

Lecture 10 - Direct and Inverse Images, Stalks, and Sheafification

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1 Generalities on Abelian Categories and Limits

In this lecture we define and explore the categories of sheaves and presheaves and functors between them. As such we begin by reviewing some basics that will come in handy later. This section exists primarily for completeness and ease of reference, and may(should) be skipped on a first reading.

1.1 Abelian Categories

Definition. *An Abelian category is a Category C with the following properties:*

- *It has an initial object $Z \in C$. That is, every other object has a unique morphism from Z , and a unique morphism to Z .*
- *All hom-sets are equipped with the structure of an abelian group such that the composition of morphisms is bilinear.*
- *All finite objects have a biproduct. That is, for every set A_1, \dots, A_k of objects there exists an object B with maps p_m to A_m and i_m from A_m such that $p_m \circ i_k$ is the zero morphism for $m \neq k$, and the identity for $m = k$, such that the morphisms p_m make B the coproduct of the A_k and the morphisms i_k make B the product of the A_k . One should think of B as the direct sum AND the direct product of the A_k .*
- *C has all kernels and cokernels. Recall that for a morphism $\phi : A \rightarrow B$ a kernel of ϕ is a morphism $k : K \rightarrow A$ such that $\phi \circ k = 0$ and any other morphism $k' : K' \rightarrow A$ such that $\phi \circ k' = 0$ factors through k . A cokernel is the dual notion.*

- Every monomorphism (morphism whose kernel is the zero map) is the kernel of its cokernel, and every epimorphism (map whose cokernel is the zero map) is the cokernel of its kernel.

The prototypical example of an abelian category is the category $R\text{-mod}$ of left modules over a ring R . In fact, Mitchell's embedding theorem says that every abelian category is a full subcategory of $R\text{-mod}$ for an appropriate R ¹.

Given an abelian category we can talk about a sequence being exact, and we will be able to develop a cohomology theory.

1.2 Limits

Let I be a category, and suppose we have a covariant functor F from I to Ab — the category of abelian groups. That is, for each element $i \in I$ we get an Abelian group F_i and for every map $i \rightarrow j$ we get a transition map $F_i \rightarrow F_j$. We say that M is the *direct limit* of F if there are morphisms $\phi_i : F_i \rightarrow M$ commuting with all the transition maps, and such that given any abelian group N and maps $\psi_i : F_i \rightarrow N$ commuting with all the transition maps, there is a unique morphism $t : M \rightarrow N$ such that $\psi_i = t \circ \phi_i$. It is easy to see that direct limits, if they exist, are unique up to unique isomorphism. We write $M = \varinjlim F$.

Lemma 1.1. *Define $M' := \bigoplus_{i \in I} F_i$. For each morphism $\phi : i \rightarrow j$ in I define $A_\phi := F_i$ and consider the map $\phi_0 : A_\phi \rightarrow M'$ which sends f to the element of M which is 0 everywhere, except its f in the F_i component and $-\phi_i(f)$ in the F_j component. Finally, let K be $\bigoplus_\phi A_\phi$ and define M to be the cokernel of the natural map $K \rightarrow M'$. Then $M \cong \varinjlim F$. In particular, direct limits exist.*

Proof. For every abelian group N and maps $\psi_i : F_i \rightarrow N$ as above we get a natural map $M' \rightarrow N$, and the compatibility of the ψ_i with the transition maps show that this map kills the image of K in M' . The claim follows. \square

Corollary 1.2. *Direct limits preserve right exactness. I.e. if F^1, F^2, F^3 are 3 functors with maps $F^1 \rightarrow F^2 \rightarrow F^3$ such that $F_i^1 \rightarrow F_i^2 \rightarrow F_i^3 \rightarrow 0$ is exact for every $i \in I$, then $\varinjlim F^1 \rightarrow \varinjlim F^2 \rightarrow \varinjlim F^3 \rightarrow 0$ is exact.*

¹this means that one can make arguments with element chasing in general abelian categories

Proof. Form M^1, M^2, M^3 and K^1, K^2, K^3 as in the proof of lemma 1.1. Then $0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow 0$ and $0 \rightarrow K^1 \rightarrow K^2 \rightarrow K^3 \rightarrow 0$ are clearly exact. The result thus follows from the snake lemma. \square

Definition. We say that a category I is filtered if ever j, j' admit morphisms to some k , and if for every pair of morphisms $f : i \rightarrow j$ and $g : i \rightarrow j'$ there exists an object k and a morphism $h : i \rightarrow k$ such that h can be factored through both f and g .

Corollary 1.3. Filtered direct limits preserve right exactness. I.e. if F^1, F^2, F^3 are 3 functors with maps $F^1 \rightarrow F^2 \rightarrow F^3$ such that $0 \rightarrow F_i^1 \rightarrow F_i^2 \rightarrow F_i^3 \rightarrow 0$ is exact for every $i \in I$, then $0 \rightarrow \varinjlim F^1 \rightarrow \varinjlim F^2 \rightarrow \varinjlim F^3 \rightarrow 0$ is exact.

Proof. In view of corollary 1.2 it suffices to prove the injectivity. So form M^1, M^2, K^1, K^2 as in the proof of lemma 1.1, and write C^1, C^2 for the cokernels of $K^1 \rightarrow M^1$ and $K^2 \rightarrow M^2$. So suppose $c \in C^1$ is such that its image in C^2 is 0. Since I is filtered, we can lift c to an element m in M^1 which lies entirely in F_i^1 for some $i \in I$. Consider the image $\phi(m)$ of m in F_i^2 . Then as $\phi(m)$ reduces to 0 in C^2 it must be in the image of K^2 . So write $\phi(m) = \sum_{j \in I} (a_j - f_j(a_j))$ where $a_j \in F_j$ and $f_j : j \rightarrow j'$ is a morphism in I , so that $f(a_j) \in F_{j'}$. Let S denote the set of indices occurring in $j, f(j)$ such that $a_j \neq 0$. Then since I is filtered we have maps from each element of S to some object k commuting with f . It thus follows that the image of $\phi(m)$ in F_k^2 equals 0. Thus, the image of m in F_k^1 must be 0, since F_k^1 injects into F_k^2 . It follows that $c = 0$, as desired. \square

2 Direct and Inverse Images of Pre-Sheaves

Recall that if X, Y are topological spaces and $f : Y \rightarrow X$ is a continuous map, then $U \rightarrow f^{-1}(U)$ induces a functor between the category of open sets on X to that of open sets on Y . Likewise, a morphism of schemes $Y \rightarrow X$ induces a *morphism of sites* $(E/X)_E \rightarrow (E'/Y)_{E'}$ if for any $Z \rightarrow X$ in X_E , $Z \times_X Y \rightarrow Y$ is in $Y_{E'}$. We shall refer to this as a *continuous* map $Y_{E'} \rightarrow X_E$.

Suppose now that P is a presheaf on $Y_{E'}$ and $\pi : Y_{E'} \rightarrow X_E$ is a continuous map. Then we can associate a *direct image* presheaf $\pi_*(P)$ on X_E by defining $\pi_*P(U) := P(U \times_X Y)$, with the natural transition maps. The following exercise is an immediate corollary of the fact that pre-images of coverings are coverings.

Exercise 2.1. Prove that if P is a sheaf, so is π_*P .

From now on we define $P(X_E)$ to be the category of presheaves on X_E and $S(X_E)$ to be the category of sheaves on $S(X_E)$. We also wish to define an inverse image functor π^p to be the left adjoint to $\pi_* : P(Y_{E'}) \rightarrow P(X_E)$. For each E' morphism $U \rightarrow Y$ consider the set of commutative squares S of the form

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where $W \rightarrow X$ is an E morphism. Given two such squares S, S' we define a map $S \rightarrow S'$ to be a map $W \rightarrow W'$ such that the resulting diagram commutes in the obvious way. Let I be the category of such squares, and let I^{op} be its opposite category. Note that I^{op} is filtered as E is closed under fiber products. Then any presheaf $P \in P(X_E)$ gives us a functor $P_I : I^{op} \rightarrow Ab$ given by $P_I(S) = P(W)$.

Definition. We define $\pi^p P$ to be $\varinjlim P_I$. Note that π^p is exact by lemma 1.3.

Lemma 2.2. π_* is right adjoint to π^p .

Proof. Let $F \in P(X_E)$ and $G \in P(Y_{E'})$. We must show that $\text{Hom}_{P(Y_{E'})}(\pi^p F, G) \cong \text{Hom}_{P(X_E)}(F, \pi_* G)$. By definition, a map $\phi : \pi^p F \rightarrow G$ amounts to maps

$$\phi_U : \pi^p F(U) \rightarrow G(U)$$

for each E' morphism $U \rightarrow Y$ compatible with the restriction maps. By definition of the limit, this amounts to maps

$$\phi_S : F(W) \rightarrow G(U)$$

for each square S compatible with the restriction maps. But now for each W , if we look at all squares S containing W there is a final object corresponding to $U = W \times_X Y$. Thus, it suffices to give maps

$$\psi_W : F(W) \rightarrow G(W \times_Y X)$$

compatible with the restriction maps. But this exactly amounts to a morphism $\psi : F \rightarrow \pi_* G$, as desired. \square

2.1 Presheaves as an Abelian Category

Theorem 2.3. $P(X_E)$ is an abelian category.

Proof. It is clear that the zero presheaf gives a zero object in $P(X_E)$, and that the product of the presheaves P_1, \dots, P_n is the presheaf $\prod_{i=1}^n P_i$ whose value on U is $\prod_{i=1}^n P_i(U)$. Moreover, if $\phi : P \rightarrow P'$ is a morphism, we can define $\ker \phi(U) := \ker \phi_U$ and $\operatorname{coker} \phi(U) := \operatorname{coker} \phi_U$. It is easy to see that these define presheaves, and to verify that these are the kernel and cokernel objects respectively.

Finally, if ϕ is a monomorphism then ϕ_U is injective for every U and thus ϕ is the kernel of $\operatorname{coker} \phi$. Likewise for epimorphisms. This completes the proof. □

Lemma 2.4. π_* and π^p are exact.

Proof. For π_* the statement is obvious from the definition, as $0 \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow 0$ is exact iff $0 \rightarrow P(U) \rightarrow P'(U) \rightarrow P''(U) \rightarrow 0$ is exact for every U .

For π^p the statement follows from lemma 1.3 — the fact that filtered direct limits preserve exactness in Ab. □

3 Sheafification

We now describe the process of *sheafification*, wherein associate a sheaf to a presheaf in a natural way. This is done in a very similar way for sites as it is for topological spaces. Namely, given a presheaf P we first mod out by all sections that vanish on some cover. Then, we add sections that should exist because on some cover there are sections that agree on fiber products. This is naturally done in one step by making the following definition:

Definition. Given a presheaf P on $P(X_E)$, we define the presheaf P^+ in the following way: For an E -morphism $U \rightarrow X$ and an E -covering $(V_i \rightarrow U)_{i \in I}$ we define $P^U(V)$ to be an element $s \in \prod_{i \in I} P(V_i)$ such that s is in the kernel of $\prod_{i \in I} P(V) \rightrightarrows \prod_{i, j \in I} P(V_i \times_U V_j)$. We then define $P^+(U)$ to be the direct limit of $P^U(V)$ over the category of all covers $V \rightarrow U$. The transition maps are given by refinements of covers.

Excercise 3.1. Prove that if P is a sheaf, then the maps in the direct limit are all isomorphisms and thus $P \cong P^+$.

It turns out that P^+ is not always a sheaf, so we define $aP = P^{++}$, and it turns out this IS always a sheaf. the obstruction is captured by the following definition:

Definition. We say that a presheaf P is separated if for all covers $(V_i \rightarrow U)_{i \in I}$ the map $P(U) \rightarrow \prod_{i \in I} P(U_i)$ is injective.

Theorem 3.2. If P is a presheaf, then P^+ is a separated presheaf. If P is a separated presheaf, then P^+ is a sheaf. thus, aP is a sheaf, and moreover for any sheaf F and a map of pre-sheaves $f : P \rightarrow F$ we get a factorization $P \rightarrow aP \rightarrow F$.

Proof. Suppose $s \in P^+(U)$ and $(V_i \rightarrow U)_{i \in I}$ is a cover such that s restricts to 0 in $P^+(V_i)$ for all $i \in I$. Then by definition, there is a cover $(W_j \rightarrow U)_{j \in J}$ and a section $s' \in \prod_j P(W_j)$ which is in the kernel of $\prod_j P(W_j) \rightrightarrows \prod_{j,j'} P(W_j \times_U W_{j'})$ such that s' represents s in $P^+(U)$. Thus letting $W' = V \times_U W$ we see that the element $s'_{W'}$ vanishes when considered as an element of $P^+(V)$. This means that there is a refinement of covers $W'' \rightarrow W'$ such that $s'_{W''}$ vanishes as an element of $P(W'')$. Since $W'' \rightarrow U$ is a cover it follows that s vanishes as an element of $P^+(U)$.

Now suppose that P is separated. To prove that P^+ is a sheaf it remains to prove that if $(V_i \rightarrow U)_{i \in I}$ is a cover such that $s \in P^+(V)$ is in the kernel of $P^+(V) \rightrightarrows P^+(V \times_U V)$, then s descends to $P^+(U)$. By definition, there is a refinement of covers $V' \rightarrow V$ and a section $s' \in P^V(V')^2$ restricting to $s \in P^+(V)$. By assumption, it follows that there is a cover $W' \rightarrow V' \times_U V'$ such that s' is in the kernel of

$$P(V') \rightrightarrows P(V' \times_U V') \rightarrow P(W')$$

. But as P is separated the map $P(V' \times_U V') \rightarrow P(W')$ is an injection. Thus, s' represents an element in $P^{V'}(U)$ and thus in $P^+(U)$, which restricts to s in $P^+(V)$, as desired.

For the second part of the theorem, consider a map $U \rightarrow X$, and suppose $(V_i \rightarrow U)$ is a covering. Then if s is in the kernel of $P(V) \rightrightarrows P(V \times_U V)$, then $f(s)$ is in the kernel of $F(V) \rightrightarrows F(V \times_U V)$, and thus since F is a sheaf there is an element $f(s)' \in F(U)$ which restricts to $f(s)$ on V . It is easy to check that this is compatible with the direct limit defining $P^+(U)$ and thus defines a map of presheaves $P^+ \rightarrow F$. Repeating, we get a map of sheaves $aP \rightarrow F$ and so we get a factorization $P \rightarrow aP \rightarrow F$, as desired. \square

²Here we employ the natural abuse of notation

Lemma 3.3. *If $F \rightarrow F'$ is a morphism of sheaves on X_E , then $F_0(U) := \ker(F(U) \rightarrow F'(U))$ is a sheaf, and is the kernel of $F \rightarrow F'$. $a(F'/F)$ is the cokernel.*

Proof. The first part of the lemma follows trivially from the sheaf axioms. Suppose that $F' \rightarrow F''$ is a map of sheaves such that the composition $F \rightarrow F''$ is the 0 map. Then as F'/F is the cokernel in the category of presheaves, we get a map of presheaves $F'/F \rightarrow F''$. By lemma 3.2 it follows that we get a map $a(F'/F) \rightarrow F''$. This completes the proof. \square

It follows that $S(X_E)$ is an abelian category. Let $i : S(X_{et}) \rightarrow P(X_{et})$ be the natural map.

Lemma 3.4. *a is the left adjoint to i . Moreover, a is exact and i is left exact.*

Proof. Let P be a presheaf and \mathfrak{F} a sheaf. Then

$$\mathrm{hom}(P, i\mathfrak{F}) \cong \mathrm{hom}(aP, \mathfrak{F})$$

by theorem 3.2, proving the adjointness.

To prove that a is exact, suppose that $0 \rightarrow P_1 \rightarrow P_2 \rightarrow P_3$ is exact in $P(X_E)$. Then consider the sequence $0 \rightarrow aP_1 \rightarrow aP_2 \rightarrow aP_3 \rightarrow 0$. As a is a filtered direct limit, it follows from lemma 1.3 that for every U , $0 \rightarrow aP_1(U) \rightarrow aP_2(U) \rightarrow aP_3(U) \rightarrow 0$ is exact. This makes the exactness trivial except possibly the surjectivity of $aP_2 \rightarrow aP_3$. To see this, by lemma 3.3 it suffices to show that $a(iaP_2/iaP_1) \cong aP_3$. But this is obvious as

$$a(iaP_2/iaP_1) \cong a(iaP_3) \cong aP_3$$

as the sheafification of a sheaf is itself.

That i is left exact is a consequence of the fact that every right adjoint functor is left exact. \square

3.1 Inverse Images of Sheaves

For a map $\pi : Y \rightarrow X$ Define the inverse image functor $\pi^* = a \circ \pi^p$. Then it follows easily from lemma 3.4 and lemma 2.2 that π^* is the left adjoint to π_* in the category of sheaves. Note that π^* is exact, being the composition of 2 exact functors.

Corollary 3.5. $\pi_*\pi'_* = (\pi\pi')_*$ and $\pi'^*\pi^* = (\pi\pi')^*$.

Proof. The first part is obvious from the definition. For the second part, note that $\pi'^*\pi^*$ is left adjoint to $\pi_*\pi'_*$ and $(\pi\pi')^*$ is left adjoint to $(\pi\pi')_*$. Since left adjoints are unique up to unique isomorphism, the claim follows. \square

4 Stalks

Recall that we define a geometric point of X to be a map from the spectrum of a separably closed field, $\bar{x} : \text{Spec } \bar{k} \rightarrow X$. If $x \in X$ is a point in the sense of schemes, then we write \bar{x} for a separable point above x . We write this shorthand as $\bar{x} \in X_{et}$. As we saw in our discussions of the étale fundamental group, geometric points for the étale site are a good analogue of points in topology. This analogy carries over into our study of cohomology.

Definition. Given a presheaf P on X_{et} and a point $\bar{x} \in X_{et}$, we define the stalk $P_{\bar{x}}$ to be the direct limit of $P(U)$ for all based étale covers $(U, \bar{u}) \rightarrow (X, \bar{x})$ where \bar{u} is a point in U_{et} mapping to X_{et} . If $U \rightarrow X$ is étale and $\bar{u} \in U_{et}$ then we get a natural isomorphism $(P|U)_{\bar{u}} \cong P_{\bar{x}}$.

Exercise 4.1. Prove that $\bar{x}^*P(\text{Spec } \bar{k}) \cong P_{\bar{x}}$.

Lemma 4.2. If F is a sheaf then a section $s \in F(X)$ is trivial iff s maps to 0 in $F_{\bar{x}}$ for each point $\bar{x} \in X_{et}$.

Proof. The “only if” direction is trivial, so suppose s maps to 0 in each stalk. This implies that for each point $x \in X$ there is an étale cover $f : U \rightarrow X$ such that $x \in f(U)$, and $s_U = 0$. Taking all such U gives us an étale cover on which s vanishes, and thus $s = 0$ since F is a sheaf. \square

Lemma 4.3. A map $\phi : F \rightarrow F'$ of sheaves in $S(X_E)$ is surjective iff for every $s \in F'(U)$ there exists a covering $(V_i \rightarrow U)_{i \in I}$ such that s_{V_i} is in the image of $F(V_i)$ for every i .

Proof. Consider the cokernel P of $i\phi : iF \rightarrow iF'$. Then ϕ is surjective iff $aP = 0$. Now, for any separated sheaf P , it is easy to see that P injects into P^+ . Thus, $aP = 0$ iff $P^+ = 0$, which is equivalent to saying that every section of $P(U)$ becomes trivial on some covering of U , which is equivalent to the statement of the lemma. \square

Theorem 4.4. Let $\mathfrak{F}, \mathfrak{F}', \mathfrak{F}''$ be sheaves in $S(X_{et})$. Then a complex $0 \rightarrow \mathfrak{F} \rightarrow \mathfrak{F}' \rightarrow \mathfrak{F}'' \rightarrow 0$ is left exact (resp. right exact, left exact) iff for every geometric point \bar{x} , the complex of abelian groups $0 \rightarrow \mathfrak{F}_{\bar{x}} \rightarrow \mathfrak{F}'_{\bar{x}} \rightarrow \mathfrak{F}''_{\bar{x}} \rightarrow 0$ is exact (resp. right exact, left exact). In other words, exactness can be checked on stalks.

Proof. Consider $\bar{x} : \text{Spec } \bar{k} \rightarrow X$. Since pullbacks are exact, the functor $\mathfrak{F} \rightarrow \bar{x}^* \mathfrak{F}$ is exact. Moreover, the functor $\mathfrak{F} \rightarrow \Gamma(\text{Spec } \bar{k}, \mathfrak{F})$ is exact on $S(\text{Spec } \bar{k}_{et})$. Thus, the functor $\mathfrak{F} \rightarrow \mathfrak{F}_{\bar{x}}$ is exact. Thus, to prove the lemma, it suffices to check that \mathfrak{F} is 0 iff $\mathfrak{F}_{\bar{x}} = 0$ for all geometric points \bar{x} . But this follows from lemma 4.2. \square