Lecture 1: Weil conjectures and motivation

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1 The Zeta function of a curve

We begin by motivating and introducing the Weil conjectures, which was bothy historically fundamental for the development of Etale cohomology, and also constitutes one of its greatest successes.

It has long been known that there is a strong analogy between rings of integers in number fields, and smooth projective curves over finite fields. As such, let us begin with the usual Riemann Zeta functions. The Riemann zeta function is most commonly defined as follows:

$$\text{for } \Re(s) > 1, \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

Using the unique factorization theorem, we can also rewrite the above sum as a product:

$$\text{for } \Re(s) > 1, \zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \tag{2}$$

The Riemann Zeta function enjoys the following properties:

- While only defined initially for $\Re(s) > 1$, $\zeta(s)$ can be mermorphicaly continued to the entire complex plane, with only a simple pole at $s = 1$.
- There is a functional equation satisfied by $\zeta(s)$, given by

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}} = \zeta(1-s)\gamma(\frac{1-s}{2})\pi^{s-1}2.$$

- (Riemann Hypothesis: Only conjectural!) The zeroes of $\zeta(s)$ all lie on the line $\Re(s) = \frac{1}{2}$, with the exception of the 'trivial' zeroes that occur at the negative even integers.

Remark. *The above suggests that the function* $\xi(s) = \zeta(s)\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}$ *is more natural to work with then the Zeta function, as it satisfies a nicer functional equation* $\xi(s) = \xi(1-s)$ *and eliminates the 'trivial' zeroes. The reason for this is that it is natural to consider the 'Archimedean prime' at* ∞ *in the product formula* (2)*, and it turns out that* $\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}$ *is the natural factor at that prime. We will not go into the justification of this heuristic, which can be found within Arakelov theory or the theory of automorphic forms.*

To try and make an analogy with finite fields, we think geometrically. Thus we form the scheme spec $\mathbb{Z}$. The closed points of spec $\mathbb{Z}$ are precisely given by the prime ideals of $\mathbb{Z}$, which are in bijection with the primes. Thus, the closed points are simply spec $\mathbb{F}_p \hookleftarrow$ spec $\mathbb{Z}$, and the prime numbers p are simply the sizes of the residue fields spec $\mathbb{F}_p$.

Now, we are ready to formulate a geometric analogue. Let q be a prime power, and X a smooth, projective curve over $\mathbb{F}_q$. What do the closed points of X look like? Well, each closed point $x \in X$ has residue field some finite field of the form $\mathbb{F}_{q^n}$. Let us write $\deg(x) = n$ and $N(x)$ for q^n, the size of the residue field $k(x)$ at X. Thus, we make the following definition:

$$\zeta(X, s) := \prod_{x \in X} (1 - N(x)^{-s})^{-1}. \tag{3}$$

We see that this definition is exactly analogous to (2). What about the representation as a sum as in (1)? The analogous notion of an integer here is that of a *positive divisor*, which on a curve is just a finite formal sum of points with non-negative coefficients. For $D = \sum_i a_i x_i$, we define $N(D) = \prod_i N(x_i)^{a_i}$. Expanding the product as with the Zeta function, we get

$$\zeta(X, s) = \sum_D N(D)^{-s}.$$

Since we are now in the world of geometry, we can also rewrite the Zeta function in a third way, by counting points in field extension; that is, using the quantities $X(\mathbb{F}_{q^n})$.

Specifically, if $\deg(x) = d$, then x contributes d points to $X(\mathbb{F}_{q^n})$ if $d|n$, and no points otherwise. Geometrically, one can think of it as follows: a point $y \in X(\mathbb{F}_{q^n})$ is a map $y :$ spec $\mathbb{F}_{q^n} \to X$. The image of y is some point $x \in X$, and thus we can factor the map as spec $\mathbb{F}_{q^n} \to$ spec $k(x)$spec X. Now, a map from spec $\mathbb{F}_{q^n}$ to spec $k(x)$ is—by definition—an embedding of fields $k(x) \hookrightarrow \mathbb{F}_{q^n}$, and since all finite field extensions are Galois, there are either $N(x)$ such extensions if $d|n$ or 0 otherwise.

Using the power series expansion for log, we can now write

$$\begin{aligned}\log \zeta(X,s) &= \sum_{x\in X}\sum_{d=1}^{\infty}\frac{N(x)^{-ds}}{d}\\ &= \sum_{n=1}^{\infty}\sum_{N(x)|n}\frac{N(x)q^{-ns}}{n}\\ &= \sum_{n=1}^{\infty}\frac{X(\mathbb{F}_{q^n})}{n}q^{-ns}\end{aligned}$$

Exponentiating, we have

$$Z(X,s) = \exp\left(\sum_{n=1}^{\infty}\frac{X(\mathbb{F}_{q^n})}{n}q^{-ns}\right). \tag{4}$$

So the Zeta function also records the number of points of a variety in extension field, and these are extraordinarily interesting.

Let us do an example. Consider the case of $X = \mathbb{P}^1/\mathbb{F}_q$. We see that $X(\mathbb{F}_{q^n}) = q^n + 1$, since we have q^n elements $(1 : t)$ with $t \in \mathbb{F}_{q^n}$ together with the point 'at infinity' $(0 : 1)$. Thus, using (4) we calculate

$$\begin{aligned}Z(X,s) &= \exp\left(\sum_{n=1}^{\infty}\frac{q^n+1}{n}q^{-ns}\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{q^{n(1-s)}}{n}\right)\cdot\exp\left(\sum_{n=1}^{\infty}\frac{q^{-ns}}{n}\right)\\ &= (1-q^{1-s})^{-1}(1-q^{-s})^{-1}.\end{aligned}$$

For X a curve of higher genus, it is no longer so easy to count points. One might wonder how to even proceed with computing the Zeta function. It turns out the Riemann-Roch formua can help. Recall equation (2). Now a divisor D gives us a line bundle $\ell(D)$.Moreover, given a line bundle ℓ, it has $H^0(X,\ell) - 1$ many sections, and up to the action of $\mathbb{F}_q^\times$ they each give a different divisor giving rise to ℓ. Thus, writing $\mathrm{Pic}(\mathbb{F}_{q^n})$ for the set of line bundles of degree n, we can rewrite

$$Z(X,s) = \sum_{n\geq 0} q^{-ns} \sum_{\ell \in \mathrm{Pic}(\mathbb{F}_q)} \frac{H^0(X,\ell)-1}{q-1}.$$

Moreover, by the Riemann-Roch theorem, if $n \geq 2g_X - 1$ then $H^0(X,\ell) = q^{n-g+1}$. Thus, we can write the above as

$$Z(X,s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where $P(T)$ is a polynomial of degree $2g_X$. Moreover, it turns out that $Z(X,s)$ satisfies a functional equation $Z(X,s) = \pm Z(X,1-s)q^{(1-g_X)(1/2-s)}$, and by a theorem of Weil, all the roots of $P(T)$ have absolute value $q^{1/2}$, which translates to the zeroes of $Z(X,s)$ all being on the line $\Re(s) = \frac{1}{2}$; that is, the Riemann hypothesis holds!

At this point, it is natural to ask what happens if we go to higher dimensions. So suppose X is a smooth, projective variety over $\mathbb{F}_q$. Then we can define the Zeta function of X exactly as in (3). Moreover, by the same analysis, this will be identical to the representation in (4)[1].The one hiccup is that divisors are no longer collections of points, and so the representation (2) is no longer applicable[2].

As an example, one can compute the zeta function in the case $X = \mathbb{P}^n$ to be

$$Z(X,s) = \frac{1}{(1-q^{-s})(1-q^{1-s})\dots(1-q^{n-s})}.$$

2 Statement of the Weil Conjectures

At this point we are ready to state the Weil conjectures. These were Made after Weil after he computed a plethora of examples - a feat in itself, as computing points over finite fields is note easy.

Theorem 2.1. *(Weil Conjectures) Suppose X is a smooth projective variety of dimension n over $\mathbb{F}_q$. Then the Zeta function of X satisfies the following properties:*

1. *(Rationality) The Zeta function $Z(X,s)$ is a rational function of q^{-s}.*

[1] To avoid confusion, let me clarify that this equality has nothing to do with either the smoothness or the projectivity assumption.

[2] Of course, we can just replace the word 'divisor' with '0-cycle' and it will hold. However, the Riemann-Roch theorem is no longer applicable, and so this representation is less useful.

2. *(Functional equation) there is an integer E such that $Z(X, n-s) = \pm q^{E(n/2-s)} Z(X, s)$.*

3. *(Riemann Hypothesis) The Zeta function can be written as an alternating product*

$$Z(X,s) = \frac{P_1(q^{-s})P_3(q^{-s})\dots P_{2n-1}(q^{-s})}{P_0(q^{-s})P_2(q^{-s})\dots P_{2n}(q^{-s})}$$

where each $P_i(T)$ is an integral polynomial all of whose roots have absolute value $q^{-m/2}$. Moreover, $P_0(T) = 1-T$ and $P_{2n}(T) = 1-q^nT$.

4. *(Betti Numbers) Suppose X is a "good reduction" of a characteristic zero variety. That is, there is a smooth projective morphism $\tilde{X} \to Y$ such that the base change w.r.t one of the spec $\mathbb{F}_q$-valued points of Y is X, and the base change to one of the spec $\mathbb{C}$-valued points of Y is a smooth projective complex variety X_0. Then the degree of the i'th polynomial P_i is the i'th betti number of the space of the topological space $Y(\mathbb{C})$.*

Note in particular the Riemann Hypothesis - called such because it places the zeroes and poles of $Z(X, s)$ on nice vertical lines in the complex plane. The weil conjectures, as we sketch next section, led to the development of Etale cohomology, as (4) above suggests that a certain cohomology theory is lurking in the background, and Grothendieck realized that a suitable cohomology theory would be very useful in proving the Weil conjectures. We should mention that the rationality of the Zeta function was first proven by Dwork before the development of Etale cohomology, though his proof did not give nearly as much information.

3 Cohomology of manifolds and Grothendieck's Dream

Let's recall how 'ordinary' topological Cech cohomology works, and then we'll see why an appropriate analogue would be useful in proving the Weil conjectures.

So suppose M is an n-dimensional compact real manifold, and T is a triangulation of M into simplices. Let T_i be the i-dimensional simplices in T.Let C_i denote the set of maps from T_i to $\mathbb{Q}$. Finally, let d_m be the map

$C_m \to C_{m+1}$ defined as follows:

$$d_m(\phi)(v_0, v_1, \ldots, v_m) = \sum_{i=0}^{m} (-1)^i \phi(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m).$$

Then it is easy to verify that $d_{i+1}d_i = 0$, and so we get a *complex*

$$C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2 \ldots \xrightarrow{d_{n-1}} C_n.$$

Then we define the Cech Cohomology groups to be $H^i(M, Q) := \ker d_i / \mathrm{im} d_{i+1}$. It is true (though not obvious) that given any two triangulations of M, their cohomology groups can be naturally identified. Moreover, we have the following wonderful properties:

- The groups $H^i(M, \mathbb{Q})$ are finite dimensional. Moreover, if M is a complex algebraic algebraic variety, then $H^j(M, \mathbb{Q}) = 0$ for $j > 2 \dim_{\mathbb{C}} M$.
- (Functoriality) For any continuous map $\phi : M \to N$, we have induced maps $\phi_i : H^i(M, \mathbb{Q}) \to H^i(N, \mathbb{Q})$ compatible with compositions.
- (Poincare Duality) The groups $H^i(M, \mathbb{Q})$ and $H^{n-i}(M, \mathbb{Q})$ are canonically dual. Moreover, $H^n(M, \mathbb{Q})$ is one dimensional, and there is a natural perfect pairing $H^i(M, \mathbb{Q}) \times H^{n-i}(M, \mathbb{Q}) \to H^n(M, \mathbb{Q})$.
- (Lefschetz trace formua) Suppose $\phi : M \to M$ is a continuous map with only simple,[3] isolated fixed points. Then

$$\#\{\text{fixed points of } \phi\} = \sum_{i=0}^{n} (-1)^i \mathrm{tr}(\phi_i).$$

Now, suppose for a second that we had a way to define a cohomology theory for proper,smooth varieties X over finite fields satisfying some version of the above properties. The reason this is useful, is that if X is a variety over $\mathbb{F}_q$, then we have a natural map $X \to X$ known as the *Absolute Frobenius* morphism. If $X = \mathrm{spec}\ A$ then this is induced by the map of rings $A \to A$ given by[4] $a \to a^q$, and otherwise its defined by gluing[5]. Then it is not hard to see that the fixed points of F^m on $X(\overline{\mathbb{F}_q})$ are exactly $X(\mathbb{F}_{q^m})$. So we

[3] This is a bit technical to define. But if M is a smooth manifold then its enough to say that the graph of ϕ in $M \times M$ is transverse to the diagonal

[4] Verify this is a map of rings!

[5] Check that this glues!

could hope that some version of the Lefschetz Trace formula would imply that

$$\#X(\mathbb{F}_{q^m}) = \sum_{i=0}^{2\dim X} (-q)^i \mathrm{tr} F_{|H^i(X,\mathbb{Q})}.$$

Combining this with the formal identity of matrices

$$-\log\det(1-TM) = \sum_{i=1}^{\infty} \frac{\mathrm{tr}M^i}{i}$$

we would deduce that

$$Z(X,s) = \prod_{i=0}^{2\dim X} \det(1-q^{-s}F_{|H^i(X,\mathbb{Q})}).$$

This would imply the rationality of the Zeta function immediately. Moreover, one can see that an appropriate version of Poincare Duality would yield the functional equation, and the compatibility with reduction from characteristic 0 would follow from some sort of compatibility with regular cohomology.

This also strongly suggests that the sought for polynomials in (3) of 2.1 are $P_i(T) = \det(1-TF)_{|H^i(X,\mathbb{Q})}$ and reformulates the Riemann Hypothesis as saying that the eigenvalues of F on the i'th cohomology group are of size $q^{i/2}$ - this is the only part of the Weil conjectures that would not follow 'formally' from the Weil conjectures, but it still provides some insight into whats going on[6].This sketch is what we will justify using Etale Cohomology. It turns out that we cannot have our coefficient group be $\mathbb{Q}$ — we need to use a profinite group such as $\mathbb{Q}_l$— but the basic ideas remain the same.

As a final comment, we point out that in trying to define a cohomology theory to satisfy all of the above, the Zariski topology is grossly inadequate. For instance, the Zariski topology on any two curves is identical(prove this)!

[6]And is essential to Deligne's eventual resolution of the Riemann Hypothesis