y})$ of $(X, \overline{s})$ the base change $Y \times_X U$ is connected, and hence is the spectrum of a separable field extension L of K. Moreover, Letting $\mathcal{O}_L$ be the normalization of A in L, $\overline{y}$ corresponds to a maximal ideal in the semi-local ring $\mathcal{O}_L$ above $\mathfrak{M}$ which is unramified, since we assume Y was etale over X. Thus to compute stalks, we have to take an inverse limit over such covers. We define some subgroups of the Galois group to help with this procedure:

Consider the primes $(\mathfrak{M}_i)_{1 \in I}$ in $\mathcal{O}_{K^{sep}}$ above $\mathfrak{M}$. One can consider these as inverse limits of primes in finite extensions. G_K acts transitively on these. Fix such a prime $\mathfrak{M}_1$, and let D be the stabilizer of $\mathfrak{M}_1$. $(K^{sep})^D$ is the largest field extension of K in which $\mathfrak{M}_1$ is unramified over $\mathfrak{M}$ with the same residue field. Define $I \subset D$ to be the normal subgroup which acts trivially on the residue field of $\mathfrak{M}_1$. Then $(K^{sep})^I$ is the largest field extension of K in which $\mathfrak{M}_1$ is unramified over $\mathfrak{M}$. It follows that $(j_*\mathfrak{F})_{\overline{s}}$ can be identified with M^I, once one chooses a lift of $\overline{s}$ to $\mathcal{O}_{K^{sep}}$. Moreover, D/I acts on the residue field $\overline{k}$ of $\mathfrak{M}_1$ in a way which identifies it with the Galois group G_k. Thus, the action of G_k on M^I is via D/I.

As a result, we have the following theorem:

Theorem 4.1. *Let $X = Spec\ A$. The category $S(X_{et})$ is equivalent to the category whose objects are (M, N, ϕ) where M is a G_K module, N is a G_k module, and $N \to M^I$ is a homomorphism of G_k-modules. The morphisms are the obvious ones making everything commute.*