Lecture 5: Etale Morphisms

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1 Standard Etale Morphsms

We begin by finally defining etale morphisms:

Definition. *A morphism is* etale *if it is flat, separable and unramified.*

Lemma 1.1. *If $f : X \to S$ is unramified, and $g : Y \to X$ is such that $f \circ g : Y \to S$ is etale, then g is etale.*

It is not hard to see that compositions of etale maps are etale, and base changes of etale maps are etale.

Proof. Write $g = p_2 \circ \Gamma_g$ where $\Gamma_g : Y \to Y \times_S X$ is the graph of g and $p_2 : Y \times_S X \to X$ is the projection. We shall show that Γ_g and p_2 are base changes of etale maps. To prove this, consider the following cartesian squares:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\Gamma_g"] & X \arrow[d, "\Delta"] \\ Y \times_S X \arrow[r, "(g,id)"] & X \times_S X \end{tikzcd}$$

and

$$\begin{tikzcd} Y \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"] & Y \arrow[d, "f \circ g"] \\ X \arrow[r, "g"] & S \end{tikzcd}$$

.

Since Δ is an open immersion, it is etale, its base change Γ_g is etale, and since $f \circ g$ is etale, p_2 is also etale. The claim follows.

$\square$

As an important example, consider a ring A, and an extension $B = A[T]/P(T)$ where $P(T) \in A[T]$ is a monic polynomial. Let Q be a prime ideal of A. In order for the morphism $\phi : \text{Spec } B \to \text{Spec } A$ to be unramified over Q, by our discussions in the previous lecture it is equivalent for $k(Q)[T]/\bar{P}(T)$ be a separable extension of $k(Q)$, where $k(Q)$ is the fraction field of A/Q, and we write $\bar{P}$ for the reduction of P mod Q. This is equivalent to $\bar{P}'(T)$ being a unit in $k(Q)[T]/\bar{P}(T)$.

More generally, if we factor $\bar{P}(T) = \prod_{i=1}^{r} P_i(T)^{e_i}$ into powers of irreducible factors, then the prime ideals of $k(Q)[T]/\bar{P}(T)$ are $Q_i = (P_i(T))$, and ϕ is unramified at Q_i if $P'(T)$ is a unit in B_{Q_i}. Thus, if we pick $b \in B$ such that $P'(T)$ is a unit in B_b, the map $\phi_b : \text{Spec } B_b \to \text{Spec } A$ is unramified. Moreover, ϕ_b is separated since any map of affine schemes is separated, and its flat since B is free over A. We call this a standard etale morphism.

Definition. *For a monic polynomial $P(T) \in A[T]$ and an element $b \in B = A[T]/P(T)$ such that $P'(T)$ is a unit in B_b, we call the map ϕ_b : Spec $B_b \to$ Spec A a* standard etale morphism.

The importance of such morphisms comes from their simplicity, as well as the following theorem:

Theorem 1.2. *Any etale morphism $\phi : Y \to X$ is locally standard. In other words, for any $y \in Y$ there are open subset V containing y and U containing X such that $\phi \mid V : V \to U$ is a standard etale morphism.*

Before we prove the theorem, we shall need some preparation.

Lemma 1.3. *Let M be a finite module over a Noetherian ring A. TFAE*

- *M is flat over A.*
- *For all prime ideals P of A, M_P is free over A_P.*
- *There are finitely many elements $(a_1, \ldots, a_n)$ generating the unit ideal in A, such that N_{a_i} is free over A_{a_i} for $i = 1, \ldots, n$.*
- *(If A is an integral domain:) The number $\dim_{k(P)}(M \otimes_A k(P))$ is constant over all prime ideals P in A.*

Proof. (1)$\leftrightarrow$ (2): We have seen in lecture 2 that M is flat over A iff M_P if flat over A_P for each prime ideal P. Thus, it suffices to prove that if A is local with maximal ideal P and M is finite flat over A, then M is free.

Pick a map $\phi : A^n \to M$ such that the reduction mod P is an isomorphism. Then ϕ is surjective by Nakayama's lemma. Let K be the kernel of ϕ. We claim that for any A-module N, the sequence

$$0 \to K \otimes AN \to N^n \to M \otimes_A N$$

is an isomorphism[1].To prove this, pick a surjection $\psi : F \to N$ where F is a free module and let K' be the kernel.

Then we have the following exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & K \otimes_A K' \arrow[r] \arrow[d] & K'^n \arrow[r] \arrow[d] & M \otimes_A K' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \otimes_A F \arrow[r] \arrow[d] & F^n \arrow[r] \arrow[d] & M \otimes_A F \arrow[r] \arrow[d] & 0 \\
 & K \otimes_A N \arrow[r] \arrow[d] & N^n \arrow[r] \arrow[d] & M \otimes_A N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now a diagram chase(exercise!) show that the bottom left horizontal map is injective.

Now pick $N = A/P$. Then we see that $K/PK = 0$. Since A is noetherian, K is finitely generated, and hence $K = 0$ by Nakayama's lemma, as desired.

(3)$\to$(2): For each prime P, there exists an $a_i \notin P$. Then A_P is a localization of A_{a_i}, and the claim follows.

(2)$\to$(3): For a prime ideal P, pick an isomorphism $\phi : A_P^n \to M_P$. Since M is finite over A, one can express ϕ as a matrix using finitely many elements, and thus there is some $a \notin P$ and a lifting $\phi_a :_a: A_a^n \to M_P$ which reduces to ϕ_P. Moreover, let K_a, C_a be the kernel and cokernel of ϕ_a respecively. Their localization at P is 0, and they are finite over A_a since A_a is Noetherian. By similar reasoning to before, one can pick an $a_P \notin P$ such that ϕ_{a_P} is an isomorphism.

[1]This follows from standard properties of the Tor functor, but as we have not yet covered any cohomology we give a direct proof - which is really an unwinding of the usual proof.

Now, the a_P generated an ideal not contained in any prime ideal, hence they must generated the unit ideal. Thus a finite subset of them generates the unit ideal. the claim follows.

(3)$\rightarrow$ (4): Obvious, since integral domains have a unique minimal prime.

(4)$\rightarrow$ (2): For any prime ideal P, pick a map $\phi_P : A_P^n \rightarrow M_P$ for $n = \dim_{k(P)}(M \otimes_A k(P))$ such that the reduction mod P is an isomorphism. Let K be the kernel of ϕ. Then ϕ is surjective by Nakayama's lemma. For any prime ideal Q of A_P, the map $\phi_Q : A_Q^n \rightarrow M_Q$ is surjective, hence by dimension counting the reduction mod Q must be an isomorphism. Thus, $K \subset Q$ for any prime ideal Q, and thus K is in the nilradical of A, which is 0. Thus ϕ is an isomorphism and the claim follows.

□

We shall also need the following input(recall that a morphism is *quasi-finite* if the pre-image of each point is a finite set).

Theorem 1.4. *(Zariski's Main Theorem)*

Let $\phi : Y \rightarrow X$ be a quasi-finite morphism of finite-type between Noetherian schemes. Then ϕ can be factored as

$$Y \xrightarrow{f} Y' \xrightarrow{g} X$$

where f is an open immersion and g is finite.

For a proof, see [Raynaud, Anneaux Locaux Henseliens, Chapter IV].

Proof. of theorem 1.2:[2]

Since the statement is obviously local on X and Y, we set $X = \text{Spec } A$. Moreover, by Zariski's Main Theorem we may pick a Y' over X and an open immersion $Y \rightarrow Y'$ such that Y' is finite over X. Set $Y' = \text{Spec } C$. Let Q be the prime ideal corresponding to y, and $\mathfrak{P}$ the prime ideal corresponding to $f(y)$. We must show that there is a standard etale A-algebra B_b such that $B_b = C_c$ for $c \notin Q$. Because everything is finite over A, it suffices to do this with A replaced by A_P by the same arguments as in lemma 1.3. Thus we assume that A is local with unique maximal ideal $\mathfrak{P}$.

Now, consider $C/\mathfrak{P}C$. This is an artinian ring as it is finite over $k(\mathfrak{P})$, and thus it is a direct sum of its localizations. As $Y \rightarrow X$ is etale in a neighborhood of y, we have that $C/\mathfrak{P}C = k(Q) \times C'$ for some $k(Q)$-algebra C'. Now pick $t \in C$ such that $\bar{t} \in k(Q) \times 0$ and $k(\mathfrak{P})[\bar{t}] = k(Q)$, and let

[2]This proof is almost identical to [Milne, Etale Cohomology, I,3.14], with some tiny additions in exposition

Q' be the intersection of Q with $A[t]$. Consider C as a finite $A[t]$-algebra. It is easy to see that the only prime ideal of C over Q' is Q, as it suffices to do this after quotienting out by Q. Thus, $C_{Q'} := C \otimes_{A[t]} A[t]_{Q'}$ is a finite extension of a local ring, such that $C_{Q'}/Q'$ is artinian. It follows by Nakayama's lemma that any element of $C_{Q'}$ not in Q is a unit, and thus that $C_{Q'}$ is a local ring with maximal ideal generated by Q. Thus, we must have that $C_{Q'} = C_Q$.

Now, the map $A[t]_{Q'} \to C_{Q'}$ is injective since localization is an exact functor, and on the other hand is surjective by Nakayama's lemma (and the fact that it is an isomorphism on residue fields by construction). Thus $A[t]_{Q'} \cong C_Q$. As $A[t]$ is finite over A, this isomorphism extends to $A[t]_a \cong C_c$ for some $a \notin Q'$ and $c \notin Q$,so wlog we may replace C by $A[t]$. That is, we may assume that C is generated by a single element t.

Now, let $n = [k(Q) : k(P)]$. Then by Nakayama's lema, $1, t, \ldots, t^{n-1}$ generate $A[t]$ over A. Thus there is a monic polynomial $P(t)$ of degree n and a surjection ϕ of the form $B := A[t]/P(t) \to C$, which is an isomorphism when reduced modulo $\mathfrak{P}$. Since $P(t)$ is the characteristic polynomial of t over A and $k(Q)/k(P)$ is separable, it follows that $P'(t)$ is a unit modulo Q and thus if we let $b = P'(t)$, then B_b is a standard etale algebra over $C_{\phi(b)}$. Possibly localizing further, we get etale algebras B_b, C_c over A, so it follows by lemma 1.1 that C_c is etale over B_b, and in particular flat. By lemma 1.3, we can cover Spec B_b by finitely many open sets of the form Spec B_{b_i} s.t C_{cb_i} is free over B_{b_i}. But being a quotient, it must be free of rank 1. Thus, the map Spec $C_c \to$ Spec B_b is an open immersion, which completes the proof. [3] □

Using this theorem, it is easy to show that many properties are preserved under etale morphisms, such as dimension, regularity, and normality (see [Milne, EC, I, 3.17]).

1.1 Etale morphisms are "local isomorphisms"

We give a final justification of our heuristic assertion that Etale maps are akin to local isomorphisms.

Definition. *If R is a ring with an ideal I, the we define the completion $\widehat{R}_I$ of R I to be $\varprojlim R/I^n$. If R is a local ring, we simply right $\widehat{R}$ to be the completion of R at its maximal ideal.*

[3] since on the one hand $C_{Q'}$ has fewer elements inverted than C_Q, and on the other hand every element not in Q is invertible in $C_{Q'}$.

Lemma 1.5. *If R is a Noetherian, local ring, then $\widehat{R}$ is a Noetherian ring, and is faithfully flat over $\widehat{R}$.*

For the proof, see [Atiyah-Macdonald, Introduction to Commutative Algebra, 10.14 and 10.26]. Heuristically, though Spec $\mathcal{O}_{X,x}$ is a small neighborhood around the point $x \in X$, Spec $\widehat{\mathcal{O}_{X,x}}$ should be viewed as zooming in to an even smaller neighborhood a so-caed 'formal neighborhood' around x.

Theorem 1.6. *Let $\phi : Y \to X$ be a finite-type morphism of schemes of finite type over Spec k, where k is algebraically closed. Then TFAE*

1. *ϕ is etale at some closed point $y \in Y(k)$ (equivalently, at a neighborhood of y, see exercise at the end).*
2. *$\widehat{\phi_y} : \widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$ is an isomorphism.*

Proof. $(1) \to (2)$: Since the statement is local on both X and Y, by theorem 1.2 we can assume $X =$ Spec A and $Y =$ Spec B_b where $B = A[t]/P(t)$ and $P(t)$ is monic, with $P'(t)$ being a unit in B_b. Let y correspond to the maximal ideal Q. Now, since X is finite type over k we can write $A = k[x_1, \dots, x_n]/I$ for some ideal I. Then the point $\phi(y)$ corresponds to some maximal ideal $M = (x_1 - a_1, \dots, x_n - a_n)$ containing I. The point y likewise corresponds to an ideal $N = (x_1 - a_1, \dots, x_n - a_n, t - c)$ containing $P(t)$. Now, consider the image of $P(t)$ modulo this ideal. One gets a polynomial $P_{a_1,\dots,a_n}(t)$, which must vanish at $t = c$. Moreover, since ϕ is unramified at y, it follows that the root $(t - c)$ must be simple. Now, to complete B at y we may first complete it at $\phi(y)$, so that $\mathcal{O}_{Y,y}$ is the completion of $\widehat{A}_M[t]/P(t)$ at the maximal ideal which is the pullback of $(t - b)$. Now, by Hensels lemma [Atiyah-Macdonald, 10, Ex. 9], since $P(t)$ modulo M factors into $(t-c)\bar{Q}(t)$ where $(t - b, \bar{Q}(t)) = 1$, we also have a factorization $P(t) = (t - c')(Q(t)$ which reduces to the factorization mod M. Thus, when we localize at N, $Q(t)$ becomes a unit, and so we have

$$\widehat{\mathcal{O}_{Y,y}} \cong \widehat{A}_M[t]/(t - b) \cong \widehat{A}_M \cong \mathcal{O}_{X,\phi(x)}$$

as desired.

$(2) \to (1)$: Let $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}_{Y,y}$. Then by assumption $\widehat{A} \to \widehat{B}$ is an isomorphism. We prove B is a flat A module. To see this, suppose $M \to N$ is an injection of A modules. Then by lemma 1.5 $M \otimes_A \widehat{B}$ injects into $N \otimes_A \widehat{B}$. But $\widehat{B}$ is faithfully flat over B, and thus $M \otimes_A \to N \otimes_B$ is an injection. Thus B is flat over A.

It remains to prove unramifiedness. Consider $(M_A)B$ and M_B. Both of these ideals generate the maximal ideal in $\widehat{B}$. But since $\widehat{B}$ is faithfully flat over B, it follows the ideals are the same. Thus $A \to B$ is unramified.

□

2 Invariance under nilpotent thickenings

Definition. *We define et/X to be the category of Etale X-schemes, with morphisms between them being X-morphisms. That is, the objects are etale morphisms $Y \to X$, and the maps are morphisms $Y \to Z$ commuting with the structure morphisms to X.*

Theorem 2.1. *Let $X_0 \to X$ be a closed immersion identifying X_0 with the reduced subscheme of X. Then the natural functor $F : et/X \to et/X_0$ given by base change is an equivalence of categories.*

Proof. We first prove that F is fully faithful, that is for any X-schemes Y, Z with base changes Y_0, Z_0 we have an isomorphism $\hom_X(Y, Z) \cong \hom_{X_0}(Y_0, Z_0)$. To prove this, note that giving a morphisms of X schemes is equivalent to giving the graph $\Gamma : Y \to Y \times_X Z$, or equivalently a section of the projection map p_1. As we saw last lecture, sections of separated unramified morphisms are open immersions. Thus it is equivalent to finding open and closed subschemes $U \subset Y \times_X Z$ on which the restriction of p_1 is an isomorphism. There is a clear map $U \to U_0$ which is injective. It suffices to show that this map is surjective. So let U_0 be such a subscheme. Since quotienting out by nilpotents does not alter the topology, we let U be the open subscheme of $Y \times_X Z_0$ whose base change to X_0 is U_0. We claim that $\phi : U \to Y$ is an isomorphisms iff the base change to X_0 is. It is here where flatness is crucial, as $\mathcal{O}_U$ is locally free over $\mathcal{O}_Y$ of some rank n by lemma 1.3, and hence $\mathcal{O}_{U_0}$ is locally free over $\mathcal{O}_{Y_0}$ of rank n. Thus $n = 1$, and ϕ must be an isomorphism.

It remains to show that F is essentially surjective, so that every etale X_0-scheme Y_0 comes from some etale X-scheme Y. By the uniqueness above, it suffices to construct Y locally and glue. But locally, by theorem 1.2 etale morphisms are standard, and so we may simply lift the coefficients of the defining polynomial. This concludes the proof.

□

Corollary 2.2. *For any etale S-scheme X, and any S-scheme Y with reduced subscheme Y_0, there is a bijection $\hom_S(Y, X) \cong \hom_S(Y_0, X)$.*

Proof. As in the previous proof, to give a morphism from Y to X is the same as to give a section of $Y \times_S X \to Y$. Note that this is an etale map, whose base change to Y_0 is $Y_0 \times_S X$. Thus the lemma follows from the above theorem applied to $et/Y \cong et/Y_0$.

$\square$

In fact, it is possible to show that the above property characterizes etale morphisms. See http://www.math.ias.edu/ bhattb/math/etalestcksproj.pdf.

Exercises:

- For Noetherian A, and finite type map $\phi : A \to B$, and $P \subset A$ a prime ideal, suppose that $\phi_P : A_P \to B_p$ is an isomorphism. Prove that for some $a \in A \backslash P$, $\phi_a : A_a \to B_a$ is an isomorphism. In other words, the property of being an isomorphism 'spreads out' for morphisms of finte type.
- Prove that if $\phi : Y \to X$ is of finite type, then if ϕ is etale at a point y, then it is etale in a neighborhood of y. Use the above exercise and the characterization of being etale in terms of the diagonal morphism.