Lecture 4: Unramified morphisms

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1 Separable maps of rings

Recall that the Jacobson radical of a ring R, written $J(R)$, is defined to be the interection of all of the maximal ideals of R.

Definition. *Let k be a field. A k-algebra A is called* separable *if $A \otimes_k \overline{k}$ has trivial Jacobson Radical.*

Recall the following version of the Chinese Remainder Theorem:

Lemma 1.1. [Chinese Remainder Theorem]: For any finite set of maximal ideals $M_i, 1 \leq i \leq n$, the map

$$f : R \to \oplus_i R/M_i$$

is surjective.

Proof. For $i \neq j$, consider an element $x_{ij} \in M_i, x \notin M_j$. Then by scaling we can make x_{ij} to reduce to the identity modulo M_j. Now let

$$y_j = \prod_{i \neq j} x_{ij}.$$

It is clear that $f(y_j)$ give us the set of idempotents that we need.

□

Lemma 1.2. *Let A be a finite k-algebra. TFAE:*

1. *A is separable over k*
2. *$\overline{A} := A \otimes_k \overline{k}$ is isomorphic to a finite product of $\overline{k}$.*
3. *A is isomorphic to a finite product of separable field extensions of k.*

4. *The discriminant of A over k is non-zero (i.e. the trace pairing is non-degenerate).*

Proof. $1 \Rightarrow 2$ Consider the set M_i of maximal ideals of $\overline{A}$. By lemma 1.1 there are finitely many such ideals, and it follows that A is a direct sum of its residue fields. Since $\overline{k}$ is algebraically closed, 2 follows.

$2 \Rightarrow 3$ Consider $\mathrm{Hom}_k(A, \overline{k})$. Clearly, the image of such a homomorphism is a subfield and thus the kernel is a maximal ideal, which must therefore contain the Jacobson radical I_r of A. Now by lemma 1.1 A/I_r is a direct sum of fields k_i, each of which is clearly a finite extension of k. thus $\mathrm{Hom}_k(A, \overline{k})$ has

$$\sum_i [k_i : k]_s$$

elements. But this set is also $\mathrm{Hom}_{\overline{k}}(\overline{A}, \overline{k})$ which we know has $[A : k]$ elements. Thus we must have that all the k_i are separable over k and $I_r = 0$.

$3 \Rightarrow 4$ This follows because a field extension being separable is equivalent to its discriminant being non-zero.

$4 \Rightarrow 1$ The discriminants of $\overline{A}$ and A are the same. Thus, $\overline{A}$ has no nilpotents (As all their multiples have trace 0). But all prime ideals of $\overline{A}$ have quotient rings which are finite domains over a field, hence are fields themselves. Thus, all prime ideals are maximal, which means that the Jacobson radical is Nilradical. The claim follows.

□

2 Unramfied morphisms

Definition. *A morphism of schemes $f : Y \to X$ is said to be* unramified *it is locally of finite type and if for all $y \in Y$, the map*

$$f^{\#} : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$$

satisfies

$$\mathfrak{m}_{Y,y} = f^{\#}(\mathfrak{m}_{X,f(y)})\mathcal{O}_{Y,y}$$

and $k(y)$is a finite separable field extension of $k(x)$.

Excercise 2.1. *Prove that closed and open immersions are unramified. Prove that unramified morphisms are closed under base change.*

Lemma 2.2. *Let $f : Y \to X$ be a morphism of finite type. TFAE*

1. *f is unramified.*

2. *For all $x \in X$, the fiber $Y_x \to x$ is unramified.*

3. *For all $x \in X$, the fiber Y_x has an open covering by spectra of finite, separable $k(x)$ algebras*

4. *For all $x \in X$, the fiber Y_x is a disjoint union of spectra of finite, separable $k(x)$-extensions.*

5. *For all* geometric points *of X, that is, maps Spec $k \to X$ where k is an algebraically closed field, the base change $Y \times_X$ Spec k is unramified over Spec k*

6. *The diagonal morphism $Y \to Y \times_X Y$ is an open immersion.*

See [Milne, Etale Cohomology, I, 3.2-3.5] for a slightly different approach.

Proof. (1)$\Leftrightarrow$(2) is easy. For (2$\Rightarrow$ (3), let $U = \text{Spec } B$ be an affine open subset of Y_x. Since Y is locally of finite type, it follows that B is finitely generated over $k(x)$ and thus noetherian. Let $Q \subset B$ be a prime ideal. Then by assumption, B_Q is a finite, separable $k(x)$ extension. Thus $B_Q = B/Q$ and Q is a maximal ideal. It follows that B is artinian, and hence a finite direct sum of its localizations. The result follows. (3)$\Leftrightarrow$(4)$\Leftrightarrow$(5) follow from lemma 1.2 and the (3)$\Rightarrow$(2) is obvious.

To see (3)$\Rightarrow$(6), note first that it is true for X the spectrum of a field. Since the statement is local on both Y and X it is sufficient to consider $Y = \text{Spec } B$ and $X = \text{Spec } A$. Let I be the ideal defining Y in $Y \times_X Y$. Note that since B is finitely generated over A by finitely many elements b_i and and I is generated by the elements $b_i \otimes 1 - 1 \otimes b_i$ we see that I is finitely generated. Consider the B-algebra I/I^2. Since we already saw that (3)$\Rightarrow$(5) for field spectra, it follows that for any maximal ideal $M \subset B$ that $0 = (I + M)/(M + I^2) = (I/I^2)_M/M/$ Since I/I^2 is finitely generated, it follows from Nakayama's lemma that $(I/I^2)_M = 0$. It follows that $I = I^2$ in a neighborhood of Spec $B/M \in Y$ and thus on a neighborhood of Y. Hence, since I is finitely generated, it follows that I is 0 in a neighborhood of Y. It follows that $(Y, \mathcal{O}_Y)$ is an open subscheme of $Y \times_X Y$.

Finally, to see (6)$\Rightarrow$(5) we may reduce to the case where $X = \text{Spec } k$ where $k = \overline{k}$ is algebraically closed. Consider $y \in Y$ a closed point. since k is algebraically closed, there exists a section $\sigma : x \to Y$ whose image is y. Note that the fiber product of $\Delta : Y \to Y \times_X Y$ and $(1, \sigma \circ f) : Y \to Y \times_X Y$ is y. Since Δ is an open immersion, we see that y is open in Y. Moreover, we must have that Spec $\mathcal{O}_y \to$ Spec $\mathcal{O}_y \otimes_k \mathcal{O}_y$ is an open immersion. But $\mathcal{O}_y$ has only one prime ideal, and thus is a local artin ring. Hence Spec $\mathcal{O}_y \otimes_k \mathcal{O}_y$ only has 1 point and so the map $\mathcal{O}_y \times_k \mathcal{O}_y \to \mathcal{O}_y$ must be an isomorphism. A dimension count over k reveals that $\mathcal{O}_y = k$. The theorem follows. □

Unramified maps are to schemes the analogue of immersions to topology. The following two lemmas are justification to that statement:

Lemma 2.3. *Suppose X is connected, $f : Y \to X$ is separated and unramified, and $s : X \to Y$ is a section of f. Then s is an isomorphism onto a connected component of Y.*

Proof. Since f is separated and unramified, $Y \to Y \times_X Y$ is a closed and open immersion by lemma 2.2. Base changing this by the map $(1, s \circ f) : Y \to Y \times_X Y$ we get that $s : X \to Y$ is also a closed and open immersion. Since X is connected, the claim follows. □

The next corollary justifies our intuition of 'lifting a morphism uniquely to a covering space.

Corollary 2.4. *Suppose X is connected, $Y \to X$ is sepa rated and unramified and Y' is an X-scheme with 2 morphisms f, g to Y. Then if there exists a point $y' \in Y'$ such that $f(y') = g(y') = y$ and the maps induced by f, g on $k(y') \to k(y)$ agree, then $f = g$.*

Proof. Since being unramified is preserved under base change, $Y' \times_X Y \to Y'$ is unramified and the graphs Γ_f, Γ_g give us sections which agree at y' by hypothesis. Thus lemma 2.3 implies that $\Gamma_f = \Gamma_g$ and so $f = g$ as desired. □

Definition. *A morphism is* etale *if it is flat and unramified*