ON THE EXISTENCE OF ADMISSIBLE SUPERSINGULAR REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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Abstract. Suppose that $\mathbf{G}$ is a connected reductive group over a finite extension $F/\mathbb{Q}_p$, and that C is a field of characteristic p. We prove that the group $\mathbf{G}(F)$ admits an irreducible admissible supercuspidal, or equivalently supersingular, representation over C.

Contents

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1. Introduction

Suppose that F is a non-archimedean field of residue characteristic p and that $\mathbf{G}$ is a connected reductive algebraic group over F. There has been a growing interest in understanding the smooth representation theory of the p-adic group $G := \mathbf{G}(F)$ over a field C of characteristic p, going back to the work of Barthel–Livné [BL94] and Breuil [Bre03] in the case of $\mathbf{G} = \mathbf{GL}_2$. The latter work in particular demonstrated the relevance of the mod p representation theory of p-adic reductive groups to the p-adic Langlands program.

The results of [AHHV17] (when C is algebraically closed) and [HV19] (for a general C of characteristic p) give a classification of irreducible admissible representations in terms of supercuspidal C-representations of Levi subgroups of G. Here, an irreducible admissible smooth representation π is said to be *supercuspidal* if it does not occur as subquotient of any parabolic induction $\mathrm{Ind}_P^G \sigma$, where P is a proper parabolic subgroup of G and σ an irreducible admissible representation of the Levi quotient of P. Unfortunately, so far, the supercuspidal representations themselves remain mostly mysterious, outside anisotropic groups, $\mathrm{GL}_2(\mathbb{Q}_p)$ ([BL94], [Bre03]), and some related cases ([Abd14], [Che13], [Koz16], [KX15]). If $F/\mathbb{Q}_p$ is a non-trivial unramified extension, then irreducible supercuspidal representations of $\mathrm{GL}_2(F)$ were first constructed by Paškūnas [Paš04]; however, it seems hopelessly complicated to classify them [BP12], [Hu10]. One additional challenge in constructing supercuspidal representations is that irreducible smooth representations need not be admissible in general (unlike what happens over $\mathbb{C}$), as was shown recently by Daniel Le [Le].

There are two ways to characterize supercuspidality in terms of Hecke actions. The first description assumes C is algebraically closed and uses weights and Hecke eigenvalues for any fixed choice K of special parahoric subgroup (a weight is then an irreducible representation of K). It was shown to be equivalent to supercuspidality in [AHHV17]. The second description uses the center of the pro-p Iwahori–Hecke algebra. The equivalence between the second Hecke description and supercuspidality was shown in [OV18] (when C is algebraically closed) and [HV19] (for a general C of characteristic p). In either description, supercuspidality is characterized by the vanishing of certain Hecke operators, which is why supercuspidal representations are also called *supersingular*.

Our main theorem is the following:

Theorem A. *Suppose F is of characteristic 0, $\mathbf{G}$ is any connected reductive algebraic group over F, and C any field of characteristic p. Then G admits an irreducible admissible supersingular, or equivalently supercuspidal, representation over C.*

This theorem is new outside the low rank cases mentioned above. Note also that the analogous theorem for supercuspidal representations over $\mathbb{C}$ was proved by Beuzart-Plessis [BP16].

We now briefly explain our argument, which uses several completely different ideas. First, in Section 3 we reduce to the cases where C is finite and $\mathbf{G}$ is absolutely simple adjoint. If $\mathbf{G}$ is moreover anisotropic, then G is compact and any irreducible smooth representation of G is finite-dimensional (hence admissible) and supercuspidal. If $\mathbf{G}$ is isotropic, we distinguish three cases.

For most groups $\mathbf{G}$ we show in Section 4 that there exists a discrete series representation π of G over $\mathbb{C}$ that admits invariants under an Iwahori subgroup $\mathfrak{B}$, and that has moreover the following property: the module $\pi^{\mathfrak{B}}$ of the Iwahori–Hecke algebra $H(G,\mathfrak{B})$ admits a $\mathbb{Z}[q^{1/2}]$-integral structure whose reduction modulo the maximal ideal of $\mathbb{Z}[q^{1/2}]$ with residue field $\mathbb{F}_p$ is supersingular. The Hecke modules $\pi^{\mathfrak{B}}$ are constructed either from characters (using [Bor76]) or reflection modules (using [Lus83] and [GS05]; the latter is needed to handle unramified non-split forms of $\mathbf{PSO}_8$).

Suppose from now on that F is of characteristic zero, i.e. that $F/\mathbb{Q}_p$ is a finite extension. The p-adic version of the de George–Wallach limit multiplicity formula ([DKV84, App. 3] plus [Kaz86, Thm. K]) implies that the representation π above embeds in $C^\infty(\Gamma\backslash G,\mathbb{C})$ for some discrete cocompact subgroup Γ of G (as $\operatorname{char} F = 0$). By construction we deduce that the Hecke module $C^\infty(\Gamma\backslash G/\mathfrak{B},\mathbb{F}_p) = C^\infty(\Gamma\backslash G,\mathbb{F}_p)^{\mathfrak{B}}$ of $\mathfrak{B}$-invariants admits a supersingular submodule. Crucially, by cocompactness of Γ we know that $C^\infty(\Gamma\backslash G,\mathbb{F}_p)$ is an admissible representation of G. Picking any non-zero supersingular vector $v \in C^\infty(\Gamma\backslash G/\mathfrak{B},\mathbb{F}_p)$, the G-subrepresentation of $C^\infty(\Gamma\backslash G,\mathbb{F}_p)$ generated by v admits an irreducible quotient, which is admissible (as $\operatorname{char} F = 0$) and supersingular.

Unfortunately, this argument does not work for all groups $\mathbf{G}$. We have the following exceptional cases:

(i) $\mathrm{PGL}_n(D)$, where $n \geq 2$ and D a central division algebra over F;
(ii) $\mathrm{PU}(h)$, where h is a split hermitian form in 3 variables over a ramified quadratic extension of F or a non-split hermitian form in 4 variables over the unramified quadratic extension of F.

Note that for the group $\mathrm{PGL}_n(D)$ with $n \geq 2$ the only discrete series representations π having $\mathfrak{B}$-invariant vectors are the unramified twists of the Steinberg representation (by Proposition 4.1.4(i) and the classification of Bernstein–Zelevinsky and Tadić), but then $\pi^{\mathfrak{B}}$ is one-dimensional with non-supersingular reduction.

In the second exceptional case, where $G \cong \mathrm{PU}(h)$ for certain hermitian forms h, we use the theory of coefficient systems and diagrams, building on ideas of Paškūnas [Paš04]. See Section 5. Note that $\mathbf{G}$ is of relative rank 1, so the adjoint Bruhat–Tits building of G is a tree, and our method works for all such groups. In order to carry it out, we may apply the reductions in Section 3 and assume that $\mathbf{G}$ is absolutely simple and simply connected. Given a supersingular module Ξ for the *pro-p* Iwahori–Hecke algebra of G, we naturally construct a G-equivariant coefficient system (or cosheaf) $\mathcal{D}_\Xi$ on the Bruhat–Tits tree of G. The homology of $\mathcal{D}_\Xi$ admits a smooth G-action, and any irreducible admissible quotient will be supersingular (by Proposition 3.1.3). To construct such a quotient, we define an auxiliary coefficient system $\mathcal{D}'$, which is built out of injective envelopes of representations of certain parahoric subgroups, along with a morphism $\mathcal{D}_\Xi \to \mathcal{D}'$. The image of the induced map on homology is admissible, and admits an irreducible quotient π' which is itself admissible (as $\operatorname{char} F = 0$) and supersingular.

In the first exceptional case, where $G \cong \mathrm{PGL}_n(D)$, we use a global method (see Section 6). We find a totally real number field $\widetilde{F}^+$ and a compact unitary group $\underline{G}$ over $\widetilde{F}^+$ such that $\underline{G}(\widetilde{F}^+_v)$ is isomorphic to $\mathrm{GL}_n(D)$ for a suitable place $v|p$ of $\widetilde{F}^+$. Then, fixing a level away from v and taking the limit over all levels at v, the space S of algebraic automorphic forms of $\underline{G}(\mathbb{A}^\infty_{\widetilde{F}^+})$ over $\overline{\mathbb{F}}_p$ affords an admissible smooth action of $\underline{G}(\widetilde{F}^+_v)$. Using automorphic induction and descent we construct an automorphic representation π of $\underline{G}(\mathbb{A}_{\widetilde{F}^+})$ whose associated Galois representation r_π has the property that its reduction modulo p is irreducible locally at v. From π we get a maximal ideal $\mathfrak{m}$ in the Hecke algebra (at good places outside p), and we claim that any irreducible subrepresentation of the localization $S_\mathfrak{m}$ is supercuspidal.

To prove the claim, we use the pro-p Iwahori–Hecke criterion for supercuspidality and argue by contradiction. If one of the relevant Hecke operators has a non-zero eigenvalue, we lift to characteristic zero by a Deligne–Serre argument and construct an automorphic representation π' with Galois representation $r_{\pi'}$ having the same reduction as r_π modulo p. Using local-global compatibility at p for $r_{\pi'}$ and some basic p-adic Hodge theory we show that the non-zero Hecke eigenvalue in characteristic p implies that $r_{\pi'}$ is reducible locally at v, obtaining the desired contradiction.

For our automorphic base change and descent argument we require results going slightly beyond [Lab11], since our group $\underline{G}$ is typically not quasi-split at all finite places. In the appendix, Sug Woo Shin explains the necessary modifications.

Finally, we remark that we would expect Theorem A to be true even when $\operatorname{char} F = p$. So far this only seems to be known for the groups $\mathrm{GL}_2(F)$ [Paš04], outside trivial cases. We crucially use that $\operatorname{char} F = 0$ in (at least) the following ways:

(i) the existence of discrete cocompact subgroups, which fails for most groups if $\operatorname{char} F = p$ [BH78, §3.4], [Mar91, Cor. IX.4.8(iv)],
(ii) admissibility is preserved under passing to a quotient representation,
(iii) the automorphic method in case of the group $\mathrm{PGL}_n(D)$.

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1.2. **Notation.** Fix a prime number p, and let F be a non-archimedean local field of residue characteristic p (we will later assume that $\operatorname{char} F = 0$, i.e., that F is a finite extension of $\mathbb{Q}_p$). The field F comes equipped with ring of integers $\mathcal{O}_F$ and residue field k_F of cardinality q, a power of p. We fix a uniformizer ϖ, and let val_F and $|\cdot|_F$ denote the normalized valuation and normalized absolute value of F, respectively.

If $\mathbf{H}$ is an algebraic F-group, we denote by H its group of F-points $\mathbf{H}(F)$.

Let $\mathbf{G}$ be a connected reductive F-group, $\mathbf{T}$ a maximal F-split subtorus of $\mathbf{G}$, $\mathbf{B}$ a minimal F-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{T}$, and x_0 a special point of the apartment of the adjoint Bruhat–Tits building defined by $\mathbf{T}$. We associate to x_0 and the triple $(\mathbf{G}, \mathbf{T}, \mathbf{B})$ the following data:

- the center $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$,
- the root system $\Phi \subset X^*(T)$,
- the set of simple roots $\Delta \subset \Phi$,
- the centralizer $\mathbf{Z}$ of $\mathbf{T}$,
- the normalizer $\boldsymbol{\mathcal{N}}$ of $\mathbf{T}$,
- the unipotent radical $\mathbf{U}$ of $\mathbf{B}$ (hence $\mathbf{B} = \mathbf{Z}\mathbf{U}$), and the opposite unipotent radical $\mathbf{U}^{\mathrm{op}}$,
- the triples $(\mathbf{G}^{\mathrm{sc}}, \mathbf{T}^{\mathrm{sc}}, \mathbf{B}^{\mathrm{sc}})$ and $(\mathbf{G}^{\mathrm{ad}}, \mathbf{T}^{\mathrm{ad}}, \mathbf{B}^{\mathrm{ad}})$, corresponding to the simply-connected covering of the derived subgroup and the adjoint group of $\mathbf{G}$,
- the apartment $\mathscr{A} := X_*(T)/X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to $\mathbf{T}$ in the adjoint Bruhat–Tits building,
- the alcove $\mathcal{C}$ of $\mathscr{A}$ with vertex x_0 lying in the dominant Weyl chamber with vertex x_0,
- the Iwahori subgroups $\mathfrak{B}$ and $\mathfrak{B}^{\mathrm{sc}}$ of G and G^{sc}, respectively, fixing $\mathcal{C}$ pointwise,
- the pro-p-Sylow subgroup $\mathfrak{U}$ of $\mathfrak{B}$.

Given a field L, we denote by $\overline{L}$ a fixed choice of algebraic closure. We fix a field C of characteristic $c \in \{0, 2, 3, 5, 7, \ldots\}$, which will serve as the field of coefficients for the modules and representations appearing below. In our main result we will assume $c = p$.

Suppose K is a compact open subgroup of G, and R is a commutative ring. We define the Hecke algebra associated to this data to be the R-algebra

$$H_R(G, K) := \mathrm{End}_G\, R[K\backslash G].$$

If $R = \mathbb{Z}$, we simply write $H(G, K)$. In our applications below, we will often assume that $K = \mathfrak{B}$ or $K = \mathfrak{U}$.

Given a module or algebra X over some ring R and a ring map $R \to R'$, we let $X_{R'} := X \otimes_R R'$ denote the base change.

Other notation will be introduced as necessary in subsequent sections.

2. Iwahori–Hecke algebras

In this section we review some basic facts concerning Iwahori–Hecke algebras and their (supersingular) modules. We will use these algebras extensively in our construction of supercuspidal G-representations. See [Vig16], [Vig14], and [Vig17] for references.

2.1. Definitions. Recall that we have defined the Iwahori–Hecke ring as

$$H(G, \mathfrak{B}) = \mathrm{End}_G\, \mathbb{Z}[\mathfrak{B}\backslash G].$$

We have an analogous ring $H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}})$ for the simply-connected group. The natural ring homomorphism $H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}}) \to H(G, \mathfrak{B})$ (induced by the covering $G^{\mathrm{sc}} \to G$ of the derived subgroup) is injective, so we identify $H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}})$ with a subring of $H(G, \mathfrak{B})$. We first discuss presentations for these rings.

There is a canonical isomorphism

$$j^{\mathrm{sc}} : H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}}) \xrightarrow{\sim} H(W, S, q_s),$$

where $H := H(W, S, q_s)$ is the Hecke ring of an affine Coxeter system (W, S) with parameters $\{q_s := q^{d_s}\}_{s\in S}$. The d_s are positive integers, which we will abusively also refer to as the parameters of G. Thus, $H(W, S, q_s)$ is a free $\mathbb{Z}$-module with basis $\{T_w\}_{w\in W}$, satisfying the braid and quadratic relations:

$$\begin{aligned} T_w T_{w'} &= T_{ww'} \quad && \text{for } w, w' \in W,\ \ell(w) + \ell(w') = \ell(ww'), \\ (T_s - q_s)(T_s + 1) &= 0 && \text{for } s \in S. \end{aligned}$$

Here $\ell : W \to \mathbb{Z}_{\geq 0}$ denotes the length function with respect to S. We identify $H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}})$ with H via j^{sc}.

In order to describe $H(G, \mathfrak{B})$, we require a larger affine Weyl group. We define the *extended affine Weyl group* to be

$$\widetilde{W} := \mathcal{N}/Z_0,$$

where Z_0 is the unique parahoric subgroup of Z. The group $\widetilde{W}$ acts on the apartment $\mathscr{A}$, and permutes the alcoves of $\mathscr{A}$ transitively. We let Ω denote the subgroup of $\widetilde{W}$ stabilizing $\mathcal{C}$. The affine Weyl group W is isomorphic to a normal subgroup of $\widetilde{W}$, and permutes the alcoves *simply* transitively. We therefore have a semidirect product decomposition

$$\widetilde{W} = W \rtimes \Omega.$$

The function ℓ extends to $\widetilde{W}$ by setting $\ell(uw) = \ell(wu) = \ell(w)$ for $u \in \Omega, w \in W$. In particular, we see that Ω is the group of length-zero elements of $\widetilde{W}$.

Let Σ denote the reduced root system whose extended Dynkin diagram Dyn is equal to the Dynkin diagram of (W, S), and let Dyn' denote the Dynkin diagram Dyn decorated with the parameters $\{d_s\}_{s\in S}$. The quotient of Ω by the pointwise stabilizer of $\mathcal{C}$ in Ω is isomorphic to a subgroup Ψ of $\mathrm{Aut}(W, S, d_s)$, the group of automorphisms of Dyn'. Thus, Ω acts on Dyn' and consequently on $H(W, S, q_s)$, and the isomorphism j^{sc} extends to an isomorphism

$$j : H(G, \mathfrak{B}) \xrightarrow{\sim} \mathbb{Z}[\Omega] \mathbin{\widetilde{\otimes}} H(W, S, q_s), \tag{2.1.1}$$

where $\widetilde{\otimes}$ denotes the twisted tensor product. The generalized affine Hecke ring $\widetilde{H} := \mathbb{Z}[\Omega] \mathbin{\widetilde{\otimes}} H(W, S, q_s)$ as above is the free $\mathbb{Z}$-module with basis $\{T_w\}_{w\in\widetilde{W}}$, satisfying the braid and quadratic relations:

$$T_w T_{w'} = T_{ww'} \quad \text{for } w, w' \in \widetilde{W},\ \ell(w) + \ell(w') = \ell(ww'), \tag{2.1.2}$$

$$(T_s - q_s)(T_s + 1) = 0 \qquad \text{for } s \in S. \tag{2.1.3}$$

Thus, we see that the Iwahori–Hecke ring $H(G, \mathfrak{B})$ is determined by the type of Σ, the parameters $\{d_s\}_{s\in S}$, and the action of Ω on Dyn'.

The group $\widetilde{W}$ forms a system of representatives for the space of double cosets $\mathfrak{B}\backslash G/\mathfrak{B}$. Under the isomorphism j, the element $T_w \in \widetilde{H}$ for $w \in \widetilde{W}$ corresponds to the endomorphism sending the characteristic function of $\mathfrak{B}$ to the characteristic function of $\mathfrak{B}n\mathfrak{B}$, where $n \in \mathcal{N}$ lifts w.

Finally, let $\mathbf{P} = \mathbf{MN}$ denote a standard parabolic F-subgroup of $\mathbf{G}$ (meaning $\mathbf{B} \subset \mathbf{P}$), and suppose that $\mathbf{M}$ contains $\mathbf{T}$. Then the group $M \cap \mathfrak{B}$ is an Iwahori subgroup of M, and we may form the algebra

$$H(M, M \cap \mathfrak{B}) = \mathrm{End}_M \mathbb{Z}[(M \cap \mathfrak{B})\backslash M].$$

It is *not* a subalgebra of $H(G, \mathfrak{B})$ in general. The basis of $H(M, M \cap \mathfrak{B})$ will be denoted T_w^M, where w is an element of the extended affine Weyl group associated to M.

2.2. **Dominant monoids.** The subgroup

$$\Lambda := Z/Z_0$$

of $\widetilde{W} = \mathcal{N}/Z_0$ is commutative and finitely generated, and its torsion subgroup is equal to $\widetilde{Z}_0/Z_0$, where $\widetilde{Z}_0$ denotes the maximal compact subgroup of Z. (When the group $\mathbf{G}$ is F-split or semisimple and simply connected, we have $Z_0 = \widetilde{Z}_0$.) The short exact sequence

$$1 \to \Lambda \to \mathcal{N}/Z_0 \to \mathcal{N}/Z \to 1$$

splits, identifying the (finite) Weyl group $W_0 := \mathcal{N}/Z$ of Σ with $\mathrm{Stab}_W(x_0)$. We therefore obtain semidirect product decompositions

$$\Lambda \rtimes W_0 = \widetilde{W}$$

and

$$\Lambda^{\mathrm{sc}} \rtimes W_0 = W,$$

where $\Lambda^{\mathrm{sc}} := \Lambda \cap W$.

Given a subgroup J of Z, we define

$$\Lambda_J := JZ_0/Z_0 \subset \Lambda.$$

We analyze Λ_J for various groups J presently.

Let T_0 denote the maximal compact subgroup of T, and note that $T_0 = Z_0 \cap T$. This implies that the inclusion $T \hookrightarrow TZ_0$ induces an isomorphism $T/T_0 \xrightarrow{\sim} TZ_0/Z_0 = \Lambda_T$, and therefore the map

$$\begin{aligned} X_*(T) &\xrightarrow{\sim} \Lambda_T \\ \mu &\longmapsto \lambda_\mu := \mu(\varpi)Z_0/Z_0 \end{aligned} \tag{2.2.1}$$

is a W_0-equivariant isomorphism.

Recall that we have a unique homomorphism

$$\nu : Z \to \mathscr{A},$$

determined by the condition

$$\langle \alpha, \nu(t) \rangle = -\mathrm{val}_F(\alpha(t))$$

for $t \in T$ and $\alpha \in \Phi$. We claim that the kernel of ν is the saturation of $Z(G)\widetilde{Z}_0$ in Z, i.e., the set of all elements $z \in Z$ such that $z^n \in Z(G)\widetilde{Z}_0$ for some $n \geq 1$. Indeed, the kernel of ν contains $Z(G)$ and $\widetilde{Z}_0$, and the group $Z/Z(G)\widetilde{Z}_0$ is commutative and finitely generated. This gives an induced map

$$\nu : Z/Z(G)\widetilde{Z}_0 \to \mathscr{A}. \tag{2.2.2}$$

We note the following three facts: (1) the image of T^{ad} in $Z/Z(G)\widetilde{Z}_0$ is of finite index (cf. comments following (16) in [Vig16]); (2) the $\mathbb{Z}$-span of the coroots $\Phi^\vee$ is of finite index in $X_*(T^{\mathrm{ad}})$; (3) $\nu(\alpha^\vee(\varpi^{-1})) = \alpha^\vee$ for any coroot $\alpha^\vee \in X_*(T)$. Combining these, we see that the image of (2.2.2) has the same rank as $Z/Z(G)\widetilde{Z}_0$, which is equal to the rank of $X_*(T^{\mathrm{ad}})$. Therefore, the kernel of (2.2.2) is exactly the torsion subgroup of $Z/Z(G)\widetilde{Z}_0$. This gives the claim.

Since Z_0 is contained in the kernel of ν, the group Λ acts by translation on $\mathscr{A}$ via ν. Therefore, $\Lambda_{\ker\nu}$ is the pointwise stabilizer of $\mathcal{C}$ in Λ. Similarly, one easily checks that $\Lambda_{\ker\nu}$ is the pointwise stabilizer of $\mathcal{C}$ in Ω. (In fact, we have $\Lambda \cap \Omega = \Lambda_{\ker\nu}$, cf. [Vig16, Cor. 5.11].) Hence, we obtain

$$\Omega/\Lambda_{\ker\nu} \xrightarrow{\sim} \Psi, \tag{2.2.3}$$

and the embeddings of Λ and Ω into $\widetilde{W}$ induce

$$\Lambda/(\Lambda_{\ker\nu} \times \Lambda^{\mathrm{sc}}) \xrightarrow{\sim} \widetilde{W}/(\Lambda_{\ker\nu} \times W) \xleftarrow{\sim} \Omega/\Lambda_{\ker\nu}. \tag{2.2.4}$$

An element $\lambda \in \Lambda$ is called *dominant* (and λ^{-1} is called *anti-dominant*), if

$$z(U \cap \mathfrak{B})z^{-1} \subset U \cap \mathfrak{B}$$

for any $z \in Z$ which lifts λ. We let Λ^+ denote the monoid consisting of dominant elements of Λ, and similarly for any subgroup $\Lambda' \leq \Lambda$ we define $\Lambda'^+ := \Lambda' \cap \Lambda^+$. Using the isomorphism (2.2.1), we say a cocharacter $\mu \in X_*(T)$ is *dominant* if λ_μ is, and let $X_*(T)^+$ denote the monoid consisting of dominant elements of $X_*(T)$. The group of invertible elements in the dominant monoid Λ^+ is exactly the subgroup $\Lambda_{\ker\nu}$, and the invertible elements of $\Lambda^{\mathrm{sc},+}$ are trivial.

Lemma 2.2.5. *The subgroup $\Lambda_{Z(G)} \times \Lambda^{\mathrm{sc}}$ (resp. Λ_T) of Λ is finitely generated of finite index. The submonoid $\Lambda_{Z(G)} \times \Lambda^{\mathrm{sc},+}$ (resp. Λ_T^+) of the dominant monoid Λ^+ is finitely generated of finite index.*

Here, we say that a submonoid N of a commutative monoid M has finite index if $M = \bigcup_{i=1}^n (N + x_i)$ for some $x_i \in M$. If M is finitely generated, then dM is of finite index in M for all $d \geq 1$.

Proof. The groups $\ker\nu/Z(G)\widetilde{Z}_0$ and $\widetilde{Z}_0/Z_0$ are finite, and equations (2.2.3) and (2.2.4) imply that $\Lambda/(\Lambda_{\ker\nu} \times \Lambda^{\mathrm{sc}})$ is isomorphic to the finite group Ψ. Thus, we see that the commutative group $\Lambda_{Z(G)} \times \Lambda^{\mathrm{sc}}$ is a finitely generated, finite index subgroup of Λ. Similarly, Λ_T is finite free and it is well known that it is of finite index in Λ. Gordan's lemma implies the second assertion (as in the proof of [HV15, 7.2 Lem.]). ☐

2.3. Bernstein elements.

Let $w \in \widetilde{W}$, and let $w = us_1 \cdots s_n$ be a reduced expression, with $u \in \Omega, s_i \in S$. We set $q_w := q_{s_1} \cdots q_{s_n}$, and define

$$T_s^* := T_s - q_s + 1 \quad \text{and} \quad T_w^* := T_u T_{s_1}^* \cdots T_{s_n}^*.$$

Then $T_w T_{w^{-1}}^* = q_w$, and the linear map defined by $T_w \mapsto (-1)^{\ell(w)} T_w^*$ is an automorphism of $\widetilde{H}$.

For $\mu \in X_*(T)$, we let $\mathcal{O}_\mu \subset \Lambda$ denote the W_0-orbit of λ_μ. We then define

$$z_\mu := \sum_{\lambda \in \mathcal{O}_\mu} E_\lambda,$$

where E_λ are the integral Bernstein elements of $\widetilde{H}$ corresponding to the spherical orientation induced by Δ ([Vig16, Cor. 5.28, Ex. 5.30]). Precisely, they are characterized by the relations

$$E_\lambda = \begin{cases} T_\lambda & \text{if } \lambda \text{ is anti-dominant,} \\ T_\lambda^* & \text{if } \lambda \text{ is dominant,} \end{cases} \tag{2.3.1}$$

$$E_{\lambda_1} E_{\lambda_2} = (q_{\lambda_1} q_{\lambda_2} q_{\lambda_1\lambda_2}^{-1})^{1/2} E_{\lambda_1\lambda_2} \quad \text{if } \lambda_1, \lambda_2 \in \Lambda, \tag{2.3.2}$$

where we take the positive square root. (If λ_1, λ_2 are both dominant (or anti-dominant), then $E_{\lambda_1} E_{\lambda_2} = E_{\lambda_1\lambda_2}$.) The elements z_μ are central in $\widetilde{H}$, and when $\mu \in X_*(T^{\mathrm{sc}})$, z_μ lies in H.

We let $\mathcal{A}$ denote the commutative subring of the generalized affine Hecke ring $\widetilde{H}$ with $\mathbb{Z}$-basis $\{E_\lambda\}_{\lambda\in\Lambda}$. When $G = Z$, we have $\widetilde{H} = \mathcal{A} \cong \mathbb{Z}[\Lambda]$, but $\mathcal{A}$ is not isomorphic to $\mathbb{Z}[\Lambda]$ in general. The rings $\mathcal{A}, \widetilde{H}$, and the center of $\widetilde{H}$ are finitely generated modules over the central subring with basis $\{z_\mu\}_{\mu\in X_*(T)}$, which is itself a finitely-generated ring.

2.4. **Supersingular modules.** We now discuss supersingular Hecke modules.

Recall that C is our coefficient field of characteristic c. We define $H_C := H \otimes C$ and $\widetilde{H}_C := \widetilde{H} \otimes C$, which are isomorphic to the Iwahori–Hecke algebras $H_C(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}})$ and $H_C(G, \mathfrak{B})$, respectively.

Definition 2.4.1 (cf. [OV18, §5.1(3)])**.** Let M be a non-zero right $\widetilde{H}_C$-module. A non-zero element $v \in M$ is called *supersingular* if $v \cdot z_\mu^n = 0$ for all $\mu \in X_*(T)^+$ such that $-\mu \notin X_*(T)^+$, and all sufficiently large n. The $\widetilde{H}_C$-module M is called *supersingular* if all its non-zero elements are supersingular. We make a similar definition for modules over H_C, using the monoid $X_*(T^{\mathrm{sc}})^+$.

We remark that the definition of a supersingular module differs slightly from that of [Vig17, Def. 6.10]. There it was required that $c = p$, and that $M \cdot z_\mu^n = 0$ for all $\mu \in X_*(T)^+$ such that $-\mu \notin X_*(T)^+$ and n sufficiently large.

Lemma 2.4.2.

(i) *Any simple $\widetilde{H}_C$-module is finite dimensional, and is semisimple as an H_C-module.*

(ii) *If $c \nmid p|W_0|$, then $\widetilde{H}_C$ does not admit any simple supersingular modules.*

(iii) *If $c = p$, a simple $\widetilde{H}_C$-module is supersingular if and only if its restriction to H_C is supersingular.*

Proof. (i) The first statement follows from [Vig07, §5.3]. For the second part, note that there exists a finite index subgroup Ω' of Ω which acts trivially on H (for example, we may take $\Omega' = \Lambda_{\ker\nu}$). Set $H'_C := C[\Omega'] \otimes_C H_C$. Any simple H_C-module N extends trivially to an H'_C-module N', and the restriction of $N' \otimes_{H'_C} \widetilde{H}_C$ to H_C is a finite direct sum $\bigoplus_{u\in\Omega/\Omega'} N^u$ of (simple) conjugates N^u of N by elements $u \in \Omega$. If M is a simple $\widetilde{H}_C$-module and N is contained in $M|_{H_C}$, then M is a quotient of $N' \otimes_{H'_C} \widetilde{H}_C$ (and thus the restriction of M is semisimple).

(ii) It suffices to assume C is algebraically closed. Let M denote a simple supersingular module. Since the center of $\widetilde{H}$ is commutative and M is finite dimensional, there exists an eigenvector $v \in M$ with eigenvalues χ for the action of the center. Supersingularity then implies

$$0 = v \cdot z_{\mu'} = \chi(z_{\mu'})v \tag{2.4.3}$$

for any $\mu' \in X_*(T)^+$ such that $-\mu' \notin X_*(T)^+$.

Choose $\mu \in X_*(T)^+$ lying in the interior of the dominant Weyl chamber, so in particular $-\mu \notin X_*(T)^+$, and let $w_\circ \in W_0$ denote the longest element. We claim that

$$z_\mu z_{-w_\circ(\mu)} = q_{\lambda_\mu}|W_0|z_0 + \sum_{\substack{\mu' \in X_*(T)^+ \\ \ell(\lambda_{\mu'})>0}} a_{\mu'} z_{\mu'} \tag{2.4.4}$$

for some $a_{\mu'} \in \mathbb{Z}$. To see this, note that the product of the orbits $\mathcal{O}_\mu \cdot \mathcal{O}_{-w_\circ(\mu)}$ consists of elements of the form $\lambda_{w(\mu)}\lambda_{-w'w_\circ(\mu)}$, where $w, w' \in W_0$. If the length of $\lambda_{w(\mu)}\lambda_{-w'w_\circ(\mu)}$ is 0, then [Vig16, Cor. 5.11] implies $w(\mu) - w'w_\circ(\mu)$ is orthogonal to every simple root. Since this element is also a sum of coroots, we conclude that $w(\mu) - w'w_\circ(\mu) = 0$, which implies $w = w'w_\circ$, as the W_0-stabilizer of μ is trivial. The product formula (2.3.2) then gives equation (2.4.4).

Now, for $\mu' \in X_*(T)^+$, the condition $-\mu' \notin X_*(T)^+$ is equivalent to $\ell(\lambda_{\mu'}) > 0$. Applying χ to both sides of (2.4.4) and using (2.4.3) (for varying μ') gives $q_{\lambda_\mu}|W_0| = 0$, a contradiction.

(iii) This follows from [Vig17, Cor. 6.13] and part (i). □

Remark 2.4.5. The conditions in part (ii) of the above lemma are necessary: when $c \neq p$ divides $|W_0|$, there exist non-zero supersingular modules. For an example, suppose $\mathbf{G} = \mathbf{SL}_2$, q is odd, and $c = 2$. Then $H_C = \widetilde{H}_C$ admits a unique character χ, which sends T_s to 1 for each $s \in S$. If we let $\mu := (1, -1) \in X_*(T)^+$, then

$$z_\mu = T_{s_1}T_{s_2} + T_{s_2}T_{s_1},$$

where $S = \{s_1, s_2\}$. Thus, we have $\chi(z_\mu) = 0$. By induction, and using the assumption $c = 2$, we see that the element $z_{k\mu}$ lies in the ideal of the center generated by z_μ, for every $k \geq 1$. From this, we conclude that χ is supersingular.

3. On supercuspidal representations

The aim of this section is to collect various results concerning supercuspidal representations. We first state Proposition 3.1.3, which gives a convenient criterion for checking that a given irreducible admissible representation is supercuspidal when $\operatorname{char} C = p$. Propositions 3.2.1 and 3.3.2 allow us to make further reductions: in order to prove that $\mathbf{G}(F)$ admits an irreducible admissible supercuspidal C-representation when $\operatorname{char} F = 0$ and $\operatorname{char} C = p$, it suffices to assume that C is finite and $\mathbf{G}$ is absolutely simple, adjoint, and isotropic.

3.1. Supercuspidality criterion.

We begin with a definition.

Definition 3.1.1. Let R be a subfield of C. We say that a C-representation π of G *descends to* R if there exists an R-representation τ of G and a G-equivariant C-linear isomorphism

$$\varphi : C \otimes_R \tau \xrightarrow{\sim} \pi.$$

We call φ (and more often τ) an *R-structure* of π, or a *descent of* π *to* R.

We now describe the scalar extension of an irreducible admissible C-representation π of G [HV19]. Given such a π, the commutant $D := \mathrm{End}_C(\pi)$ is a division algebra of finite dimension over C. Let E denote the center of D, E_s/C the maximal separable extension contained in E/C and δ the reduced degree of D/E. Let $\overline{L}$ be an algebraically closed field containing E and $\pi_{\overline{L}}$ the scalar extension of π from C to $\overline{L}$.

Proposition 3.1.2 ([HV19, Thms. I.1, III.4])**.** *The length of $\pi_{\overline{L}}$ is $\delta[E : C]$ and*

$$\pi_{\overline{L}} \cong \bigoplus_{i \in \mathrm{Hom}_C(E_s, \overline{L})} \pi_i^{\oplus \delta}$$

where each π_i is indecomposable with commutant $\overline{L} \otimes_{i,E_s} E$, descends to a finite extension of C, has length $[E : E_s]$, and its irreducible subquotients are pairwise isomorphic, say to ρ_i. The ρ_i are admissible, with commutant $\overline{L}$, $\mathrm{Aut}_C(\overline{L})$-conjugate, pairwise non-isomorphic, and descend to a finite extension of C. Any descent of ρ_i to a finite extension C'/C, viewed as C-representation of G, is π-isotypic of finite length.

Proof. By [HV19, Thms. I.1, III.4], it suffices to prove that if ρ_i descends to a C'-representation ρ_i' with C'/C finite, then ρ_i' is π-isotypic of finite length. Then $(\rho_i')_{\overline{L}}$ injects into $\pi_{\overline{L}}$, and so ρ_i' injects into $\pi_{C'}$ by [HV19, Rk. II.2], which implies the claim. □

In particular, any irreducible admissible C-representation π with commutant C is absolutely irreducible in the sense that its base change π_L is irreducible for any field extension L/C. For example, this holds when C is algebraically closed.

Given an irreducible admissible C-representation π, the space $\pi^{\mathfrak{U}}$ of $\mathfrak{U}$-invariants comes equipped with a right action of the pro-p Iwahori–Hecke algebra $H_C(G, \mathfrak{U})$. This algebra has a similar structure to that of $H_C(G, \mathfrak{B})$. In particular, we have analogous definitions of the Bernstein elements E_λ and the central elements z_μ, as well as an analogous notion of supersingularity for right $H_C(G, \mathfrak{U})$-modules (cf. Definition 2.4.1). We say an irreducible admissible C-representation π is *supersingular* if the right $H_C(G, \mathfrak{U})$-module $\pi^{\mathfrak{U}}$ is supersingular.

Finally, recall that an irreducible admissible C-representation π of G is said to be *supercuspidal* if it is not a subquotient of $\mathrm{Ind}_P^G \tau$ for any parabolic subgroup $P = MN \subsetneq G$ and any irreducible admissible representation τ of the Levi subgroup M.

Proposition 3.1.3 (Supercuspidality criterion)**.** *Assume $c = p$. Suppose that π is an irreducible admissible C-representation of G. The following are equivalent:*

(i) *π is supercuspidal;*
(ii) *π is supersingular;*
(iii) *$\pi^{\mathfrak{U}}$ contains a non-zero supersingular element;*
(iv) *every subquotient of $\pi^{\mathfrak{U}}$ is supersingular;*
(v) *some subquotient of $\pi^{\mathfrak{U}}$ is supersingular.*

Proof. We have (i)⇔(ii)⇔(iii) by [HV19, Thms. I.13, III.17]. Since (ii)⇒(iv)⇒(v), it suffices to show that (v)⇒(ii). Let $\overline{C}$ denote an algebraic closure of C. Say $\pi^{\mathfrak{U}}$ has supersingular subquotient M. Then $(\pi^{\mathfrak{U}})_{\overline{C}} \cong (\pi_{\overline{C}})^{\mathfrak{U}}$ has subquotient $M_{\overline{C}}$, and $M_{\overline{C}}$ is clearly supersingular. By Proposition 3.1.2 there exists an irreducible admissible constituent ρ of $\pi_{\overline{C}}$ such that the $H_C(G, \mathfrak{U})$-module $\rho^{\mathfrak{U}}$ shares an irreducible constituent with $M_{\overline{C}}$. In particular, $\rho^{\mathfrak{U}}$ has a supersingular subquotient, and [OV18, Thm. 3] implies ρ is supersingular. Then [HV19, Lem. III.16 2)] implies that π is supersingular. $\square$

We now discuss how supercuspidality behaves under extension of scalars. We require a preliminary lemma.

Lemma 3.1.4. *Suppose that C'/C is a finite extension and that π' is an irreducible admissible C'-representation of G. Then $\pi'|_{C[G]} \cong \pi^{\oplus n}$ for some irreducible admissible C-representation π of G and some $n \geq 1$.*

Proof. Let $\overline{C}$ be an algebraic closure of C. Then the finite-dimensional $\overline{C}$-algebra $A := C' \otimes_C \overline{C}$ is of finite length over itself. The simple A-modules are given by $\overline{C}$ with C' acting via the various C-embeddings $C' \to \overline{C}$. It follows that $\pi'|_{C[G]} \otimes_C \overline{C} \cong \pi' \otimes_{C'} A$ is of finite length as $\overline{C}$-representation by Proposition 3.1.2. So $\pi'|_{C[G]}$ is of finite length. If π denotes an irreducible submodule, then $\sum_i \lambda_i \pi = \pi'|_{C[G]}$, where $\{\lambda_i\}_{i=1}^m$ is a basis of C'/C. It follows that $\pi'|_{C[G]} \cong \pi^{\oplus n}$ for some $n \leq m$. Moreover π is admissible, as $\pi'|_{C[G]}$ is. $\square$

Proposition 3.1.5. *Let $\overline{L}$ denote an algebraically closed field containing C. If $c \neq p$, we assume that $\overline{L} = \overline{C}$ is an algebraic closure of C.*

A C-representation π is supercuspidal if and only if some irreducible subquotient ρ of $\pi_{\overline{L}}$ is supercuspidal, if and only if every irreducible subquotient ρ of $\pi_{\overline{L}}$ is supercuspidal.

Proof. If $c = p$, we note that π is supercuspidal if and only if π is supersingular by Proposition 3.1.3. This is equivalent to some/every subquotient of $\pi_{\overline{L}}$ being supersingular [HV19, Lem. III.16 2)], or equivalently supercuspidal (again by [HV19, Thm. I.13]).

Now suppose that $c \neq p$ and $\overline{L} = \overline{C}$. Recall that parabolic induction Ind_P^G is exact, and commutes with scalar extensions and restrictions [HV19, Prop. III.12(i)]. If π is not supercuspidal, then π is a subquotient of $\mathrm{Ind}_P^G \tau$ for some proper parabolic $P = MN$ and irreducible admissible C-representation τ of M. Then $\pi_{\overline{C}}$ is a subquotient of $(\mathrm{Ind}_P^G \tau)_{\overline{C}} \cong \mathrm{Ind}_P^G(\tau_{\overline{C}})$. In particular, each irreducible (admissible) subquotient π' of $\pi_{\overline{C}}$ is a subquotient of $\mathrm{Ind}_P^G \tau'$ for some irreducible (admissible) subquotient τ' of $\tau_{\overline{C}}$. Hence none of the π' are supercuspidal.

For the converse, suppose by contradiction that $\pi_{\overline{C}}$ has an irreducible subquotient ρ that is not supercuspidal, i.e. ρ is a subquotient of $\mathrm{Ind}_P^G \tau$ for some proper parabolic $P = MN$ and irreducible admissible $\overline{C}$-representation τ of M. By [Vig96, II.4.7] (as $c \neq p$), respectively by Proposition 3.1.2, we can choose a finite extension C'/C with $C' \subset \overline{C}$ such that τ, respectively all irreducible constituents of $\mathrm{Ind}_P^G \tau$ and $\pi_{\overline{C}}$, can be defined over C'. Write $\tau \cong (\tau')_{\overline{C}}$ for some C'-representation τ'. Say the irreducible subquotients of $\mathrm{Ind}_P^G \tau'$ are $\sigma_1, \ldots, \sigma_n$. So by our choice of C', we know that $\rho \cong (\sigma_i)_{\overline{C}}$ for some i. As σ_i is a subquotient of $\mathrm{Ind}_P^G \tau'$, we see that $\sigma_i|_{C[G]}$ is a subquotient of $\mathrm{Ind}_P^G(\tau'|_{C[M]})$. But $\sigma_i|_{C[G]}$ is π-isotypic by Proposition 3.1.2 and $\tau'|_{C[M]}$ has finite length by Lemma 3.1.4, so π is a subquotient of $\mathrm{Ind}_P^G \tau''$ for some irreducible (admissible) subquotient τ'' of $\tau'|_{C[M]}$. $\square$

3.2. **Change of coefficient field.**

This section contains the proof of the following result.

Proposition 3.2.1 (Change of coefficient field)**.**

(i) *If G admits an irreducible admissible supercuspidal representation over some* finite *field of characteristic p, then G admits an irreducible admissible supercuspidal representation over any field of characteristic p.*
(ii) *If G admits an irreducible admissible supercuspidal representation over some field of characteristic $c \neq p$, then G admits an irreducible admissible supercuspidal representation over any algebraic extension of the prime field of characteristic c.*

Proof. Let F_c be the prime field of characteristic c (so that $F_0 = \mathbb{Q}$ and $F_c = \mathbb{F}_c$ if $c \neq 0$).

Step 1: We show that, if $c \neq p$ and G admits an irreducible admissible supercuspidal C-representation π, then G admits one over a finite extension of F_c.

Indeed, by Proposition 3.1.5 we can suppose C is algebraically closed. We claim that we may twist π by a C-character χ of G, so that the central character of $\pi \otimes \chi$ takes values in $\overline{F_c}$. To see this, we first note that there exists a subgroup ${}^\circ G$ of G such that (1) $G/{}^\circ G \cong \mathbb{Z}^r$ for some $r \geq 0$; (2) the restriction to $Z(G)$ of the map $u : G \twoheadrightarrow \mathbb{Z}^r$ has image of finite index; (3) $\ker(u|_{Z(G)}) = Z(G) \cap {}^\circ G$ is compact. (For all of this, see [Ber84, §1.12, 2.3].) Let $\mathfrak{L} := \mathrm{im}(u|_{Z(G)}) \subset \mathbb{Z}^r$ denote the image of $u|_{Z(G)}$. Since C is algebraically closed, the restriction map

$$\mathrm{Hom}(\mathbb{Z}^r, C^\times) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\mathfrak{L}, C^\times)$$

is surjective. Let ω_π denote the central character of the irreducible admissible C-representation π, and note that $\omega_\pi|_{Z(G)\cap {}^\circ G}$ takes values in $\overline{F_c}$ (since π is smooth and $Z(G) \cap {}^\circ G$ is compact). Choose a splitting v of the surjection $u : Z(G) \twoheadrightarrow \mathfrak{L}$, and let $\chi'' \in \mathrm{Hom}(\mathfrak{L}, C^\times)$ denote the character $\omega_\pi^{-1} \circ v$. We then let $\chi' \in \mathrm{Hom}(\mathbb{Z}^r, C^\times)$ denote any preimage of χ'' under res, and let $\chi : G \to C^\times$ be the inflation of χ' to G via u. Using that $\omega_{\pi\otimes\chi} = \omega_\pi \chi$ and $\omega_{\pi\otimes\chi}|_{Z(G)\cap {}^\circ G} = \omega_\pi|_{Z(G)\cap {}^\circ G}$, the construction of χ implies $\omega_{\pi\otimes\chi}(z) \in \overline{F_c}$ for all $z \in Z(G)$.

We may therefore assume that the central character of π takes values in $\overline{F_c}$. As $c \neq p$, by [Vig96, II.4.9] the representation π descends to a finite extension F'_c/F_c. Since descent preserves irreducibility, admissibility and supercuspidality, we obtain an irreducible admissible supercuspidal F'_c-representation of G.

Step 2: We show that if G admits an irreducible admissible supercuspidal representation over a finite extension of F_c then G admits such a representation over F_c.

Suppose C/F_c is a finite field extension and π an irreducible admissible C-representation of G. By Lemma 3.1.4, $\pi|_{F_c[G]}$ contains an irreducible admissible F_c-representation π'. By adjunction, π is a quotient of the scalar extension π'_C of π' from F_c to C.

We now show that if π is supercuspidal, then π' is also supercuspidal. Assume that π' is not supercuspidal, so that it is a subquotient of $\mathrm{Ind}_P^G \tau'$, where P is a proper parabolic subgroup of G and τ' is an irreducible admissible F_c-representation of the Levi subgroup M of P. Since parabolic induction is compatible with scalar extension from F_c to C, the representation π'_C is a subquotient of $\mathrm{Ind}_P^G \tau'_C$, and therefore the same is true of π. The C-representation τ'_C of M has finite length and its irreducible subquotients are admissible by [HV19, Thm. III.4]. Hence, π is a subquotient of $\mathrm{Ind}_P^G \rho$

for some irreducible admissible subquotient ρ of τ'_C, and we conclude that π is not supercuspidal.

Step 3: We show that if G admits an irreducible admissible supercuspidal $\mathbb{F}_p$-representation (resp., F_c-representation, where $c \neq p$), then G does so over any field of characteristic p (resp., any algebraic extension of F_c). More generally we show that if L/C is any field extension, assumed to be algebraic if $c \neq p$, and G admits an irreducible admissible supercuspidal C-representation then the same is true over L.

Let L/C be a field extension as above, and choose compatible algebraic closures $\overline{L}/\overline{C}$. Suppose π is an irreducible admissible supercuspidal C-representation of G, and let τ be an irreducible subquotient of the scalar extension π_L of π from C to L. By [HV19, Lem. III.1(ii)], τ is admissible. The scalar extension $\tau_{\overline{L}}$ of τ from L to $\overline{L}$ is a subquotient of the scalar extension $\pi_{\overline{L}}$ of π_L from L to $\overline{L}$ (the latter being equal to the scalar extension of π from C to $\overline{L}$). By Propositions 3.1.2 and 3.1.5, $\pi_{\overline{L}}$ has finite length and its irreducible subquotients are admissible and supercuspidal. Therefore, the same is true of $\tau_{\overline{L}}$. By Proposition 3.1.5 this implies that τ is supercuspidal. □

We now use extension of scalars to prove the following lemma, which will be used in the proof of Prop. 3.3.9.

Lemma 3.2.2. *Let π be an irreducible admissible C-representation of G and H a finite commutative quotient of G. Then the representation $\pi \otimes_C C[H]$ of G, with the natural action of G on $C[H]$, has finite length and its irreducible subquotients are admissible.*

Proof. The scalar extension of the C-representation π (resp. $C[H]$) to $\overline{C}$ has finite length with irreducible admissible quotients π_i (resp. χ_j, of dimension 1). Therefore $(\pi \otimes_C C[H])_{\overline{C}} \cong \pi_{\overline{C}} \otimes_{\overline{C}} \overline{C}[H]$ has finite length with irreducible admissible subquotients (namely, the $\pi_i \otimes_{\overline{C}} \chi_j$), implying the same for $\pi \otimes_C C[H]$. □

3.3. **Reduction to an absolutely simple adjoint group.** As is well known, the adjoint group $\mathbf{G}^{\mathrm{ad}}$ of $\mathbf{G}$ is F-isomorphic to a finite direct product of connected reductive F-groups

$$\mathbf{G}^{\mathrm{ad}} \cong \mathbf{H} \times \prod_i \mathrm{Res}_{F'_i/F}(\mathbf{G}'_i), \tag{3.3.1}$$

where $\mathbf{H}$ is anisotropic, the F'_i/F are finite separable extensions, and $\mathrm{Res}_{F'_i/F}(\mathbf{G}'_i)$ are scalar restrictions from F'_i to F of isotropic, absolutely simple, connected adjoint F'_i-groups $\mathbf{G}'_i$.

Proposition 3.3.2. *Assume that the field C is algebraically closed or finite, and that $\operatorname{char} F = 0$. If, for each i, the group $\mathbf{G}'_i(F'_i)$ admits an irreducible admissible supercuspidal C-representation, then G admits an irreducible admissible supercuspidal C-representation.*

The proposition is the combination of Propositions 3.3.3, 3.3.6, 3.3.8, and 3.3.11 below, corresponding to the operations of finite product, central extension, and scalar restriction (all when C algebraically closed or finite). We also note that if $\mathbf{G}$ is anisotropic, then G is compact and any irreducible smooth representation of G is finite-dimensional (hence admissible) and supercuspidal.

3.3.1. *Finite product.* Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two connected reductive F-groups, and σ and τ irreducible admissible C-representations of G_1 and G_2, respectively.

Proposition 3.3.3. *Assume that C is algebraically closed.*

(i) *The tensor product $\sigma \otimes_C \tau$ is an irreducible admissible C-representation of $G_1 \times G_2$.*
(ii) *Every irreducible admissible C-representation of $G_1 \times G_2$ is of this form.*
(iii) *The C-representation $\sigma \otimes_C \tau$ determines σ and τ (up to isomorphism).*
(iv) *The C-representation $\sigma \otimes_C \tau$ is supercuspidal if and only if σ and τ are supercuspidal.*

Proof. Note first that $\sigma \otimes_C \tau$ is admissible: for compact open subgroups K_1 of G_1 and K_2 of G_2, we have a natural isomorphism ([Bou12, §12.2 Lem. 1])

$$\operatorname{Hom}_{K_1}(\mathbf{1}_{K_1}, \sigma) \otimes_C \operatorname{Hom}_{K_2}(\mathbf{1}_{K_2}, \tau) \xrightarrow{\sim} \operatorname{Hom}_{K_1 \times K_2}(\mathbf{1}_{K_1} \otimes_C \mathbf{1}_{K_2}, \sigma \otimes_C \tau),$$

where $\mathbf{1}_{K_i}$ denotes the trivial representation of K_i. Thus, the admissibility of σ and τ implies the admissibility of $\sigma \otimes_C \tau$.

Suppose now C algebraically closed.

(i) Proposition 3.1.2 implies that the commutant of σ is C. Irreducibility of $\sigma \otimes_C \tau$ then follows from [Bou12, §12.2 Cor. 1].

(ii) Let π be an irreducible admissible C-representation of $G_1 \times G_2$, and let K_1, K_2 be any compact open subgroups of G_1, G_2, respectively, such that $\pi^{K_1 \times K_2} \neq 0$.

If $c = p$, the C-representation of G_1 given by $\pi^{1 \times K_2}$ is admissible (since $\pi^{K_1' \times K_2}$ is finite dimensional for any K_1'). By [HV12, Lemma 7.10], it contains an irreducible admissible C-subrepresentation σ. Set $\tau := \operatorname{Hom}_{G_1}(\sigma, \pi) \neq 0$, with the natural action of G_2. The representation $\sigma \otimes_C \tau$ embeds naturally in π. As π is irreducible, it is isomorphic to $\sigma \otimes_C \tau$, and τ is irreducible. As π is admissible, τ is admissible as well. (This proof is due to Henniart.)

If $c \neq p$, the space $\pi^{K_1 \times K_2}$ is a simple right $H_C(G_1 \times G_2, K_1 \times K_2)$-module ([Vig96, I.4.4, I.6.3]), and we have

$$H_C(G_1 \times G_2, K_1 \times K_2) \cong H_C(G_1, K_1) \otimes_C H_C(G_2, K_2).$$

By [Bou12, §12.1 Thm. 1], the finite-dimensional simple $H_C(G_1, K_1) \otimes_C H_C(G_2, K_2)$-modules factor, meaning $\pi^{K_1 \times K_2} \cong \sigma^{K_1} \otimes_C \tau^{K_2}$ for irreducible admissible C-representations σ, τ of G_1, G_2, respectively (this uses [Vig96, I.4.4, I.6.3] again). Thus, we obtain $\pi \cong \sigma \otimes_C \tau$.

(iii) As a C-representation of G_1, $\sigma \otimes_C \tau$ is σ-isotypic. Similarly, as a C-representation of G_2, $\sigma \otimes_C \tau$ is τ-isotypic. The result follows.

(iv) The parabolic subgroups of $G_1 \times G_2$ are products of parabolic subgroups of G_1 and of G_2. Let P, Q be parabolic subgroups of G_1, G_2, respectively, with Levi subgroups M, L, respectively, and let π' be an irreducible admissible C-representation of the product $M \times L$. By part (ii), the C-representation π' factors, say $\pi' = \sigma' \otimes_C \tau'$ for irreducible admissible C-representations σ' of M and τ' of L. We then obtain a natural isomorphism

$$\operatorname{Ind}_P^{G_1} \sigma' \otimes_C \operatorname{Ind}_Q^{G_2} \tau' \xrightarrow{\sim} \operatorname{Ind}_{P \times Q}^{G_1 \times G_2} \pi'.$$

Since the inductions on the left-hand side have finite length, we conclude that the irreducible subquotients of $\mathrm{Ind}_{P\times Q}^{G_1\times G_2}\pi'$ are tensor products of the irreducible subquotients of $\mathrm{Ind}_P^{G_1}\sigma'$ and of $\mathrm{Ind}_Q^{G_2}\tau'$, which gives the result. □

We assume from now until the end of §3.3.1 that C is a finite field.

Proposition 3.3.4. *Assume that C is finite. Let π be an irreducible admissible C-representation of G. The commutant of π is a finite field extension D of C and the scalar extension π_D of π from C to D is isomorphic to*

$$\pi_D \cong \bigoplus_{i\in \mathrm{Gal}(D/C)} \pi_i,$$

where the π_i are irreducible admissible D-representations of G. Moreover, the π_i each have commutant D, are pairwise non-isomorphic, form a single $\mathrm{Gal}(D/C)$*-orbit, and, viewed as C-representations, are isomorphic to π.*

Proof. The commutant D of π is a division algebra of finite dimension over C. Since the Brauer group of a finite field is trivial, D is a finite Galois extension of C. The result now follows from [HV19, Thms. I.1, III.4] by taking $R' = D$. (Note also that as a C-representation, π_D is π-isotypic of length $[D : C]$.) □

Recall that we have fixed irreducible admissible C-representations σ and τ of G_1 and G_2, respectively. Their respective commutants D_σ and D_τ are finite extensions of C of dimensions d_σ and d_τ, respectively. We embed them into $\overline{C}$, and consider:

- the field D generated by D_σ and D_τ, which has C-dimension $\mathrm{lcm}(d_\sigma, d_\tau)$,
- the field $D' := D_\sigma \cap D_\tau$, which has C-dimension $\gcd(d_\sigma, d_\tau)$.

The fields D_σ, D_τ are linearly disjoint over D', we have $D_\sigma \otimes_{D'} D_\tau \cong D$ and

$$D_\sigma \otimes_C D_\tau \cong \prod_{k=1}^{[D':C]} D. \tag{3.3.5}$$

Proposition 3.3.6. *Assume that C is finite. The C-representation $\sigma\otimes_C\tau$ of $G_1\times G_2$ is isomorphic to*

$$\sigma\otimes_C\tau \cong \bigoplus_{k=1}^{\gcd(d_\sigma,d_\tau)} \pi_k,$$

where the π_k are irreducible admissible C-representations with commutant D, which are pairwise non-isomorphic. The C-representations σ and τ are supercuspidal if and only if all the π_k are supercuspidal, if and only if some π_k is supercuspidal.

Proof. By Proposition 3.3.4, we have

$$\sigma_D \cong \bigoplus_{i\in\mathrm{Gal}(D_\sigma/C)} \sigma_i, \qquad \tau_D \cong \bigoplus_{j\in\mathrm{Gal}(D_\tau/C)} \tau_j,$$

where the σ_i (resp. τ_j) are irreducible admissible D-representations of G_1 (resp. G_2) with commutant D, which are pairwise non-isomorphic, form a single $\mathrm{Gal}(D/C)$-orbit, descend to D_σ (resp. D_τ) and their descents, viewed as C-representations, are isomorphic to σ (resp. τ). The C-representation $\sigma \otimes_C \tau$ of $G_1 \times G_2$ is admissible, and its scalar extension from C to D is equal to

$$(\sigma \otimes_C \tau)_D \;\cong\; \sigma_D \otimes_D \tau_D \cong \bigoplus_{(i,j)\in \mathrm{Gal}(D_\sigma/C)\times \mathrm{Gal}(D_\tau/C)} \sigma_i \otimes_D \tau_j. \tag{3.3.7}$$

The D-representation $\sigma_i \otimes_D \tau_j$ of $G_1 \times G_2$ is admissible and has commutant $D \otimes_D D = D$ ([Bou12, §12.2 Lem. 1]). Hence, $\sigma_i \otimes_D \tau_j$ is absolutely irreducible and equation (3.3.7) implies $(\sigma \otimes_C \tau)_D$ is semisimple. By [Bou12, §12.7 Prop. 8], this implies that $\sigma \otimes_C \tau$ is semisimple; its commutant is isomorphic to $D_\sigma \otimes_C D_\tau$ by [Bou12, §12.2 Lem. 1]. From equation (3.3.5) we see that $\sigma \otimes_C \tau$ has length $[D' : C] = \gcd(d_\sigma, d_\tau)$, its irreducible constituents π_k are admissible and pairwise non-isomorphic with commutant D.

Applying Proposition 3.3.3 over $\overline{C}$ and Proposition 3.1.5 (several times), we see that σ and τ are supercuspidal if and only if some/every σ_i and some/every τ_j are supercuspidal, if and only if some/every $\sigma_i \otimes_D \tau_j$ is supercuspidal. From Proposition 3.1.5 again, this is also equivalent to π_k being supercuspidal for some/every k. □

3.3.2. *Central extension.* Recall that we have a short exact sequence of F-groups

$$1 \to \mathbf{Z}(\mathbf{G}) \to \mathbf{G} \xrightarrow{i} \mathbf{G}^{\mathrm{ad}} \to 1,$$

which induces an exact sequence between F-points

$$1 \to Z(G) \to G \xrightarrow{i} G^{\mathrm{ad}} \to H^1(F, \mathbf{Z}(\mathbf{G})).$$

The image $i(G)$ of G is a closed cocompact normal subgroup of G^{ad} and $H^1(F, \mathbf{Z}(\mathbf{G}))$ is commutative.

Until the end of §3.3.2, we assume that $\operatorname{char} F = 0$. The group $H^1(F, \mathbf{Z}(\mathbf{G}))$ is then finite ([PR94, Thm. 6.14]), implying that $i(G)$ is an *open* normal subgroup of G^{ad} and the quotient $G^{\mathrm{ad}}/i(G)$ is finite and commutative. Our next task will be to prove the following:

Proposition 3.3.8. *Suppose that* $\operatorname{char} F = 0$*. Then* G^{ad} *admits an irreducible admissible supercuspidal* C*-representation if and only if* G *admits such a representation such that moreover* $Z(G)$ *acts trivially.*

Inflation from $i(G)$ to G identifies representations of $i(G)$ with representations of G having trivial $Z(G)$-action; this inflation functor respects irreducibility and admissibility. The composite functor

$$(\text{inflation from } i(G) \text{ to } G) \circ (\text{restriction from } G^{\mathrm{ad}} \text{ to } i(G))$$

from C-representations of G^{ad} to representations of G trivial on $Z(G)$ will be denoted by $- \circ i$.

Suppose $\widetilde{\rho}$ is an irreducible admissible C-representation of G with trivial action of $Z(G)$. Then $\widetilde{\rho}$ is the inflation of a representation ρ of the open, normal, finite-index subgroup $i(G)$ of G^{ad}. The C-representation ρ of $i(G)$ is irreducible and admissible, and therefore the induced representation $\operatorname{Ind}_{i(G)}^{G^{\mathrm{ad}}} \rho$ of G^{ad} is admissible of finite length. Any irreducible quotient π of $\operatorname{Ind}_{i(G)}^{G^{\mathrm{ad}}} \rho$ is admissible (if $c = p$, this uses the assumption

$\operatorname{char} F = 0$; see [Hen09, §4, Thm. 1]). By adjunction, $\pi|_{i(G)}$ contains a subrepresentation isomorphic to ρ and, by inflation from $i(G)$ to G, $\widetilde{\rho}$ is isomorphic to a subquotient of $\pi \circ i$.

Conversely, suppose π is an irreducible admissible C-representation of G^{ad}. The restriction $\pi|_{i(G)}$ of π to $i(G)$ is semisimple of finite length, and its irreducible constituents ρ are G^{ad}-conjugate and admissible (see [Vig96, I.6.12]; note that the condition that the index is invertible in C is not necessary and not used in the proof). Hence, the C-representation $\pi \circ i$ of G is semisimple of finite length, and its irreducible constituents are the inflations $\widetilde{\rho}$ of the irreducible constituents ρ of $\pi|_{i(G)}$.

Proposition 3.3.8 now follows from:

Proposition 3.3.9. *Suppose that* $\operatorname{char} F = 0$ *and let* π, ρ *and* $\widetilde{\rho}$ *be as above. Then* π *is supercuspidal if and only if some* $\widetilde{\rho}$ *is supercuspidal, if and only if all* $\widetilde{\rho}$ *are supercuspidal.*

Proof. We first check first the compatibility of parabolic induction with $- \circ i$. The parabolic F-subgroups of $\mathbf{G}$ and of $\mathbf{G}^{\mathrm{ad}}$ are in bijection via the map i ([Bor91, 22.6 Thm.]). If the parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$ corresponds to the parabolic F-subgroup $\mathbf{Q}$ of $\mathbf{G}^{\mathrm{ad}}$, then i restricts to an isomorphism between their unipotent radicals, and sends a Levi subgroup $\mathbf{M}$ of $\mathbf{P}$ onto a Levi subgroup $\mathbf{L}$ of $\mathbf{Q}$. Further, we have an exact sequence between F-points:

$$1 \to Z(G) \to M \xrightarrow{i} L \to H^1(F, \mathbf{Z}(\mathbf{G})).$$

We have $G^{\mathrm{ad}} = Qi(G)$ and $Q \cap i(G) = i(P) = i(M)U$, where $i(M)$ is an open normal subgroup of L having finite commutative quotient, and U is the unipotent radical of Q. Thus, if σ is a smooth C-representation of L, the Mackey decomposition implies $(\operatorname{Ind}_Q^{G^{\mathrm{ad}}} \sigma)|_{i(G)} \cong \operatorname{Ind}_{i(P)}^{i(G)}(\sigma|_{i(M)})$ and, by inflation from $i(G)$ to G, we obtain:

$$(\operatorname{Ind}_Q^{G^{\mathrm{ad}}} \sigma) \circ i \cong \operatorname{Ind}_P^G(\sigma \circ i). \tag{3.3.10}$$

We may now proceed with the proof. It suffices to prove:

(i) if π is non-supercuspidal, then all $\widetilde{\rho}$ are non-supercuspidal,
(ii) if some $\widetilde{\rho}$ is non-supercuspidal, then π is non-supercuspidal.

To prove (i), let π be an irreducible admissible non-supercuspidal C-representation of G^{ad}, which is isomorphic to a subquotient of $\operatorname{Ind}_Q^{G^{\mathrm{ad}}} \sigma$ for $Q \subsetneq G^{\mathrm{ad}}$ and σ an irreducible admissible C-representation of L. Therefore, $\pi \circ i$ is isomorphic to a subquotient of $(\operatorname{Ind}_Q^{G^{\mathrm{ad}}} \sigma) \circ i$, and by equation (3.3.10), each $\widetilde{\rho}$ is isomorphic to a subquotient of $\operatorname{Ind}_P^G \widetilde{\tau}$ for some irreducible subquotient $\widetilde{\tau}$ of $\sigma \circ i$ (depending on ρ). Since $\widetilde{\tau}$ is admissible and $P \subsetneq G$, all the $\widetilde{\rho}$ are non-supercuspidal.

To prove (ii), let π be an irreducible admissible C-representation of G^{ad} such that some irreducible constituent $\widetilde{\rho}$ of $\pi \circ i$ is non-supercuspidal. Suppose $\widetilde{\rho}$ is isomorphic to a subquotient of $\operatorname{Ind}_P^G \tau'$ for $P \subsetneq G$ and τ' an irreducible admissible C-representation of M. The central subgroup $Z(G)$ acts trivially on $\widetilde{\rho}$, and hence also on τ'. Therefore $\tau' = \widetilde{\tau}$ for some irreducible subquotient τ of $\sigma|_{i(G)}$, where σ is an irreducible admissible C-representation of L. The representation $\widetilde{\rho}$ is isomorphic to a subquotient of $\operatorname{Ind}_P^G(\sigma \circ i)$. By equation (3.3.10) and exactness of parabolic induction, $\operatorname{Ind}_{i(G)}^{G^{\mathrm{ad}}}(\rho)$, and hence its quotient π, is isomorphic to a subquotient of $\operatorname{Ind}_{i(G)}^{G^{\mathrm{ad}}}((\operatorname{Ind}_Q^{G^{\mathrm{ad}}} \sigma)|_{i(G)})$. This representation is isomorphic to

$$\mathrm{Ind}_{i(M)U}^{G^{\mathrm{ad}}}(\sigma|_{i(M)}) \cong \mathrm{Ind}_{Q}^{G^{\mathrm{ad}}}\left(\mathrm{Ind}_{i(M)}^{L}(\sigma|_{i(M)})\right) \cong \mathrm{Ind}_{Q}^{G^{\mathrm{ad}}}(\sigma \otimes_C C[i(M)\backslash L]).$$

By Lemma 3.2.2, the C-representation $\sigma \otimes_C C[i(M)\backslash L]$ of L has finite length and its irreducible subquotients ν are admissible. Therefore π is isomorphic to a subquotient of $\mathrm{Ind}_Q^{G^{\mathrm{ad}}} \nu$ for some ν and some $Q \subsetneq G^{\mathrm{ad}}$, and therefore π is non-supercuspidal. $\square$

3.3.3. *Scalar restriction.* Now let F'/F be a finite separable extension, $\mathbf{G}'$ a connected reductive F'-group and $\mathbf{G} := \mathrm{Res}_{F'/F}(\mathbf{G}')$ the scalar restriction of $\mathbf{G}'$ from F' to F. As topological groups, $G' := \mathbf{G}'(F')$ is equal to $G := \mathbf{G}(F)$. By [BT65, 6.19. Cor.], G' and G have the same parabolic subgroups. Hence:

Proposition 3.3.11. *G' admits an irreducible admissible supercuspidal C-representation if and only if G does.*

4. Proof of the main theorem for most simple groups

4.1. **Discrete Iwahori–Hecke modules.** Let $\mathrm{Rep}_C(G, \mathfrak{B})$ denote the category of C-representations of G generated by their $\mathfrak{B}$-invariant vectors, and let $\mathrm{Mod}(H_C(G, \mathfrak{B}))$ denote the category of right $H_C(G, \mathfrak{B})$-modules. The functor of $\mathfrak{B}$-invariants

$$\begin{aligned} \mathrm{Rep}_C(G, \mathfrak{B}) &\to \mathrm{Mod}(H_C(G, \mathfrak{B})) \\ \pi &\mapsto \pi^{\mathfrak{B}} \end{aligned}$$

admits a left adjoint

$$\begin{aligned} \mathfrak{T} : \mathrm{Mod}(H_C(G, \mathfrak{B})) &\to \mathrm{Rep}_C(G, \mathfrak{B}) \\ M &\mapsto M \otimes_{H_C(G,\mathfrak{B})} C[\mathfrak{B}\backslash G]. \end{aligned}$$

Proposition 4.1.1. *When $c \neq p$, the functor $\pi \mapsto \pi^{\mathfrak{B}}$ induces a bijection between the isomorphism classes of irreducible C-representations π of G with $\pi^{\mathfrak{B}} \neq 0$ and isomorphism classes of simple right $H_C(G, \mathfrak{B})$-modules* ([Vig96, I.4.4, I.6.3])*. When $C = \mathbb{C}$, the functors are inverse equivalences of categories* (*cf.* [Ber84, Cor. 3.9(ii)]*; see also* [Mor99, Thms. 4.8, 4.4(iii)])*.*

When $C = \mathbb{C}$, the Bernstein ring embedding $H_{\mathbb{C}}(Z, Z_0) \xrightarrow{\theta} \widetilde{H}_{\mathbb{C}}$ is the linear map defined by sending T_λ^Z to $\theta_\lambda := q_\lambda^{-1/2} E_\lambda$ for $\lambda \in \Lambda$. Its image is equal to $\mathcal{A}_{\mathbb{C}}$. Note that if $\lambda \in \Lambda$ is anti-dominant and $z \in Z$ lifts λ, we have $q_\lambda = \delta_B(z)$, where δ_B denotes the modulus character of B.

We now recall some properties of the category $\mathrm{Rep}_{\mathbb{C}}(G, \mathfrak{B})$, including Casselman's criterion of square integrability modulo center, before giving the definition of a discrete simple right $H_{\mathbb{C}}(G, \mathfrak{B})$-module. Recall that π_U denotes the space of U-coinvariants (i.e., the unnormalized Jacquet module) of a representation π.

Lemma 4.1.2. *Suppose that π is an admissible $\mathbb{C}$-representation of G. Then the natural map $\pi \twoheadrightarrow \pi_U$ induces an isomorphism $\varphi : \pi^{\mathfrak{B}} \xrightarrow{\sim} \pi_U^{Z_0}$. Moreover, we have $\varphi(v \cdot \theta_{\lambda^{-1}}) = \delta_B^{-1/2}(t)(t \cdot \varphi(v))$ for $\lambda \in \Lambda_T$, $t \in T$ lifting λ, and $v \in \pi^{\mathfrak{B}}$.*

Proof. Recall that $\mathfrak{B}$ has an Iwahori decomposition with respect to Z, U, U^{op}. Then [Cas, Prop. 4.1.4] implies that the map $\pi \twoheadrightarrow \pi_U$ induces an isomorphism $\pi^{\mathfrak{B}} \cdot T_{\lambda^{-1}} \xrightarrow{\sim} \pi_U^{Z_0}$ for $\lambda \in \Lambda_T$ with $\max_{\alpha\in\Delta} |\alpha(\lambda)|_F$ sufficiently small. By [Vig16, Prop. 4.13(1)] the operator $T_{\lambda^{-1}}$ is invertible in $H_{\mathbb{C}}(G, \mathfrak{B})$, so $\pi^{\mathfrak{B}} \cdot T_{\lambda^{-1}} = \pi^{\mathfrak{B}}$.

To show the last statement, we may assume that $\lambda \in \Lambda_T^+$. Then, in our terminology, [Cas, Lemma 4.1.1] says that $\varphi(|\mathfrak{B}t\mathfrak{B}/\mathfrak{B}|^{-1}[\mathfrak{B}t\mathfrak{B}]\cdot v) = t\cdot\varphi(v)$, where $[\mathfrak{B}t\mathfrak{B}]$ denotes the usual double coset operator on $\pi^{\mathfrak{B}}$. Now $[\mathfrak{B}t\mathfrak{B}]\cdot v = v\cdot T_{t^{-1}}$ and $T_{t^{-1}} = E_{t^{-1}} = q_{t^{-1}}^{1/2}\theta_{t^{-1}}$. Moreover, $|\mathfrak{B}t\mathfrak{B}/\mathfrak{B}| = q_t = q_{t^{-1}} = \delta_B(t^{-1})$. Putting this all together, we obtain the claim. □

Remark 4.1.3. The lemma and its proof hold when $\mathfrak{B}$ is replaced by $\mathfrak{U}$ and Z_0 is replaced by $Z_0 \cap \mathfrak{U}$.

Proposition 4.1.4. *Let π be an irreducible $\mathbb{C}$-representation of G with $\pi^{\mathfrak{B}} \neq 0$.*

(i) *π is isomorphic to a subrepresentation of $\mathrm{Ind}_B^G \sigma$, where σ is a $\mathbb{C}$-character of Z trivial on Z_0.*

(ii) ***Casselman's criterion:** π is square integrable modulo center* (*as defined in* [Cas, §2.5]) *if and only if its central character is unitary and*

$$|\chi(\mu(\varpi))|_{\mathbb{C}} < 1$$

for all $\mu \in X_*(T)^+$ such that $-\mu \notin X_*(T)^+$, and all characters χ of T contained in $\delta_B^{-1/2}\pi_U$.

Proof. (i) Since π is irreducible and smooth, it is admissible by [Vig96, II.2.8], and [Cas, 3.3.1] implies π_U is admissible as well. By Lemma 4.1.2 and the assumption $\pi^{\mathfrak{B}} \neq 0$, we see that $\pi_U \neq 0$. The claim now follows by choosing an irreducible quotient $\pi_U \to \sigma$ for which $\sigma^{Z_0} \neq 0$ and applying Frobenius reciprocity.

(ii) This follows from [Cas, Thm. 6.5.1]. □

Definition 4.1.5. We say a simple right $H_{\mathbb{C}}(G, \mathfrak{B})$-module is *discrete* if it is isomorphic to $\pi^{\mathfrak{B}}$ for an irreducible admissible square-integrable modulo center $\mathbb{C}$-representation π of G.

Proposition 4.1.6. *A simple right $H_{\mathbb{C}}(G, \mathfrak{B})$-module M is discrete if and only if any $\mathbb{C}$-character χ of $\mathcal{A}_{\mathbb{C}}$ contained in M satisfies the following condition: the restriction of χ to $\Lambda_{Z(G)}$ is a unitary character, and*

$$|\chi(\theta_{\lambda_\mu^{-1}})|_{\mathbb{C}} < 1 \tag{4.1.7}$$

for any $\mu \in X_*(T)^+$ such that $-\mu \notin X_*(T)^+$.

Proof. Note that $M = \pi^{\mathfrak{B}}$ for an irreducible (admissible) $\mathbb{C}$-representation π of G. Then π has unitary central character if and only if $\Lambda_{Z(G)}$ acts by a unitary character on M. As any irreducible $\mathcal{A}_{\mathbb{C}}$-module is a character, by Casselman's criterion (Proposition 4.1.4) and Lemma 4.1.2, M is discrete if and only condition (4.1.7) holds. □

Remark 4.1.8. Some authors view $\pi^{\mathfrak{B}}$ as a left $H_{\mathbb{C}}(G, \mathfrak{B})$-module. One may pass between left and right modules by using the anti-automorphism $T_w \mapsto T_{w^{-1}}$; that is, we may define

$$T_w \cdot v = v \cdot T_{w^{-1}}$$

for $w \in \widetilde{W}, v \in \pi^{\mathfrak{B}}$. The space $\pi^{\mathfrak{B}}$, viewed as either a left or right $H_{\mathbb{C}}(G, \mathfrak{B})$-module, is then called discrete if π is square integrable modulo center. For left modules, the proposition above holds with "dominant" replaced by "anti-dominant," and "θ" replaced by "$\widetilde{\theta}^{+}$" (for the definition of $\widetilde{\theta}^{+}$, see the paragraph preceding Proposition 8 in [Vig05]).

Lemma 4.1.9. *For a character $\chi : \mathcal{A}_{\mathbb{C}} \to \mathbb{C}$ such that $\chi|_{\Lambda_{Z(G)}}$ is unitary, the following conditions are equivalent:*

(i) $|\chi(\theta_{\lambda_\mu^{-1}})|_{\mathbb{C}} < 1$ *for any* $\mu \in X_*(T)^+$ *such that* $-\mu \notin X_*(T)^+$,
(ii) $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} < 1$ *for any* $\lambda \in \Lambda^{\mathrm{sc},+}$ *such that* $\lambda^{-1} \notin \Lambda^{\mathrm{sc},+}$,
(iii) $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} < 1$ *for any* $\lambda \in \Lambda^{+}$ *such that* $\lambda^{-1} \notin \Lambda^{+}$.

Proof. We first recall that the invertible elements in Λ^+ consist of $\Lambda_{\ker \nu}$, so $|\chi(\theta_\lambda)|_{\mathbb{C}} = 1$ for all invertible elements of Λ^+.

As $\Lambda_T \cong X_*(T)$, we see that (iii) implies (i) and (ii). To prove that (ii) implies (iii), we need to show that $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} = 1$ for $\lambda \in \Lambda^+$ implies $\lambda^{-1} \in \Lambda^+$. By Lemma 2.2.5 pick $n \geq 1$ such that $\lambda^n \in \Lambda_{Z(G)} \times \Lambda^{\mathrm{sc},+}$. Then $\lambda^n \lambda_0 \in \Lambda^{\mathrm{sc},+}$ for some $\lambda_0 \in \Lambda_{Z(G)}$. As $|\chi(\theta_{\lambda^{-n}\lambda_0^{-1}})|_{\mathbb{C}} = 1$ we deduce from (ii) that $\lambda^n\lambda_0 \in \Lambda^{\mathrm{sc},+} \cap (\Lambda^{\mathrm{sc},+})^{-1}$, which is contained in $\Lambda^+ \cap (\Lambda^+)^{-1}$. Therefore $\lambda^n \in \Lambda^+ \cap (\Lambda^+)^{-1}$. From the definition of dominance it follows that $\lambda \in \Lambda^+ \cap (\Lambda^+)^{-1}$.

The proof that (i) implies (iii) is similar but easier. □

Proposition 4.1.10. *A simple right $H_{\mathbb{C}}(G, \mathfrak{B})$-module M is discrete if and only if $\Lambda_{Z(G)}$ acts on M by a unitary character and if its restriction to $H_{\mathbb{C}}(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}})$ is discrete.*

Proof. This follows from Proposition 4.1.6 and Lemma 4.1.9. □

4.2. **Characters.** In this section we continue to assume C is a field of characteristic c, and suppose further that $\mathbf{G}$ is absolutely simple and isotropic. We determine the characters $H = H(G^{\mathrm{sc}}, \mathfrak{B}^{\mathrm{sc}}) \to C$ which extend to $\widetilde{H} = H(G, \mathfrak{B})$. This is an exercise, which is already in the literature when $C = \mathbb{C}$ (cf. [Bor76]).

For distinct reflections $s, t \in S$, the order $n_{s,t}$ of st is finite, except if the type of Σ is A_1. In the finite case, the braid relations (2.1.2) imply

$$(T_sT_t)^r = (T_tT_s)^r \quad \text{if } n_{s,t} = 2r, \tag{4.2.1}$$

$$(T_sT_t)^rT_s = (T_tT_s)^rT_t \quad \text{if } n_{s,t} = 2r+1. \tag{4.2.2}$$

The T_s for $s \in S$ and the relations (2.1.3), (4.2.1) and (4.2.2) give a presentation of H. A presentation of $\widetilde{H}$ is given by the T_u, T_s for $u \in \Omega, s \in S$ and the relations (2.1.3), (4.2.1), (4.2.2) and

$$T_uT_{u'} = T_{uu'} \quad \text{if } u, u' \in \Omega, \tag{4.2.3}$$

$$T_uT_s = T_{u(s)}T_u \quad \text{if } u \in \Omega, s \in S, \tag{4.2.4}$$

where $u(s)$ denotes the action of Ω on S.

We have a disjoint decomposition

$$S = \bigsqcup_{i=1}^{m} S_i,$$

where S_i is the intersection of S with a conjugacy class of W. The S_i are precisely the connected components of Dyn when all multiple edges are removed (see [Bou02, VI.4.3 Th. 4] and [Bor76, 3.3]). Thus, we have

$$m = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \qquad \text{when the type of } \Sigma = \begin{cases} \mathrm{A}_\ell\ (\ell \geq 2), \mathrm{D}_\ell\ (\ell \geq 4), \mathrm{E}_6, \mathrm{E}_7, \text{or } \mathrm{E}_8; \\ \mathrm{A}_1, \mathrm{B}_\ell\ (\ell \geq 3), \mathrm{F}_4, \text{or } \mathrm{G}_2; \\ \mathrm{C}_\ell\ (\ell \geq 2). \end{cases}$$

When $m > 1$, we fix a labeling of the S_i such that $|S_1| \geq |S_2|$, and when the type of Σ is C_ℓ $(\ell \geq 2)$, we let $S_2 = \{s_2\}$ and $S_3 = \{s_3\}$ denote the endpoints of Dyn. (Note that there are two possible labelings in types A_1 and C_ℓ $(\ell \geq 2)$.) The parameters d_s are equal on each component S_i; we denote this common value by d_i.

Definition 4.2.5. The unique character $\chi : H \to C$ with $\chi(T_s) = q_s$ (resp., $\chi(T_s) = -1$) for all $s \in S$ is called the *trivial* (resp., *special*) C-character.

Lemma 4.2.6. *Suppose $\{T_s\}_{s\in S} \to C$ is an arbitrary map.*

(i) *When $c \neq p$, the above map extends to a character of H if and only if it is constant on each S_i, and takes the value -1 or q^{d_i} on each $T_s, s \in S_i$. There are 2^m characters if $q^{d_i} + 1 \neq 0$ in C for each i.*

(ii) *When $c = p$, the above map extends to a character of H if and only if its values are -1 or 0 on each $T_s, s \in S$. There are $2^{|S|}$ characters. Such a character is supersingular if and only if it is not special or trivial.*

Proof. (i) This follows from the presentation of H and the fact that the T_w are invertible (so that the map must be constant on conjugacy classes).

(ii) This follows from [Vig17, Prop. 2.2]. The claim about supersingularity follows from [Vig17, Thm. 6.15]. □

We wish to determine which characters of H extend to $\widetilde{H}$. Since the elements T_u for $u \in \Omega$ are invertible in $\widetilde{H}$, the relations (4.2.4) imply that a character $\chi : H \to C$ extends to a character of $\widetilde{H}$ if and only if $\chi(T_s) = \chi(T_{u(s)})$ for all $s \in S$ and $u \in \Omega$. For example, if the image Ψ of Ω in $\mathrm{Aut}(W, S, d_i)$ is trivial, then any character of H extends to $\widetilde{H}$. The extensions are not unique in general. By their very definition, the trivial and special characters always extend, and we also refer to their extensions as trivial and special characters.

Let $\chi : H \to C$ denote a character, and suppose $c \neq p$. By Lemma 4.2.6(i), the value of χ on T_s for $s \in S_i$ is constant for each $1 \leq i \leq m$. We define $\chi_i := \chi(T_s) \in C$ for $s \in S_i$, and identify the character χ with the m-tuple $(\chi_i)_{1\leq i\leq m}$.

Lemma 4.2.7. *Assume $c \neq p$. Let $\chi : H \to C$ denote a character of H, associated to the m-tuple $(\chi_i)_{1\leq i\leq m}$. Then χ extends to a character of $\widetilde{H}$ except in the following cases:*

- *type A_1, equal parameters $d_1 = d_2$, $\Psi \neq 1$, and $\chi_1 \neq \chi_2$;*
- *type C_ℓ $(\ell \geq 2)$, equal parameters $d_2 = d_3$, $\Psi \neq 1$, and $\chi_2 \neq \chi_3$.*

Proof. When $m = 1$, then $\chi(T_s) = \chi(T_{u(s)})$ for all $u \in \Omega$ and $s \in S$, so that χ extends to $\widetilde{H}$. We may therefore assume $m > 1$. We proceed type-by-type:

- *Type* A_1 *with equal parameters* $d_1 = d_2$. The group $\mathrm{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$ permutes s_1 and s_2. If $\Psi = 1$ or $\chi_1 = \chi_2$, then χ extends to $\widetilde{H}$, while if $\Psi \neq 1$ and $\chi_1 \neq \chi_2$, the character χ cannot extend.
- *Type* B_ℓ $(\ell \geq 3)$. In this case, $\mathrm{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$ stabilizes the sets S_1 and S_2, so that $\chi(T_s) = \chi(T_{u(s)})$ for all $u \in \Omega$ and $s \in S$. Thus χ extends to $\widetilde{H}$.
- *Type* C_ℓ $(\ell \geq 2)$ *with equal parameters* $d_2 = d_3$. The group $\mathrm{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$ permutes s_2 and s_3. If $\Psi = 1$ or if $\chi_2 = \chi_3$, then χ extends to $\widetilde{H}$, while if $\Psi \neq 1$ and $\chi_2 \neq \chi_3$, the character χ cannot extend.
- *Type* A_1 *with unequal parameters* $d_1 \neq d_2$; *Type* F_4; *Type* G_2; *Type* C_ℓ $(\ell \geq 2)$ *with unequal parameters* $d_2 \neq d_3$. In these cases, $\mathrm{Aut}(W, S, d_s)$ (and consequently Ψ) is trivial, and thus χ extends to $\widetilde{H}$.

□

Before stating the next result, we require a definition.

Definition 4.2.8. Let $R \subset \mathbb{C}$ be a subring of $\mathbb{C}$. We say a right $\widetilde{H}_{\mathbb{C}}$-module M is *R-integral* if there exists an $\widetilde{H}_R$-submodule $M^\circ \subset M$ such that the natural map

$$\mathbb{C} \otimes_R M^\circ \to M$$

is an isomorphism of $\widetilde{H}_{\mathbb{C}}$-modules. We call M° an *R-integral structure* of M. If $\mathfrak{p}$ is a maximal ideal of R, the $\widetilde{H}_{R/\mathfrak{p}}$-module $R/\mathfrak{p} \otimes_R M^\circ$ is called *reduction of M° modulo $\mathfrak{p}$*. We make similar definitions for the algebra $H_{\mathbb{C}}$.

The following proposition combines the above results.

Proposition 4.2.9.

(i) *$H_{\mathbb{C}}$ admits 2^m $\mathbb{C}$-characters. They are all $\mathbb{Z}$-integral, and their reductions modulo p are supersingular except for the special and trivial characters.*

(ii) *Suppose $\chi : H_{\mathbb{C}} \to \mathbb{C}$ is a character, associated to the m-tuple $(\chi_i)_{1 \leq i \leq m}$, and suppose we are in one of the following two cases:*
   - *type A_1, equal parameters $d_1 = d_2$, $\Psi \neq 1$, and $\chi_1 \neq \chi_2$;*
   - *type C_ℓ $(\ell \geq 2)$, equal parameters $d_2 = d_3$, $\Psi \neq 1$, and $\chi_2 \neq \chi_3$.*

   Then the $H_{\mathbb{C}}$-module $\chi \oplus \overline{\chi}$ extends to a two-dimensional, $\mathbb{Z}$-integral simple (left or right) $\widetilde{H}_{\mathbb{C}}$-module with supersingular reduction modulo p, where $\overline{\chi} = (\chi_2, \chi_1)$ in the A_1 case and $\overline{\chi} = (\chi_1, \chi_3, \chi_2)$ in the C_ℓ case.

(iii) *Suppose $\chi : H_{\mathbb{C}} \to \mathbb{C}$ is a character which does not fall into either of the two cases of the previous point. Then χ extends to a $\mathbb{Z}$-integral complex character of $\widetilde{H}_{\mathbb{C}}$, and its reduction modulo p is supersingular if χ is not special or trivial.*

Proof. The claims regarding integrality in (i) and (iii) are immediate.

(i) This follows from Lemma 4.2.6.

(ii) and (iii): Let $\chi_0 : H \to \mathbb{Z}$ denote the underlying $\mathbb{Z}$-integral structure of χ. If we are not in one of the two exceptional cases, the result follows from Lemmas 4.2.6, 4.2.7 and 2.4.2(iii). Otherwise, the character χ_0 of H extends to a character χ_0' of $H' := \mathbb{Z}[\Lambda_{\ker \nu}] \otimes H$ that is trivial on $\Lambda_{\ker \nu}$. The tensor product $\chi_0' \otimes_{H'} \widetilde{H}$ is a right $\widetilde{H}$-module that is free of rank 2 (since the subgroup $\Lambda_{\ker \nu}$ of Ω has index $|\Psi| = 2$, by (2.2.3)). If $\chi' : H' \to \mathbb{C}$ denotes the base change of χ_0' to $\mathbb{C}$, then $\chi' \otimes_{H'_{\mathbb{C}}} \widetilde{H}_{\mathbb{C}}$ is simple and its restriction to $H_{\mathbb{C}}$ is equal to $\chi \oplus \overline{\chi}$. Note that the characters χ and $\overline{\chi}$ in (ii) are neither special nor trivial, since the χ_i are unequal by assumption and therefore have supersingular reduction modulo p. We conclude by Lemma 2.4.2(iii). $\square$

4.3. **Discrete simple modules with supersingular reduction.** We continue to assume $\mathbf{G}$ is absolutely simple and isotropic. Let $\mathfrak{p}$ denote the maximal ideal of $\mathbb{Z}[q^{1/2}] \subset \mathbb{C}$ with residue field $\mathbb{F}_p$. We now discuss discrete, $\mathbb{Z}[q^{1/2}]$-integral $\widetilde{H}_{\mathbb{C}}$-modules with supersingular reduction modulo $\mathfrak{p}$.

The following is the key proposition of this section.

Proposition 4.3.1. *Suppose the type of Σ is not equal to A_ℓ with equal parameters. Then there exists a right $\widetilde{H}_{\mathbb{C}}$-module $M_{\mathbb{C}}$ such that:*

- *$M_{\mathbb{C}}$ is simple and discrete as an $\widetilde{H}_{\mathbb{C}}$-module;*
- *$M_{\mathbb{C}}$ has a $\mathbb{Z}[q^{1/2}]$-integral structure M which is furthermore free over $\mathbb{Z}[q^{1/2}]$;*
- *M has supersingular reduction modulo $\mathfrak{p}$.*

The proposition will follow from Propositions 4.3.2, 4.3.3, and 4.3.4 below. We sketch the main ideas of the proof.

Consider first the special character $\chi : H_{\mathbb{C}} \to \mathbb{C}$. It is $\mathbb{Z}[q^{1/2}]$-integral, its reduction modulo $\mathfrak{p}$ is non-supersingular, and $\mathfrak{T}(\chi)$ is equal to the Steinberg representation of G^{sc} over $\mathbb{C}$, so that χ is discrete. Any discrete, non-special character of $H_{\mathbb{C}}$ is $\mathbb{Z}[q^{1/2}]$-integral (in fact, $\mathbb{Z}$-integral) and Lemma 4.2.6 implies that its reduction modulo $\mathfrak{p}$ is supersingular (since the trivial character of $H_{\mathbb{C}}$ is not discrete). Thus, we first attempt to find a discrete non-special character of $H_{\mathbb{C}}$; these have been classified by Borel in [Bor76, §5.8]. (Note that in [Bor76], the Iwahori subgroup is the pointwise stabilizer $\widetilde{Z}_0\mathfrak{B}$ of an alcove; recall again that if $\mathbf{G}$ is F-split or semisimple and simply connected we have $\widetilde{Z}_0 = Z_0$.) We describe these characters in Proposition 4.3.2, and use Proposition 4.2.9 to determine which of these characters extend to $\widetilde{H}_{\mathbb{C}}$.

When $m = 1$, there do not exist any discrete non-special characters of $H_{\mathbb{C}}$, and we use instead a reflection module of $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$ (see Proposition 4.3.3). It is free of rank $|S|$ over $\mathbb{Z}[q^{1/2}]$ and has supersingular reduction modulo $\mathfrak{p}$. When the type is A_ℓ, this module is non-discrete, which is why we must omit this type. (We also use reflection modules in Proposition 4.3.4 to handle certain groups of type B_3 for which Proposition 4.3.2 does not apply.)

We now proceed with the required propositions.

Proposition 4.3.2. *Suppose the type of Σ is B_ℓ ($\ell \geq 4$), C_ℓ ($\ell \geq 2$), F_4, G_2, A_1 with parameters $d_1 \neq d_2$, or B_3 with parameters $(d_1, d_2) \neq (1, 2)$. Then the algebra $\widetilde{H}_{\mathbb{C}}$ admits a discrete non-special simple right module $M_{\mathbb{C}}$, induced from or extending a character of $H_{\mathbb{C}}$, which is $\mathbb{Z}[q^{1/2}]$-integral. Moreover, the dimension of $M_{\mathbb{C}}$ is 1, unless $\Psi \neq 1$ and the type is*

- *C_2 with parameters $(1,1,1), (2,1,1)$, or $(3,2,2)$;*
- *C_3 with parameters $(1,1,1), (1,2,2)$, or $(2,3,3)$;*
- *C_4 with parameters $(1,2,2)$, or $(2,3,3)$;*
- *C_5 with parameters $(1,2,2)$.*

In these cases, $M_{\mathbb{C}}$ extends the $H_{\mathbb{C}}$-module $(-1,-1,q^d) \oplus (-1,q^d,-1)$ where $d := d_2 = d_3$, and thus the dimension of $M_{\mathbb{C}}$ is 2.

Proof. When $m = 1$, the only discrete character of $H_{\mathbb{C}}$ is the special one ([Bor76, §5.7]).

Suppose $m > 1$. For each choice of irreducible root system Σ, we list in Tables 1 and 2 the possible parameters (d_1, d_2) or (d_1, d_2, d_3) for G (from the tables in [Tit79, §4]), and describe if $H_{\mathbb{C}}$ has a discrete non-special character (using [Bor76, §5.8]).

We start with $m = 2$ in Table 1. For every entry marked "Y," the given discrete non-special character extends to a character of $\widetilde{H}_{\mathbb{C}}$ using the condition of Lemma 4.2.7.

TABLE 1. $m = 2$

Σ	Parameters	$\exists$ discrete non-special character of $H_{\mathbb{C}}$?
A_1	(d, d) $(d \geq 1)$	N
	$(1, 3)$	Y
	$(2, 3)$	Y
	$(1, 2)$	Y
	$(1, 4)$	Y
	$(3, 4)$	Y
B_ℓ $(\ell \geq 3)$	$(1, 1)$	Y
	$(1, 2)$	Y (if $\ell \geq 4$), N (if $\ell = 3$)
	$(2, 1)$	Y
	$(2, 3)$	Y
F_4	$(1, 1)$	Y
	$(1, 2)$	Y
	$(2, 1)$	Y
G_2	$(1, 1)$	Y
	$(1, 3)$	Y
	$(3, 1)$	Y

We now consider $m = 3$ (that is, type C_ℓ) in Table 2. In this case, the tables in [Bor76, §5.8] show that $H_{\mathbb{C}}$ always admits a discrete, non-special character. Note also that Borel omitted the parameters $(3, 2, 2)$ for type C_2. In order to obtain this missing case, we use the criterion of [Bor76, Eqn. 5.6(2)] to see that the only discrete non-special characters of $H_{\mathbb{C}}$ are $(-1, -1, 1)$ and $(-1, 1, -1)$ (in the notation of [Bor76]). Note that the characters corresponding to parameters with $d_2 \neq d_3$ automatically extend to $\widetilde{H}_{\mathbb{C}}$, by Lemma 4.2.7.

(We have one more remark about the tables in [Bor76, §5.8]: the character $(-1, -1, 1)$ for parameters $(2, 1, 4)$ only works for $\ell \geq 3$.)

Finally, we remark that in all cases, Propositions 4.1.10 and 4.2.9 imply that the $\widetilde{H}_{\mathbb{C}}$-module $M_{\mathbb{C}}$ constructed above (either as the extension of a character of $H_{\mathbb{C}}$, or as the induction of a character from $H_{\mathbb{C}}$ to $\widetilde{H}_{\mathbb{C}}$) is discrete and $\mathbb{Z}[q^{1/2}]$-integral. $\square$

We consider now the types D_ℓ $(\ell \geq 4)$, E_6, E_7, and E_8. The tables in [Tit79, §4] imply that $\mathbf{G}$ is F-split, so that $d_s = 1$ for all $s \in S$, and for distinct $s, t \in S$, the order $n_{s,t}$ of st is 2 or 3.

TABLE 2. $m = 3$

Σ	Parameters	Condition that some discrete non-special character of $H_{\mathbb{C}}$ extends to $\widetilde{H}_{\mathbb{C}}$
C_ℓ ($\ell \geq 2$)	$(1,1,1)$	$\ell \geq 4$, or $\Psi = 1$
	$(2,1,1)$	$\ell \geq 3$, or $\Psi = 1$
	$(2,3,3)$	$\ell = 2, \ell \geq 5$, or $\Psi = 1$
	$(2,1,3)$	none
	$(1,1,2)$	none
	$(2,2,3)$	none
	$(2,1,2)$	none
	$(1,2,2)$	$\ell = 2, \ell \geq 6$, or $\Psi = 1$
	$(2,1,4)$	none
	$(2,3,4)$	none
C_2	$(3,2,2)$	$\Psi = 1$

Proposition 4.3.3. *Assume that the type of Σ is D_ℓ ($\ell \geq 4$), E_6, E_7, or E_8. Let M denote the right $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module obtained as the twist of the (left) reflection $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module by the anti-automorphism $T_w \mapsto (-1)^{\ell(w)} T^*_{w^{-1}}$. Then M is free of rank $|S|$ over $\mathbb{Z}[q^{1/2}]$, has supersingular reduction modulo $\mathfrak{p}$, and $M_{\mathbb{C}}$ is a discrete simple right $\widetilde{H}_{\mathbb{C}}$-module.*

Proof. The left reflection $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module is the free $\mathbb{Z}[q^{1/2}]$-module with basis $\{e_t\}_{t\in S}$, with $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module structure given by

$$T_s \cdot e_t = \begin{cases} -e_t & \text{if } s = t, \\ qe_t & \text{if } s \neq t,\ n_{s,t} = 2, \\ qe_t + q^{1/2} e_s & \text{if } s \neq t,\ n_{s,t} = 3, \end{cases}$$
$$T_u \cdot e_t = e_{u(t)},$$

where $s, t \in S, u \in \Omega$. Twisting this module by the automorphism $T_w \mapsto (-1)^{\ell(w)} T^*_w$ gives a left $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module M', satisfying

$$T_s \cdot e_t = \begin{cases} qe_t & \text{if } s = t, \\ -e_t & \text{if } s \neq t,\ n_{s,t} = 2, \\ -e_t - q^{1/2} e_s & \text{if } s \neq t,\ n_{s,t} = 3, \end{cases}$$
$$T_u \cdot e_t = e_{u(t)}.$$

Finally, we define M to be the right $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module obtained from M' by applying the anti-automorphism $T_w \mapsto T_{w^{-1}}$. The $\widetilde{H}_{\mathbb{C}}$-module $M_{\mathbb{C}}$ is simple (even as an $H_{\mathbb{C}}$-module, cf. [Lus83, §3.13]).

By applying Lemma 2.4.2(iii) and Proposition 4.1.10 twice, we may assume that $\mathbf{G}$ is adjoint in order to prove the required properties of M. The reduction modulo $\mathfrak{p}$ of M is the $\mathbb{F}_p$-vector space with basis $\{e_t\}_{t\in S}$, with the structure of a right $\widetilde{H}_{\mathbb{F}_p}$-module given by

$$e_t \cdot T_s = \begin{cases} 0 & \text{if } s = t, \\ -e_t & \text{if } s \neq t, \end{cases}$$
$$e_t \cdot T_u = e_{u^{-1}(t)}.$$

The restriction to $H_{\mathbb{F}_p}$ of this $\widetilde{H}_{\mathbb{F}_p}$-module is the direct sum of the characters $\{\chi_s\}_{s\in S}$, where

$$\chi_s(T_t) = \begin{cases} 0 & \text{if } s = t, \\ -1 & \text{if } s \neq t. \end{cases}$$

By Lemmas 2.4.2(iii) and 4.2.6, we deduce that $M_{\mathbb{F}_p}$ is supersingular. Further, one checks that the right action of $\theta_{\lambda_\mu^{-1}}$ ($\mu \in X_*(T)^+$) on $M_{\mathbb{C}}$ is equal to the left action of $(-1)^{\ell(\lambda_\mu)}\hat{T}^{-1}_{\lambda_\mu^{-1}}$ on (the base change to $\mathbb{C}$ of) the reflection module, where $\hat{T}_{\lambda_\mu^{-1}}$ is defined in [Lus83, §4.3] (note that with respect to our normalizations, the elements ω_i of *op. cit.* are anti-dominant). The discreteness of $M_{\mathbb{C}}$ now follows from Proposition 4.1.6 and [Lus83, §3.2, Thm. 4.7]. (See also [Lus83, §4.23].) □

Finally, we consider one of the omitted cases from Proposition 4.3.2, namely type B_3 with parameters $(1, 2)$.

Proposition 4.3.4. *Assume that the type of Σ is B_3 with parameters $(1, 2)$. Then $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$ admits a right module M, such that M is free of rank 3 over $\mathbb{Z}[q^{1/2}]$, has supersingular reduction modulo $\mathfrak{p}$, and $M_{\mathbb{C}}$ is a discrete simple right $\widetilde{H}_{\mathbb{C}}$-module.*

Proof. In this case, the group $\mathbf{G}^{\mathrm{sc}}$ is an unramified non-split form of $\mathbf{Spin}_8$, by the tables in [Tit79]. We will use the reflection module as defined in [GS05, §7].

Denote by $\widetilde{\Delta}_{\mathrm{long}}$ the subset of simple affine roots $\widetilde{\Delta}$ which are long. We define an action of $H_{\mathbb{Z}[q^{1/2}]}$ on the free $\mathbb{Z}[q^{1/2}]$-module of rank 3 with basis $\{e_\beta\}_{\beta\in\widetilde{\Delta}_{\mathrm{long}}}$ as follows. If $\alpha \in \widetilde{\Delta}_{\mathrm{long}}$, we set

$$T_{s_\alpha} \cdot e_\beta = \begin{cases} -e_\beta & \text{if } \alpha = \beta, \\ qe_\beta & \text{if } \alpha \neq \beta,\ n_{s_\alpha,s_\beta} = 2, \\ qe_\beta + q^{1/2}e_\alpha & \text{if } \alpha \neq \beta,\ n_{s_\alpha,s_\beta} = 3, \end{cases}$$

and if α is the unique short root in $\widetilde{\Delta}$, we set

$$T_{s_\alpha} \cdot e_\beta = q^2 e_\beta.$$

Twisting this reflection module by the automorphism $T_w \mapsto (-1)^{\ell(w)}T_w^*$ gives a new left $H_{\mathbb{Z}[q^{1/2}]}$-module M', with action given by

$$T_{s_\alpha} \cdot e_\beta = \begin{cases} qe_\beta & \text{if } \alpha = \beta, \\ -e_\beta & \text{if } \alpha \neq \beta,\ n_{s_\alpha,s_\beta} = 2, \\ -e_\beta - q^{1/2}e_\alpha & \text{if } \alpha \neq \beta,\ n_{s_\alpha,s_\beta} = 3, \end{cases}$$

if $\alpha \in \widetilde{\Delta}_{\mathrm{long}}$, and

$$T_{s_\alpha} \cdot e_\beta = -e_\beta$$

if $\alpha \in \widetilde{\Delta}$ is short. We extend the action of $H_{\mathbb{Z}[q^{1/2}]}$ on M' to $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$ by declaring that

$$T_u \cdot e_\alpha = e_{u(\alpha)}.$$

As the algebra $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$ is generated by $H_{\mathbb{Z}[q^{1/2}]}$ and the elements T_u, $u \in \Omega$, subject to the relations $T_{uv} = T_u T_v$ and $T_u T_{s_\alpha} T_u^{-1} = T_{s_{u(\alpha)}}$ for $u, v \in \Omega$ and $\alpha \in \widetilde{\Delta}$, we see that M' is a well-defined module of $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$. Finally, we define M to be the right $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module obtained from M' by applying the anti-automorphism $T_w \mapsto T_{w^{-1}}$. One checks directly that $M_{\mathbb{C}}$ is simple (even as an $H_{\mathbb{C}}$-module).

By Lemma 2.4.2(iii) and Proposition 4.1.10 we are now reduced to the case where $\mathbf{G}$ is simply connected. The reduction modulo $\mathfrak{p}$ of M is the $\mathbb{F}_p$-vector space with basis $\{e_\beta\}_{\beta \in \widetilde{\Delta}_{\mathrm{long}}}$, with the structure of a right $H_{\mathbb{F}_p}$-module given by

$$e_\beta \cdot T_{s_\alpha} = \begin{cases} 0 & \text{if } \alpha = \beta, \\ -e_\beta & \text{if } \alpha \neq \beta, \end{cases}$$

for $\alpha \in \widetilde{\Delta}$. Therefore $M_{\mathbb{F}_p}$ is equal to the direct sum of the characters $\{\chi_\beta\}_{\beta \in \widetilde{\Delta}_{\mathrm{long}}}$, where

$$\chi_\beta(T_{s_\alpha}) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ -1 & \text{if } \alpha \neq \beta. \end{cases}$$

for $\alpha \in \widetilde{\Delta}$. Lemma 4.2.6 therefore implies that $M_{\mathbb{F}_p}$ is supersingular.

Once again, we see that the right action of $\theta_{\lambda_\mu^{-1}}$ ($\mu \in X_*(T)^+$) on $M_{\mathbb{C}}$ is equal to the left action of $(-1)^{\ell(\lambda_\mu)} q_{\lambda_\mu}^{1/2} T_{\lambda_\mu^{-1}}^{-1}$ on (the base change to $\mathbb{C}$ of) the reflection module. Section 8.5 of [GS05] gives an explicit description of Hecke operators associated to the fundamental anti-dominant coweights in terms of T_u and the T_{s_α}. Using this description along with Proposition 4.1.6, we see that the $\widetilde{H}_{\mathbb{C}}$-module $M_{\mathbb{C}}$ is discrete. (See also [GS05, Prop. 7.11].) □

4.4. **Admissible integral structure via discrete cocompact subgroups.** Let E be a number field with ring of integers $\mathcal{O}_E$, $\mathfrak{p}$ a maximal ideal of $\mathcal{O}_E$ with residue field $k := \mathcal{O}_E/\mathfrak{p}$, and C/E a field extension.

For any extension of fields, the scalar extension functor commutes with the $\mathfrak{B}$-invariant functor and its left adjoint $\mathfrak{T}$ ([HV19, Lem. III.1(ii)]). Therefore, if τ is an E-structure of a C-representation π, then $\tau^{\mathfrak{B}}$ is an E-structure of $\pi^{\mathfrak{B}}$. Conversely, if M is an E-structure of an $\widetilde{H}_C$-module N, then $\mathfrak{T}(M)$ is an E-structure of $\mathfrak{T}(N)$.

Definition 4.4.1. We say that an admissible C-representation π of G is *$\mathcal{O}_E$-integral* if π contains a G-stable $\mathcal{O}_E$-submodule τ° such that, for any compact open subgroup K of G, the $\mathcal{O}_E$-module $(\tau^\circ)^K$ is finitely generated, and the natural map

$$\varphi : C \otimes_{\mathcal{O}_E} \tau^\circ \to \pi$$

is an isomorphism. We call φ (and more often τ°) an *$\mathcal{O}_E$-integral structure* of π. The G-equivariant map $\tau^\circ \to k \otimes_{\mathcal{O}_E} \tau^\circ$ (and more often the k-representation $k \otimes_{\mathcal{O}_E} \tau^\circ$ of G) is called the *reduction of τ° modulo* $\mathfrak{p}$. We say that τ° is *admissible* if $k \otimes_{\mathcal{O}_E} \tau^\circ$ is admissible for all $\mathfrak{p}$.

For any commutative ring R and any discrete cocompact subgroup Γ of G, we define

$$C^\infty(\Gamma\backslash G, R) := \left\{ f : G \to R \;\middle|\; f(\gamma g k) = f(g) \quad \begin{array}{l} \text{for all } \gamma \in \Gamma, g \in G, \\ \text{and } k \in K_f \end{array} \right\},$$

where K_f is some compact open subgroup of G depending on f. Letting G act on this space by right translation, we obtain a smooth R-representation ρ^Γ_R. The complex representation $\rho^\Gamma_{\mathbb{C}}$ of G has an admissible $\mathcal{O}_E$-integral structure given by $\rho^\Gamma_{\mathcal{O}_E}$: the reduction of $\rho^\Gamma_{\mathcal{O}_E}$ modulo $\mathfrak{p}$ is the admissible representation ρ^Γ_k.

Proposition 4.4.2. *Assume* $\operatorname{char} F = 0$ *and* $\mathbf{G}$ *semisimple. If* π *is a square-integrable* $\mathbb{C}$*-representation of* G*, then there exists a discrete cocompact subgroup* Γ *of* G *such that*

$$\operatorname{Hom}_G(\pi, \rho^\Gamma_{\mathbb{C}}) \neq 0.$$

Proof. Since $\operatorname{char} F = 0$, there exists a decreasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of discrete cocompact subgroups of G with trivial intersection, such that each is normal and of finite index in Γ_0. (See [BH78, Thm. A]. The construction there is global, and we obtain the required decreasing sequence by passing to congruence subgroups.) For any discrete cocompact subgroup Γ, the normalized multiplicity of π in $\rho^\Gamma_{\mathbb{C}}$ is

$$m_{\Gamma,dg}(\pi) := \operatorname{vol}_\Gamma \cdot \dim_{\mathbb{C}} \left(\operatorname{Hom}_G(\pi, \rho^\Gamma_{\mathbb{C}}) \right),$$

where $\operatorname{vol}_\Gamma$ is the volume of $\Gamma\backslash G$ for a G-invariant measure induced by a fixed Haar measure on G. By the square-integrability assumption on π and the limit multiplicity formula, the sequence $(m_{\Gamma_n,dg}(\pi))_{n\in\mathbb{N}}$ converges to a nonzero real number (see [DKV84, App. 3, Prop.] and [Kaz86, Thm. K]). $\square$

Proposition 4.4.3. *Assume* $\operatorname{char} F = 0$*. Let* π *be an irreducible* $\mathbb{C}$*-representation of* G *such that* $\pi^{\mathfrak{B}} \neq 0$ *and* Γ *a discrete cocompact subgroup of* G*.*

(i) *If* $\varphi : \mathbb{C} \otimes_E \tau \xrightarrow{\sim} \pi$ *is an* E*-structure of* π*, then the natural map*

$$\mathbb{C} \otimes_E \operatorname{Hom}_{E[G]}(\tau, \rho^\Gamma_E) \to \operatorname{Hom}_{\mathbb{C}[G]}(\pi, \rho^\Gamma_{\mathbb{C}})$$

is an isomorphism.

(ii) *Any irreducible subrepresentation* τ *of* ρ^Γ_E *admits an admissible* $\mathcal{O}_E$*-integral structure* $\tau \cap \rho^\Gamma_{\mathcal{O}_E}$*, whose reduction modulo* $\mathfrak{p}$ *contained in* ρ^Γ_k*.*

Proof. We recall a general result in algebra from [Bou12, §12.2 Lem. 1]: let C'/C be a field extension and A a C-algebra. For A-modules M, N, the natural map

$$C' \otimes_C \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{C'\otimes_C A}(C' \otimes_C M, C' \otimes_C N) \tag{4.4.4}$$

is injective, and bijective if C'/C is finite or the A-module M is finitely generated.

(i) Take $C'/C = \mathbb{C}/E$, $A = E[G]$, $(M, N) = (\tau, \rho^\Gamma_E)$. Then (4.4.4) is an isomorphism because τ is an irreducible E-representation of G.

(ii) For any compact open subgroup K of G, the $\mathcal{O}_E$-module $(\rho^\Gamma_{\mathcal{O}_E})^K$ is finite free and $\rho^\Gamma_{\mathcal{O}_E}$ contains $\tau^\circ := \tau \cap \rho^\Gamma_{\mathcal{O}_E}$ as $\mathcal{O}_E$-representations of G. Since the ring $\mathcal{O}_E$ is noetherian, these facts imply the $\mathcal{O}_E$-submodule $(\tau^\circ)^K$ of $(\rho^\Gamma_{\mathcal{O}_E})^K$ is finitely generated. The natural linear G-equivariant isomorphism

$$E \otimes_{\mathcal{O}_E} \rho^\Gamma_{\mathcal{O}_E} \xrightarrow{\sim} \rho^\Gamma_E$$

restricts to a linear G-equivariant isomorphism

$$E \otimes_{\mathcal{O}_E} \tau^\circ \xrightarrow{\sim} \tau,$$

and therefore τ° is an $\mathcal{O}_E$-integral structure of τ. It remains to verify that the injection $\tau^\circ \hookrightarrow \rho^\Gamma_{\mathcal{O}_E}$ stays injective after reduction modulo $\mathfrak{p}$. (As ρ^Γ_k is admissible, this will also imply that $k \otimes_{\mathcal{O}_E} \tau^\circ$ is admissible.) More generally, suppose that $0 \to M' \to M \to M'' \to 0$ is any exact sequence of $\mathcal{O}_E$-modules with M'' torsion-free. Then M'' is the direct limit of its finitely generated submodules, and finitely generated torsion-free modules are projective, as $\mathcal{O}_E$ is Dedekind. Hence $\mathrm{Tor}_1^{\mathcal{O}_E}(M'', k) = 0$, as Tor functors commute with direct limits, so the sequence stays exact after reduction modulo $\mathfrak{p}$. □

The above result will be used in our construction of irreducible, admissible, supersingular C-representations. It also has the following consequence, which may be of independent interest.

Corollary 4.4.5. *Assume* $\mathrm{char}\, F = 0$ *and* $\mathbf{G}$ *semisimple. Then any irreducible supercuspidal* $\mathbb{C}$*-representation admits an admissible* $\mathcal{O}_E$*-integral structure whose reduction modulo* $\mathfrak{p}$ *is contained in* ρ^Γ_k*, for some discrete cocompact subgroup* Γ *of* G*.*

Proof. When $\mathbf{G}$ is semisimple, any irreducible admissible supercuspidal $\mathbb{C}$-representation π of G descends to a number field (see [Vig96, II.4.9]). Since π is in particular square-integrable, Proposition 4.4.2 implies that π embeds into $\rho^\Gamma_{\mathbb{C}}$ for some discrete cocompact subgroup Γ of G. The claim the follows from points (i) and (ii) of Proposition 4.4.3. □

4.5. Reduction to rank 1 and $\mathrm{PGL}_n(D)$.

We now prove that most p-adic reductive groups admit irreducible admissible supersingular representations.

Theorem 4.5.1. *Assume that* $c = p$ *and* $\mathrm{char}\, F = 0$*. Suppose* $\mathbf{G}$ *is an isotropic, absolutely simple, connected adjoint* F*-group, not isomorphic to any of the following groups:*

(i) $\mathbf{PGL}_n(D)$*, where* $n \geq 2$ *and* D *a central division algebra over* F*;*
(ii) $\mathbf{PU}(h)$*, where* h *is a split hermitian form in* 3 *variables over a ramified quadratic extension of* F *or a non-split hermitian form in* 4 *variables over the unramified quadratic extension of* F*.*

Then G *admits an irreducible admissible supercuspidal* C*-representation.*

Proof. We first note by the tables in [Tit79] that the above exceptional groups are precisely the ones where Σ is of type A_ℓ with equal parameters. (In that reference our exceptional groups have names A_{m-1}, ${}^d\mathrm{A}_{md-1}$ for $m \geq 2$, $d \geq 2$ in case (i) and C-BC$_1$, ${}^2\mathrm{A}_3''$ in case (ii).)

By Proposition 4.3.1 there exists a right $\widetilde{H}_{\mathbb{Z}[q^{1/2}]}$-module M which is free over $\mathbb{Z}[q^{1/2}]$, whose base change $M_{\mathbb{C}}$ is a discrete simple $\widetilde{H}_{\mathbb{C}}$-module, and whose reduction $M_{\mathbb{F}_p}$ is supersingular. Set $E := \mathbb{Q}(q^{1/2})$, so that $\mathbb{Z}[q^{1/2}] \subset \mathcal{O}_E$. Let $\pi := \mathfrak{T}(M_{\mathbb{C}})$ denote the irreducible square-integrable $\mathbb{C}$-representation of G corresponding to $M_{\mathbb{C}}$; then $\tau := \mathfrak{T}(M_E)$ is a E-structure of π. We know by Proposition 4.4.2 that π injects into $\rho^\Gamma_{\mathbb{C}}$ for some discrete cocompact subgroup Γ of G, and therefore τ injects into ρ^Γ_E by Proposition 4.4.3(i). We identify π and τ with their images in $\rho^\Gamma_{\mathbb{C}}$ and ρ^Γ_E, respectively. Proposition 4.4.3(ii) then ensures that $\tau^\circ := \tau \cap \rho^\Gamma_{\mathcal{O}_E}$ is an admissible $\mathcal{O}_E$-integral structure of τ. In particular, we have a G-equivariant map $\mathbb{C} \otimes_{\mathcal{O}_E} \tau^\circ \xrightarrow{\sim} \pi$.

Define $M' := (\tau^\circ)^{\mathfrak{B}}$; since E is a localization of $\mathcal{O}_E$, the isomorphism above implies $\mathbb{C} \otimes_{\mathcal{O}_E} M' \xrightarrow{\sim} \pi^{\mathfrak{B}} \cong M_{\mathbb{C}}$, so that M' is an $\mathcal{O}_E$-integral structure of $M_{\mathbb{C}}$. The module M' is a finitely generated, torsion-free module over the Dedekind domain $\mathcal{O}_E$, so we can write

$$M' \cong I_1 \oplus \ldots \oplus I_r$$

as $\mathcal{O}_E$-modules, where the I_i are non-zero ideals of $\mathcal{O}_E$. Let E' denote any finite extension of E such that every ideal of $\mathcal{O}_E$ becomes principal in $\mathcal{O}_{E'}$. We identify $M'_{\mathcal{O}_{E'}} = \mathcal{O}_{E'} \otimes_{\mathcal{O}_E} M'$ with a submodule of $(\rho^{\Gamma}_{\mathcal{O}_{E'}})^{\mathfrak{B}}$. We have

$$M'_{\mathcal{O}_{E'}} \cong I_1\mathcal{O}_{E'} \oplus \ldots \oplus I_r\mathcal{O}_{E'},$$

and since each $I_i\mathcal{O}_{E'}$ is a principal ideal of $\mathcal{O}_{E'}$, we see that $M'_{\mathcal{O}_{E'}}$ is free as an $\mathcal{O}_{E'}$-module.

Both $M_{\mathcal{O}_{E'}}$ and $M'_{\mathcal{O}_{E'}}$ are $\widetilde{H}_{\mathcal{O}_{E'}}$-modules which are free over $\mathcal{O}_{E'}$ and isomorphic over $\mathbb{C}$. Thus, if we let $\mathfrak{p} \subset \mathcal{O}_{E'}$ denote any prime ideal lying over p, we see that the reductions $M_{\mathcal{O}_{E'}/\mathfrak{p}}$ and $M'_{\mathcal{O}_{E'}/\mathfrak{p}}$ agree up to semisimplification by the Brauer–Nesbitt theorem. In particular, $M'_{\mathcal{O}_{E'}/\mathfrak{p}}$ is supersingular (since the same is true of $M_{\mathcal{O}_{E'}/\mathfrak{p}}$) and, by construction, $M'_{\mathcal{O}_{E'}/\mathfrak{p}}$ is a submodule of $(\rho^{\Gamma}_{\mathcal{O}_{E'}/\mathfrak{p}})^{\mathfrak{B}}$ (this uses the final claim of Proposition 4.4.3 (ii)). Therefore we can pick a non-zero supersingular element v of $(\rho^{\Gamma}_{\mathcal{O}_{E'}/\mathfrak{p}})^{\mathfrak{B}}$. The G-representation $\rho^{\Gamma}_{\mathcal{O}_{E'}/\mathfrak{p}}$ is admissible, as Γ is cocompact, and hence so is its subrepresentation $\langle G\cdot v\rangle$ generated by v. Any irreducible quotient of $\langle G\cdot v\rangle$ (which exists by Zorn's lemma) is admissible by [Hen09, §4, Thm. 1], as F is of characteristic zero, and supersingular by Proposition 3.1.3, as it contains (the nonzero image of) v. The theorem now follows from Proposition 3.2.1. □

The two exceptional cases will be dealt with in Sections 5, 6 below. Assuming this, we can now prove our main result.

Proof of Theorem A. Suppose that $\mathbf{G}$ is a connected reductive group over F. We want to show that $G = \mathbf{G}(F)$ admits an irreducible admissible supercuspidal representation over any field C of characteristic p. By Proposition 3.2.1 we may assume that C is finite and as large as we like. Then by Proposition 3.3.2 we may assume that $\mathbf{G}$ is isotropic, absolutely simple, and connected adjoint. The result then follows from Theorem 4.5.1, Corollary 5.5.2, and Corollary 6.6.2. □

5. Supersingular representations of rank 1 groups

In this section we verify Theorem A when $\mathbf{G}$ is a connected reductive F-group of relative semisimple rank 1. In particular, this deals with the second exceptional case in Theorem 4.5.1.

5.1. **Preliminaries.** We suppose in this section that C is a finite extension of $\mathbb{F}_p$ which contains the $|G|_{p'}$-th roots of unity, where $|G|_{p'}$ denotes the prime-to-p part of the pro-order of G.

Suppose that $\operatorname{char} F = 0$. We will show that G admits irreducible, admissible, supercuspidal C-representations. By Proposition 3.3.2, it suffices to assume $\mathbf{G}$ is an absolutely simple and adjoint group of relative rank 1. We make one further reduction. Let $\mathbf{G}^{\mathrm{sc}}$ denote the simply-connected cover of $\mathbf{G}$:

$$1 \to \mathbf{Z}(\mathbf{G}^{\mathrm{sc}}) \to \mathbf{G}^{\mathrm{sc}} \to \mathbf{G} \to 1.$$

By Proposition 3.3.8, we see that G^{sc} admits an irreducible, admissible, supercuspidal representation on which $Z(G^{\mathrm{sc}})$ acts trivially if and only if G does. Therefore, we may assume that our group $\mathbf{G}$ is absolutely simple, simply connected, and has relative rank equal to 1. We will then construct irreducible, admissible, supercuspidal representations of G on which its (finite) center acts trivially.

5.2. **Parahoric subgroups.** Let $\mathscr{B}$ denote the adjoint Bruhat–Tits building of G. By our assumptions on $\mathbf{G}$, $\mathscr{B}$ is a one-dimensional contractible simplicial complex, i.e., a tree. Recall that $\mathcal{C}$ denotes the chamber of $\mathscr{B}$ corresponding to the Iwahori subgroup $\mathfrak{B}$, and let x_0 and x_1 denote the two vertices in the closure of $\mathcal{C}$. We let K_0 and K_1 denote the pointwise stabilizers of x_0 and x_1, respectively. We then have $\mathfrak{B} = K_0 \cap K_1$.

The vertices x_0 and x_1 are representatives of the two orbits of G on the set of vertices of $\mathscr{B}$, and the edge $\mathcal{C}$ is a representative of the unique orbit of G on the edges of $\mathscr{B}$. By [Ser03, §4, Thm. 6], we may therefore write the group G as an amalgamated product:

$$G \cong K_0 *_{\mathfrak{B}} K_1.$$

Since the group $\mathbf{G}$ is semisimple and simply connected, the stabilizers of vertices and edges in $\mathscr{B}$ are parahoric subgroups (see, e.g., [Vig16, §3.7]). For $i \in \{0,1\}$, we let K_i^+ denote the pro-p radical of K_i, that is, the largest open, normal, pro-p subgroup of K_i. The quotient $\mathbb{G}_i := K_i/K_i^+$ is isomorphic the group of k_F-points of a connected reductive group over k_F (see [HV15, §3.7]). Likewise, the pro-p-Sylow $\mathfrak{U}$ is the largest open, normal, pro-p subgroup of $\mathfrak{B}$, and $\mathbb{Z} := \mathfrak{B}/\mathfrak{U}$ is isomorphic to the group of k_F-points of a torus over k_F. The image of $\mathfrak{B}$ in $\mathbb{G}_i$ is equal to a minimal parabolic subgroup $\mathbb{B}_i$, with Levi decomposition $\mathbb{B}_i = \mathbb{Z}_i\mathbb{U}_i$. Thus, we identify the quotient $\mathbb{Z}$ with $\mathbb{Z}_i$.

5.3. **Pro-p Iwahori–Hecke algebras.** We work in slightly greater generality than in §2. Let

$$\mathcal{H}_C := H_C(G, \mathfrak{U}) = \mathrm{End}_G\, C[\mathfrak{U}\backslash G]$$

denote the pro-p Iwahori–Hecke algebra of G with respect to $\mathfrak{U}$. We view $\mathcal{H}_C$ as the convolution algebra of C-valued, compactly supported, $\mathfrak{U}$-bi-invariant functions on G (see [Vig16, §4] for more details). For $g \in G$, we let T_g denote the characteristic function of $\mathfrak{U}g\mathfrak{U}$. The algebra $\mathcal{H}_C$ is generated by two operators $T_{\tilde{s}_0}, T_{\tilde{s}_1}$, where $\tilde{s}_0$ and $\tilde{s}_1$ are lifts to the pro-p Iwahori–Weyl group $\mathcal{N}/(Z \cap \mathfrak{U})$ of affine reflections s_0, s_1 fixing x_0, x_1, respectively, along with operators T_z for $z \in \mathbb{Z}$. (Note that this labeling is different than the labeling in §4.2.) For $i \in \{0,1\}$, we let $\mathcal{H}_{C,i}$ denote the subalgebra of $\mathcal{H}_C$ generated by $T_{\tilde{s}_i}$ and T_z for $z \in \mathbb{Z}$; this is exactly the subalgebra of functions in $\mathcal{H}_C$ with support in K_i, i.e.,

$$\mathcal{H}_{C,i} = H_C(K_i, \mathfrak{U}) = \mathrm{End}_{K_i}\, C[\mathfrak{U}\backslash K_i].$$

The algebra $\mathcal{H}_{C,i}$ is canonically isomorphic to the finite Hecke algebra $H_C(\mathbb{G}_i, \mathbb{U}_i)$ (see [CE04, §6.1]).

Since K_i^+ is an open normal pro-p subgroup of K_i, the irreducible smooth C-representations of K_i and $\mathbb{G}_i$ are in bijection. Further, the finite group $\mathbb{G}_i$ possesses a strongly split BN pair of characteristic p ([Vig16, Prop. 3.25]). Therefore, by [CE04, Thm. 6.12], the functor $\rho \mapsto \rho^{\mathfrak{U}}$ induces a bijection between isomorphism classes of irreducible smooth C-representations of K_i and isomorphism classes of simple right $\mathcal{H}_{C,i}$-modules, all of which are one-dimensional.

We briefly recall some facts about supersingular $\mathcal{H}_C$-modules (compare Lemma 4.2.6). We refer to [Vig17, Def. 6.10] for the precise definition (which is analogous to Definition 2.4.1) and give instead the classification of simple supersingular $\mathcal{H}_C$-modules. Since $\mathbf{G}$ is simply connected, every supersingular $\mathcal{H}_C$-module is a character. The characters Ξ of $\mathcal{H}_C$ are parametrized by pairs (χ, J), where $\chi : \mathbb{Z} \to C^\times$ is a character of the finite torus and J is a subset of

$$S_\chi := \big\{ s \in \{s_0, s_1\} : \chi(c_{\tilde{s}}) \neq 0 \big\}$$

(here $c_{\tilde{s}}$ is a certain element of $C[\mathbb{Z}]$ which appears in the quadratic relation for $T_{\tilde{s}}$; note also that the definition of S_χ is independent of the choice of lift $\tilde{s} \in \mathcal{N}/(Z \cap \mathfrak{U})$ of s). The correspondence is given as follows (cf. [Vig17, Thm. 1.6]): for $z \in \mathbb{Z}$, we have $\Xi(T_z) = \chi(z)$, and for $s \in \{s_0, s_1\}$, we have

$$\Xi(T_{\tilde{s}}) = \begin{cases} 0 & \text{if } s \in J, \\ \chi(c_{\tilde{s}}) & \text{if } s \notin J. \end{cases}$$

Since $\mathbf{G}$ is simple, [Vig17, Thm. 1.6] implies that Ξ is supersingular if and only if

$$(S_\chi, J) \neq (\{s_0, s_1\}, \emptyset),\ (\{s_0, s_1\},\ \{s_0, s_1\}).$$

5.4. **Diagrams.** Since the group G is an amalgamated product of two parahoric subgroups, the formalism of diagrams used in [KX15] applies to the group G. We recall that a diagram D is a quintuple $(\rho_0, \rho_1, \sigma, \iota_0, \iota_1)$ which consists of smooth C-representations ρ_i of K_i ($i \in \{0, 1\}$), a smooth C-representation σ of $\mathfrak{B}$, and $\mathfrak{B}$-equivariant morphisms $\iota_i : \sigma \to \rho_i|_{\mathfrak{B}}$. We depict diagrams as

$$\begin{tikzcd} & \rho_0 \\ \sigma \arrow[ur, "\iota_0"] \arrow[dr, "\iota_1"'] & \\ & \rho_1 \end{tikzcd}$$

Morphisms of diagrams are defined in the obvious way (i.e., so that the relevant squares commute).

Let Ξ denote a supersingular character of $\mathcal{H}_C$, associated to a pair (χ, J). We define a diagram D_Ξ as follows:

- set $\sigma := \chi^{-1}$, which we view as a character of $\mathfrak{B}$ by inflation;
- we let $\rho_{\Xi,i}$ denote an irreducible smooth C-representation of K_i such that $\rho_{\Xi,i}^{\mathfrak{U}} \cong \Xi|_{\mathcal{H}_{C,i}}$ as $\mathcal{H}_{C,i}$-modules (by the discussion above, $\rho_{\Xi,i}$ is unique up to isomorphism);
- let ι_i denote the $\mathfrak{B}$-equivariant map given by $\sigma = \chi^{-1} \xrightarrow{\sim} \rho_{\Xi,i}^{\mathfrak{U}} \hookrightarrow \rho_{\Xi,i}|_{\mathfrak{B}}$.

Pictorially, we write

$$D_\Xi \quad = \quad \left(\begin{array}{ccc} & & \rho_{\Xi,0} \\ & \overset{\iota_0}{\nearrow} & \\ \chi^{-1} & & \\ & \underset{\iota_1}{\searrow} & \\ & & \rho_{\Xi,1} \end{array} \right)$$

We now wish to construct an auxiliary diagram D' into which D_Ξ injects. This will be done with the use of injective envelopes. Recall that if $\mathcal{G}$ is a profinite group and τ is a smooth C-representation of $\mathcal{G}$, an injective envelope consists of a smooth injective C-representation $\mathrm{inj}_{\mathcal{G}}\tau$ of $\mathcal{G}$ along with a $\mathcal{G}$-equivariant injection $j : \tau \hookrightarrow \mathrm{inj}_{\mathcal{G}}\tau$ which satisfies the following property: for any nonzero C-subrepresentation $\tau' \subset \mathrm{inj}_{\mathcal{G}}\tau$, we have $j(\tau) \cap \tau' \neq 0$. This data exists and is unique up to (non-unique) isomorphism.

Lemma 5.4.1 ([Paš04, Lem. 6.13])**.** *Let τ denote a smooth C-representation of $\mathcal{G}$, and let $j : \tau \hookrightarrow \mathrm{inj}_{\mathcal{G}}\tau$ denote an injective envelope. Let $\mathfrak{I}$ denote an injective representation of $\mathcal{G}$, and suppose we have an injection $\phi : \tau \hookrightarrow \mathfrak{I}$. Then ϕ extends to an injection $\widetilde{\phi} : \mathrm{inj}_{\mathcal{G}}\tau \hookrightarrow \mathfrak{I}$ such that $\phi = \widetilde{\phi} \circ j$.*

Lemma 5.4.2. *Suppose $\mathcal{G}$ has an open, normal subgroup $\mathcal{G}^+$. Let τ denote a smooth C-representation of $\mathcal{G}$ such that $\mathcal{G}^+$ acts trivially, and let $j : \tau \hookrightarrow \mathrm{inj}_{\mathcal{G}}\tau$ denote an injective envelope of τ in the category of C-representations of $\mathcal{G}$. Then $\tau \hookrightarrow (\mathrm{inj}_{\mathcal{G}}\tau)^{\mathcal{G}^+}$ is an injective envelope of τ in the category of C-representations of $\mathcal{G}/\mathcal{G}^+$.*

Proof. This is [Paš04, Lem. 6.14]; its proof does not require that τ be irreducible or that $\mathcal{G}^+$ be pro-p, as we assume that $\mathcal{G}^+$ acts trivially. □

We now begin constructing D'.

Lemma 5.4.3. *Let $i \in \{0,1\}$. We then have*

$$(\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}} \cong \bigoplus_{\xi} \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus |\mathbb{B}_i \backslash \mathbb{G}_i|},$$

where ξ runs over all C-characters of $\mathfrak{B}$ (or, equivalently, of $\mathbb{Z}_i$), and we have fixed choices of injective envelopes.

Proof. Consider the $\mathfrak{B}$-representation $(\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{\mathfrak{U}}$. The action of $\mathfrak{B}$ factors through the quotient $\mathfrak{B}/\mathfrak{U} \cong \mathbb{Z}$, which is commutative of order coprime to p. Therefore, we obtain a $\mathfrak{B}$-equivariant isomorphism

$$(\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{\mathfrak{U}} \cong \bigoplus_{\xi} \xi^{\oplus m_\xi} \tag{5.4.4}$$

for non-negative integers m_ξ satisfying

$$\begin{aligned} m_\xi &= \dim_C \mathrm{Hom}_{\mathfrak{B}}(\xi, \mathrm{inj}_{K_i} C[\mathbb{G}_i]) \\ &= \dim_C \mathrm{Hom}_{\mathfrak{B}}\left(\xi, (\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{K_i^+}\right) \\ &= \dim_C \mathrm{Hom}_{\mathbb{B}_i}(\xi, \mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i]) \\ &= \dim_C \mathrm{Hom}_{\mathbb{Z}_i}\left(\xi, (\mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i])^{\mathbb{U}_i}\right). \end{aligned}$$

(The third equality follows from Lemma 5.4.2.) Since $C[\mathbb{G}_i]$ is injective as a representation of $\mathbb{G}_i$, we have isomorphisms of $\mathbb{Z}_i$-representations

$$(\mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i])^{\mathbb{U}_i} \cong C[\mathbb{U}_i\backslash\mathbb{G}_i] \cong \bigoplus_{\xi} \xi^{\oplus|\mathbb{B}_i\backslash\mathbb{G}_i|},$$

so that $m_\xi = |\mathbb{B}_i\backslash\mathbb{G}_i|$.

The isomorphism (5.4.4) implies we have a $\mathfrak{B}$-equivariant injection

$$\bigoplus_{\xi} \xi^{\oplus|\mathbb{B}_i\backslash\mathbb{G}_i|} \hookrightarrow (\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}}.$$

As $\mathfrak{B}$ is open, [Vig96, §I.5.9 d)] implies that the representation on the right-hand side is injective. Lemma 5.4.1 then says that the above morphism extends to a split injection between injective $\mathfrak{B}$-representations

$$\bigoplus_{\xi} \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus|\mathbb{B}_i\backslash\mathbb{G}_i|} \hookrightarrow (\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}}.$$

Since the $\mathfrak{U}$-invariants of both representations agree, the above injection must be an isomorphism. $\square$

Lemma 5.4.5. *Set* $a := \mathrm{lcm}(|\mathbb{B}_0\backslash\mathbb{G}_0|,\ |\mathbb{B}_1\backslash\mathbb{G}_1|)$. *There exists a diagram* D' *of the form*

$$D' = \left(\begin{tikzcd} & \mathrm{inj}_{K_0} C[\mathbb{G}_0]^{\oplus a\cdot|\mathbb{B}_0\backslash\mathbb{G}_0|^{-1}} \\ \bigoplus_\xi \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus a} \arrow[ur, "\kappa_0"] \arrow[dr, "\kappa_1"'] & \\ & \mathrm{inj}_{K_1} C[\mathbb{G}_1]^{\oplus a\cdot|\mathbb{B}_1\backslash\mathbb{G}_1|^{-1}} \end{tikzcd}\right)$$

where κ_0 *and* κ_1 *are isomorphisms, and a morphism of diagrams*

$$\psi: \quad \begin{tikzcd} & \rho_{\Xi,0} \arrow[r, hook, "\psi_{K_0}"] & & \mathrm{inj}_{K_0} C[\mathbb{G}_0]^{\oplus a\cdot|\mathbb{B}_0\backslash\mathbb{G}_0|^{-1}} \\ \chi^{-1} \arrow[ur, "\iota_0"] \arrow[rr, hook, "\psi_{\mathfrak{B}}"] \arrow[dr, "\iota_1"'] & & \bigoplus_\xi \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus a} \arrow[ur, "\kappa_0"] \arrow[dr, "\kappa_1"'] & \\ & \rho_{\Xi,1} \arrow[r, hook, "\psi_{K_1}"] & & \mathrm{inj}_{K_1} C[\mathbb{G}_1]^{\oplus a\cdot|\mathbb{B}_1\backslash\mathbb{G}_1|^{-1}} \end{tikzcd}$$

in which all arrows are injections.

Proof. We fix the following injections, which are equivariant for the relevant groups:

- injective envelopes $j_\xi : \xi \hookrightarrow \mathrm{inj}_{\mathfrak{B}}\xi$ for each C-character ξ of $\mathfrak{B}$;
- injective envelopes $j_i : C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}} \hookrightarrow \mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}$ for $i \in \{0,1\}$;
- an inclusion $c : \chi^{-1} \hookrightarrow \bigoplus_\xi \xi^{\oplus a}$;
- an inclusion $c_i : \rho_{\Xi,i} \hookrightarrow C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}$ for $i \in \{0,1\}$.

Let $i \in \{0,1\}$. We first construct the κ_i. We have a $\mathfrak{B}$-equivariant sequence of maps

$$\chi^{-1} \overset{\iota_i}{\hookrightarrow} \rho_{\Xi,i} \overset{c_i}{\hookrightarrow} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}} \overset{j_i}{\hookrightarrow} \mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}$$

and thus we obtain

$$\chi^{-1} \xhookrightarrow{j_i\circ c_i\circ\iota_i} \big(\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}\big)^{\mathfrak{U}}.$$

By Lemmas 5.4.3 and 5.4.2, we have $\bigoplus_\xi \xi^{\oplus a} \cong (\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}})^{\mathfrak{U}}$. We *fix* an isomorphism $\alpha_i : \bigoplus_\xi \xi^{\oplus a} \xrightarrow{\sim} (\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}})^{\mathfrak{U}}$ such that

$$\alpha_i \circ c = j_i \circ c_i \circ \iota_i. \tag{5.4.6}$$

Now consider the maps of C-representations of $\mathfrak{B}$:

$$\bigoplus_\xi \xi^{\oplus a} \overset{\alpha_i}{\hookrightarrow} \big(\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}\big)^{\mathfrak{U}} \hookrightarrow \big(\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}\big)|_{\mathfrak{B}}.$$

By Lemma 5.4.1, the above map extends to an $\mathfrak{B}$-equivariant split injection

$$\kappa_i : \bigoplus_\xi \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus a} \hookrightarrow \big(\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}}\big)|_{\mathfrak{B}}$$

such that

$$\kappa_i \circ \Big(\bigoplus_\xi j_\xi^{\oplus a}\Big) = \alpha_i. \tag{5.4.7}$$

Since both $\bigoplus_\xi \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus a}$ and $(\mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a\cdot|\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}})|_{\mathfrak{B}}$ are injective C-representations of $\mathfrak{B}$ and κ_i induces an isomorphism between their $\mathfrak{U}$-invariants (cf. Lemma 5.4.3), we see that κ_i must in fact be an isomorphism.

We now construct the morphism of diagrams. Set $\psi_{K_i} := j_i \circ c_i$ and $\psi_{\mathfrak{B}} := (\bigoplus_\xi j_\xi^{\oplus a}) \circ c$. We have

$$\psi_{K_i} \circ \iota_i \overset{(5.4.6)}{=} \alpha_i \circ c \overset{(5.4.7)}{=} \kappa_i \circ \psi_{\mathfrak{B}},$$

and therefore we obtain the desired morphism of diagrams. $\square$

5.5. **Supersingular representations via homology.** Recall that a G-equivariant coefficient system $\mathcal{D}$ consists of C-vector spaces $\mathcal{D}_{\mathcal{F}}$ for every facet $\mathcal{F} \subset \mathscr{B}$, along with restriction maps for every inclusion of facets. This data is required to have a compatible G-action such that each $\mathcal{D}_{\mathcal{F}}$ is a smooth C-representation of the G-stabilizer of $\mathcal{F}$. The functor sending $\mathcal{D}$ to the quintuple $(\mathcal{D}_{x_0}, \mathcal{D}_{x_1}, \mathcal{D}_{\mathcal{C}}, \iota_0, \iota_1)$, where the ι_i are the natural restriction maps, is an equivalence of categories between G-equivariant coefficient systems and diagrams (cf. [KX15, §6.3]).

We let $\mathcal{D}_\Xi$ and $\mathcal{D}'$ denote the G-equivariant coefficient systems on $\mathscr{B}$ associated to D_Ξ and D', respectively. The homology of G-equivariant coefficient systems gives rise to smooth C-representations of G, and we define

$$\pi := \mathrm{im}\left(H_0(\mathscr{B}, \mathcal{D}_\Xi) \xrightarrow{\psi_*} H_0(\mathscr{B}, \mathcal{D}')\right),$$

where ψ_* denotes the map on homology induced by ψ.

Theorem 5.5.1. *Suppose* $\mathrm{char}\, F = 0$. *Then the C-representation π of G admits an irreducible, admissible, supercuspidal quotient.*

Proof. We use language and notation from [Paš04] and [KX15].

Step 1: The representation π is nonzero.

Fix a basis v for χ^{-1}. Let $\omega_{0,\iota_0(v)}$ denote the 0-chain with support x_0 satisfying $\omega_{0,\iota_0(v)}(x_0) = \iota_0(v)$ and let $\bar{\omega}_{0,\iota_0(v)}$ denote its image in $H_0(\mathscr{B}, \mathcal{D}_\Xi)$. Set $\bar{\omega} := \psi_*(\bar{\omega}_{0,\iota_0(v)}) = \bar{\omega}_{0,\psi_{K_0}\circ\iota_0(v)} \in \pi \subset H_0(\mathscr{B}, \mathcal{D}')$. This is the image in $H_0(\mathscr{B}, \mathcal{D}')$ of a $\mathcal{D}'_{x_0}$-valued 0-chain supported on x_0, and since the maps κ_0, κ_1 are isomorphisms and ψ is injective, we have $\bar{\omega} \neq 0$ [Paš04, Lem. 5.7]. We also note that therefore $\bar{\omega}_{0,\iota_0(v)} \neq 0$.

Step 2: The representation π is admissible.

Since κ_0, κ_1 are isomorphisms, [Paš04, Prop. 5.10] gives

$$\pi|_{\mathfrak{B}} \subset H_0(\mathscr{B}, \mathcal{D}')|_{\mathfrak{B}} \cong \mathcal{D}'_{\mathcal{C}} \cong \bigoplus_{\xi} \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus a},$$

which by Lemma 5.4.2 implies $\pi^{\mathfrak{U}} \hookrightarrow \bigoplus_\xi \xi^{\oplus a}$, so that π is admissible.

Step 3: The $\mathcal{H}_C$-module $\pi^{\mathfrak{U}}$ contains Ξ.

The element $\bar{\omega}_{0,\iota_0(v)} \in H_0(\mathscr{B}, \mathcal{D}_\Xi)$ is $\mathfrak{U}$-invariant and stable by the action of $\mathcal{H}_C$, and the vector space it spans is isomorphic to Ξ as an $\mathcal{H}_C$-module (for all of this, see the proof of [KX15, Prop. 7.3]). Since ψ_* is G-equivariant, the same is true for $\bar{\omega} \in \pi$.

Step 4: The vector $\bar{\omega}$ generates π.

Since $\bar{\omega}_{0,\iota_0(v)}$ generates $H_0(\mathscr{B}, \mathcal{D}_\Xi)$ as a G-representation and ψ_* is G-equivariant, $\bar{\omega}$ generates π as a G-representation.

Step 5: We construct the quotient π' and list its properties.

By the previous step, the representation π is generated by $\bar{\omega}$. Proceeding as in the end of the proof of Theorem 4.5.1, we see that any irreducible quotient of $\pi = \langle G \cdot \bar{\omega} \rangle$ is admissible (since $\mathrm{char}\, F = 0$, and such quotients exist by Zorn's lemma). Let π' be any such quotient.

Step 6: We prove π' is supercuspidal.

Since $\bar{\omega}$ generates π, its image in π' is nonzero. Thus, we obtain an injection of $\mathcal{H}_C$-modules $\Xi \cong C\bar{\omega} \hookrightarrow (\pi')^{\mathfrak{U}}$, and supercuspidality follows from Proposition 3.1.3. □

Corollary 5.5.2. *Suppose* $\mathrm{char}\, F = 0$ *and* $\mathbf{G}$ *is a connected reductive F-group of relative semisimple rank 1. Then G admits an irreducible admissible supercuspidal C-representation.*

Proof. By the reductions in §5.1, it suffices to assume $\mathbf{G}$ is absolutely simple and simply connected, and to construct a supercuspidal C-representation on which $Z(G)$ acts trivially. Since the center of G is finite, it is contained in $\mathfrak{B} \cap Z = Z_0$. Hence, taking Ξ to be associated to $(\mathbf{1}_{\mathbb{Z}}, J)$, where $\mathbf{1}_{\mathbb{Z}}$ is the trivial character of $\mathbb{Z}$ and $J \neq \emptyset, \{s_0, s_1\}$ (noting that $S_{\mathbf{1}_{\mathbb{Z}}} = \{s_0, s_1\}$), Theorem 5.5.1 produces an irreducible admissible supercuspidal C-representation π' with trivial action of the center. This gives the claim. □

Remark 5.5.3. The construction of π' above shares some similarities with the construction in §4.5. Therein, supercuspidal representations are constructed as subquotients of $C^\infty(\Gamma\backslash G, C) \cong \mathrm{Ind}_\Gamma^G \mathbf{1}_\Gamma$, where Γ is a discrete, cocompact subgroup of G and $\mathbf{1}_\Gamma$ denotes the trivial character of Γ. Taking Γ to be torsion-free, we use the Mackey formula to obtain

$$(\mathrm{Ind}_\Gamma^G \mathbf{1}_\Gamma)|_{K_i} \cong \bigoplus_{\Gamma\backslash G/K_i} \mathrm{Ind}_{\{1\}}^{K_i} \mathbf{1} \cong \mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a'_i}$$

where $a_i' = |\Gamma\backslash G/K_i|$. (The last isomorphism follows from the fact that $\mathrm{Ind}_{\{1\}}^{K_i}\mathbf{1}$ is injective, by [Vig96, §I.5.9 b)], and $(\mathrm{Ind}_{\{1\}}^{K_i}\mathbf{1})^{K_i^+} \cong C[\mathbb{G}_i]$; we may then proceed as in the proof of Lemma 5.4.3.) The construction of Theorem 5.5.1 produces supercuspidal representations as subquotients of $H_0(\mathscr{B},\mathcal{D}')$, for which we have

$$H_0(\mathscr{B},\mathcal{D}')|_{K_i} \cong \mathrm{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a_i},$$

where $a_i = a \cdot |\mathbb{B}_i\backslash\mathbb{G}_i|^{-1}$ (cf. [Paš04, Prop. 5.10]).

6. Supersingular representations of $\mathrm{PGL}_n(D)$

In this section we verify Theorem A when $\mathbf{G} = \mathbf{PGL}_n(D)$, where $n \geq 2$ and D a central division algebra over F. In particular, this deals with the first exceptional case in Theorem 4.5.1.

6.1. **Notation and conventions.** Throughout Section 6, we let $\overline{\mathbb{Q}}_p$ denote a fixed algebraic closure of $\mathbb{Q}_p$, with ring of integers $\overline{\mathbb{Z}}_p$ and residue field $\overline{\mathbb{F}}_p$. We normalize the valuation val of $\overline{\mathbb{Q}}_p$ such that $\mathrm{val}(p) = 1$.

Let D denote a central division algebra over F of dimension d^2. Let $B = ZU$ denote the upper-triangular Borel subgroup of $\mathrm{GL}_n(D)$ with diagonal minimal Levi subgroup $Z \cong (D^\times)^n$ and unipotent radical U. Let $T \cong (F^\times)^n$ denote the diagonal maximal split torus, $\mathcal{N}$ its normalizer in $\mathrm{GL}_n(D)$, and U^{op} the lower-triangular unipotent matrices.

Let $\mathcal{O}_D$ denote the ring of integers of D, $\mathfrak{m}_D$ the maximal ideal of $\mathcal{O}_D$, and k_D the residue field, so $[k_D : k_F] = d$. Let $D(1) := 1 + \mathfrak{m}_D$, so $D(1) \lhd D^\times$. Let $\mathrm{val}_D : D^\times \twoheadrightarrow \mathbb{Z}$ denote the normalized valuation of D. Let $I(1)$ denote the pro-p Iwahori subgroup

$$I(1) := \{g \in \mathrm{GL}_n(\mathcal{O}_D) : \overline{g} \in \mathrm{GL}_n(k_D) \text{ is upper-triangular unipotent}\}.$$

For any field K let Γ_K denote the absolute Galois group for a choice of separable closure. If K'/K is a finite separable extension, then $\Gamma_{K'}$ is a subgroup of Γ_K, up to conjugacy, hence the restriction of a Γ_K-representation to $\Gamma_{K'}$ is well defined up to isomorphism.

If $K/\mathbb{Q}_p$ is finite we let I_K denote the inertia subgroup of Γ_K and k_K the residue field of K. If $\rho : \Gamma_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is de Rham and $\tau : K \to \overline{\mathbb{Q}}_p$ is continuous, we let $\mathrm{HT}_\tau(\rho)$ denote multi-set of τ-Hodge–Tate weights. We normalize Hodge–Tate weights so that the cyclotomic character ε has τ-Hodge–Tate weight -1 for any τ. We let $\mathrm{WD}(\rho)$ denote the associated Weil–Deligne representation of W_K over $\overline{\mathbb{Q}}_p$ (defined by Fontaine, cf. Appendix B.1 of [CDT99]).

We normalize local class field theory so that uniformizers correspond to geometric Frobenius elements under the local Artin map. Let rec_F denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ to isomorphism classes of n-dimensional Frobenius semisimple Weil–Deligne representations of W_F over $\mathbb{C}$. (See [HT01].)

If L is a global field, we let $|\cdot|_L$ denote the normalized absolute value of $\mathbb{A}_L$.

6.2. **On the Jacquet–Langlands correspondence.** We recall some basic facts about the representation theory of $\mathrm{GL}_n(D)$ and the local Jacquet–Langlands correspondence. All representations in this section will be smooth and over $\mathbb{C}$.

For a finite-dimensional central simple algebra A let $\mathrm{Nrd} : A^\times \to Z(A)^\times$ (or Nrd_A for clarity) denote the reduced norm. Let ν denote the smooth character $|\mathrm{Nrd}|_F$ of

$\mathrm{GL}_m(D)$ for any m. If π_i are smooth representations of $\mathrm{GL}_{n_i}(D)$, let $\pi_1 \times \cdots \times \pi_r$ denote the normalized parabolic induction of $\pi_1 \otimes \cdots \otimes \pi_r$ to $\mathrm{GL}_{\sum n_i}(D)$. In particular these notions also apply to general linear groups over F (by setting $D = F$).

We will say that a representation is *essentially unitarizable* if some twist of it is unitarizable.

The Jacquet–Langlands correspondence [DKV84] is a canonical bijection JL between irreducible essentially square-integrable representations of $\mathrm{GL}_n(D)$ and irreducible essentially square-integrable representations of $\mathrm{GL}_{nd}(F)$ that is compatible with character twists and preserves central characters. (For short, we say "square-integrable" instead of "square-integrable modulo center".)

On the other hand, Badulescu [Bad08] defined a map $|\,\mathrm{LJ}_{\mathrm{GL}_n(D)}\,|$ in the other direction, from irreducible essentially unitarizable representations of $\mathrm{GL}_{nd}(F)$ to irreducible essentially unitarizable representations of $\mathrm{GL}_n(D)$ or zero, which in general is neither injective nor surjective. (More precisely, [Bad08] only considers unitarizable representations, but we can extend it by twisting.) In the split case $|\,\mathrm{LJ}_{\mathrm{GL}_n(F)}\,|$ is the identity. It follows from Thm. 2.2 and Thm. 2.7(a) in [Bad08] that $|\,\mathrm{LJ}_{\mathrm{GL}_n(D)}\,|(\mathrm{JL}(\pi)) \cong \pi$ for any essentially square-integrable representation π of $\mathrm{GL}_n(D)$.

If ρ is a supercuspidal representation of $\mathrm{GL}_m(F)$ and $\ell \geq 1$, then $Z^u(\rho, \ell)$ is by definition the unique irreducible quotient of $\rho\nu^{(1-\ell)/2} \times \rho\nu^{(3-\ell)/2} \times \cdots \times \rho\nu^{(\ell-1)/2}$. It is an essentially square-integrable representation of $\mathrm{GL}_{m\ell}(F)$. All essentially square-integrable representations of $\mathrm{GL}_n(F)$ arise in this way, for some decomposition $n = m\ell$.

If ρ' is a supercuspidal representation of $\mathrm{GL}_m(D)$, we can write $\mathrm{JL}(\rho') \cong Z^u(\rho, s)$ for some supercuspidal representation ρ and integer $s \geq 1$. Then $Z^u(\rho', \ell)$ is by definition the unique irreducible quotient of $\rho'\nu^{s(1-\ell)/2} \times \rho'\nu^{s(3-\ell)/2} \times \cdots \times \rho'\nu^{s(\ell-1)/2}$. It is an essentially square-integrable representation of $\mathrm{GL}_{m\ell}(D)$. All essentially square-integrable representations of $\mathrm{GL}_n(D)$ arise in this way, for some decomposition $n = m\ell$ (a result of Tadić, cf. [Bad08, §2.4]). Moreover, $\mathrm{JL}(Z^u(\rho', \ell)) \cong Z^u(\rho, \ell s)$ [Bad08, §3.1].

If π is a smooth representation of $\mathrm{GL}_n(D)$ let π_U denote its (unnormalized) Jacquet module. The following lemma was proved earlier, cf. Remark 4.1.3.

Lemma 6.2.1. *Suppose that π is an admissible representation of $\mathrm{GL}_n(D)$ over $\mathbb{C}$. Then the natural map $p_U : \pi \to \pi_U$ induces an isomorphism $\pi^{I(1)} \to (\pi_U)^{Z\cap I(1)}$.*

The following results will be needed in Section 6.3.

Lemma 6.2.2. *Suppose that Π is an irreducible generic smooth representation of $\mathrm{GL}_{nd}(F)$ over $\mathbb{C}$ that is essentially unitarizable and such that the representation $\pi := |\,\mathrm{LJ}_{\mathrm{GL}_n(D)}\,|(\Pi)$ of $\mathrm{GL}_n(D)$ is non-zero. If $\pi^{I(1)} \neq 0$, then there exist irreducible representations ρ_1', ..., ρ_n' of $D^\times/D(1)$ such that π is a subquotient of $\rho_1' \times \cdots \times \rho_n'$ and $\mathrm{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{i=1}^n \mathrm{rec}_F(\mathrm{JL}(\rho_i'))|_{W_F}$.*

Proof. After a twist we may assume that Π is unitarizable. As Π is moreover generic, we know that $\Pi \cong \sigma_1\nu^{\alpha_1} \times \cdots \times \sigma_r\nu^{\alpha_r}$ for some square-integrable σ_i of $\mathrm{GL}_{n_i}(F)$ and real numbers $\alpha_i \in (-\frac{1}{2}, \frac{1}{2})$ satisfying $\alpha_i + \alpha_{r+1-i} = 0$ and $\sigma_i = \sigma_{r+1-i}$ if $\alpha_i \neq 0$ (see e.g. [HT01, Lemma I.3.8]). Since $|\,\mathrm{LJ}_{\mathrm{GL}_n(D)}\,|(\Pi) \neq 0$ by assumption, it follows that $d \mid n_i$ for all i and $\pi = |\,\mathrm{LJ}_{\mathrm{GL}_n(D)}\,|(\Pi) \cong \sigma_1'\nu^{\alpha_1} \times \cdots \times \sigma_r'\nu^{\alpha_r}$, where σ_i' is the square-integrable representation of $\mathrm{GL}_{n_i/d}(D)$ such that $\mathrm{JL}(\sigma_i') \cong \sigma_i$. (See [Bad08, §3.5].) Let $n_i' := n_i/d$.

From $\pi^{I(1)} \neq 0$ and Lemma 6.2.1 it follows that the supercuspidal support of π is a tame representation of Z (up to conjugacy), so each σ_i' is of the form $Z^u(\rho_i'', n_i')$, where ρ_i'' is an irreducible representation of $D^\times/D(1)$. We write $\mathrm{JL}(\rho_i'') \cong Z^u(\rho_i, e_i)$ with ρ_i irreducible supercuspidal, so $\sigma_i \cong Z^u(\rho_i, e_i n_i')$. In particular, π is a subquotient of the normalized induction of $\bigotimes_{1\le i\le r, 0\le j\le n_i'-1} \rho_i''\nu^{\alpha_i + e_i((n_i'-1)/2-j)}$. On the other hand, Π is a subquotient of the normalized induction of $\bigotimes_{1\le i\le r, 0\le j\le e_i n_i'-1} \rho_i\nu^{\alpha_i+(e_i n_i'-1)/2-j}$. As $\mathrm{rec}_F(\Pi)|_{W_F}$ only depends on the supercuspidal support of Π (see the paragraph before Thm. VII.2.20 in [HT01]), we obtain

$$\mathrm{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{1\le i\le r, 0\le j\le e_i n_i'-1} |\cdot|_F^{\alpha_i+(e_i n_i'-1)/2-j}\mathrm{rec}_F(\rho_i)|_{W_F}.$$

Similarly, $\mathrm{rec}_F(\mathrm{JL}(\rho_i''))|_{W_F} \cong \bigoplus_{k=0}^{e_i-1} |\cdot|_F^{(e_i-1)/2-k}\mathrm{rec}_F(\rho_i)|_{W_F}$. Denoting by $\rho_1', \ldots, \rho_n'$ the representations $\rho_i''\nu^{\alpha_i+e_i((n_i'-1)/2-j)}$ in any order, a straightforward computation confirms that $\bigoplus_{i=1}^n \mathrm{rec}_F(\mathrm{JL}(\rho_i'))|_{W_F} \cong \mathrm{rec}_F(\Pi)|_{W_F}$. $\square$

We now recall a result of Bushnell–Henniart concerning explicit functorial transfers of irreducible representations of $D^\times/D(1)$. An *admissible tame pair* $(E/F,\zeta)$ consists of an unramified extension of degree f dividing d and a tamely ramified smooth character $\zeta : E^\times \to \mathbb{C}^\times$ such that all $\mathrm{Gal}(E/F)$-conjugates of ζ are distinct. In that case, $B := Z_D(E)$ is a central simple E-algebra of dimension e^2, where $e := d/f$. Define a smooth character $\Lambda : B^\times(1+\mathfrak{m}_D) \to \mathbb{C}^\times$ by declaring it to be $\zeta\circ\mathrm{Nrd}_B$ on $B^\times$ and trivial on $1+\mathfrak{m}_D$. Then define $\pi_D(\zeta) := \mathrm{Ind}_{B^\times(1+\mathfrak{m}_D)}^{D^\times}\Lambda$ is an irreducible representation of $D^\times/D(1)$ (of dimension f).

Proposition 6.2.3.

(i) *Any irreducible representation of $D^\times/D(1)$ is isomorphic to $\pi_D(\zeta)$ for some admissible tame pair $(E/F,\zeta)$.*
(ii) *The element $\varpi \in F$ acts as the scalar $\zeta(\varpi)^e$ on $\pi_D(\zeta)$.*
(iii) *If $(E/F,\zeta)$ is an admissible tame pair, then*

$$\mathrm{rec}_F(\mathrm{JL}(\pi_D(\zeta))) \cong \mathrm{Sp}_e(\mathrm{Ind}_{W_E}^{W_F}(\eta_E^{e(f-1)}\zeta)),$$

where η_E is the unramified quadratic character of $E^\times$.

We recall that the special Weil–Deligne representation $\mathrm{Sp}_e(\sigma)$, for σ an irreducible representation of W_F, is indecomposable and satisfies $\mathrm{Sp}_e(\sigma)|_{W_F} \cong \bigoplus_{k=0}^{e-1}\sigma|\cdot|_F^{\frac{e-1}{2}-k}$.

Proof. For (i), see [BH11, §1.5]. Part (ii) follows from the definition. Part (iii) is the main result of [BH11]. $\square$

6.3. **On lifting non-supersingular Hecke modules.** Let $\mathcal{H} := \mathcal{H}(\mathrm{GL}_n(D), I(1))$ the corresponding pro-p Iwahori–Hecke algebra over $\mathbb{Z}$ [Vig16] and for a commutative ring R let $\mathcal{H}_R := \mathcal{H}\otimes R$. Similarly we define $\mathcal{H}_Z := \mathcal{H}(Z, Z\cap I(1))$ and $\mathcal{H}_{Z,R} := \mathcal{H}_Z\otimes R$. Note that the pro-$p$ Iwahori subgroup $Z\cap I(1)$ is normal in Z. All Hecke modules we will consider are right modules. A finite-dimensional $\mathcal{H}_{\overline{\mathbb{Q}}_p}$-module is said to be *integral* if it arises by base change from a $\mathcal{H}_{\overline{\mathbb{Z}}_p}$-module that is finite free over $\overline{\mathbb{Z}}_p$.

Let $W(1) := \mathcal{N}/Z \cap I(1)$, $\Lambda(1) := Z/Z \cap I(1)$, and define monoids

$$Z^+ := \{\mathrm{diag}(\delta_1, \ldots, \delta_n) \in Z : \mathrm{val}_D(\delta_1) \geq \cdots \geq \mathrm{val}_D(\delta_n)\}$$

and $\Lambda(1)^+ := Z^+/Z \cap I(1)$.

We recall that $\mathcal{H}$ has an Iwahori–Matsumoto basis T_w for $w \in W(1)$ and a Bernstein basis E_w for $w \in W(1)$, which in fact depends on a choice of spherical orientation. We choose our spherical orientation such that $E_w = T_w$ for $w \in \Lambda(1)^+$. (This is possible by [Vig16, Ex. 5.30]. It is the opposite of our convention in §2.3.) Similarly, $\mathcal{H}_Z$ has basis T^Z_w for $w \in \Lambda(1)$.

For $w \in W(1)$ we have integers $q_w \in q^{\mathbb{Z}_{>0}}$, as recalled in §2.3. (Note that our base alcove $\mathcal{C}$ is the one fixed pointwise by $I(1)$.)

Lemma 6.3.1. *For $z = \mathrm{diag}(\delta_1, \ldots, \delta_n) \in Z$ with $\delta_i \in D^\times$ we have*

$$q_z = q^{d \sum_{i<j} |\mathrm{val}_D(\delta_i) - \mathrm{val}_D(\delta_j)|}.$$

Proof. As the Iwahori–Hecke algebra has equal parameters q^d we deduce that $q_z = q^{d\ell(z)}$, where ℓ is the length function relative to the alcove $\mathcal{C}$. By using the action of the finite Weyl group $\mathcal{N}/Z$ and the first length formula in [Vig16, Cor. 5.11], we may assume that $z \in Z^+$. By [Vig16, §3.9] we then have $q_z = (I(1)zI(1) : I(1)) = (I(1) : I(1) \cap zI(1)z^{-1}) = (U_0 : zU_0z^{-1})$, where $U_0 := U \cap I(1)$. Hence $q_z = q^{d \sum_{i<j}(\mathrm{val}_D(\delta_i) - \mathrm{val}_D(\delta_j))}$, as required. □

Let $W_0 \cong S_n$ denote the Weyl group of T. Recall from [Vig17, §5, §1.3] that $\mathcal{A}_0(\Lambda_T)$ is the free module with basis $E_{\mu(\varpi)}$ for $\mu \in \Lambda_T := X_*(T)$ and that the central subalgebra $\mathcal{Z}_T := \mathcal{A}_0(\Lambda_T)^{W_0}$ of $\mathcal{H}$ has a basis consisting of the sums $\sum_\mu E_{\mu(\varpi)}$ with μ running over the W_0-orbits in $X_*(T)$. For $I \subset \{1, \ldots, n\}$ let $E_I := E_{\mu_I(\varpi)}$, where $\mu_I \in X_*(T) \cong \mathbb{Z}^n$ is defined by $\mu_{I,i} = 1$ if $i \in I$ and $\mu_{I,i} = 0$ otherwise. For $1 \leq i \leq n$ let $\mathscr{X}_i := \sum_{I, |I|=i} E_I$. By induction and [Vig16, Cor. 5.28] we see that the algebra $\mathcal{Z}_T$ is generated by $\mathscr{X}_1$, $\ldots$, $\mathscr{X}_{n-1}$, $\mathscr{X}_n^{\pm 1}$.

The following lemma follows from [Vig17, Prop. 6.9].

Lemma 6.3.2. *A finite-dimensional $\mathcal{H}_{\overline{\mathbb{F}}_p}$-module M is supersingular if and only if the action of $\mathscr{X}_i$ on M is nilpotent for all $1 \leq i \leq n-1$.*

Lemma 6.3.3. *There exists a unique injective algebra homomorphism $\widetilde{\theta} : \mathcal{H}_{Z,\overline{\mathbb{Q}}_p} \to \mathcal{H}_{\overline{\mathbb{Q}}_p}$ such that $\widetilde{\theta}(T^Z_w) = T_w$ for all $w \in \Lambda(1)^+$. We have*

$$E_I = q^{d^2(\sum_{i \in I} i - \binom{|I|+1}{2})} \widetilde{\theta}(T^Z_{\mu_I(\varpi)}). \tag{6.3.4}$$

Proof. The first assertion follows from [OV18, §2.5.2, Rk. 2.20]. We claim that for any $\mu \in X_*(T)$,

$$E_{\mu(\varpi)} = q^{d^2 \sum_{r<s: \mu_r < \mu_s} (\mu_s - \mu_r)} \widetilde{\theta}(T^Z_{\mu(\varpi)}), \tag{6.3.5}$$

which implies (6.3.4) by taking $\mu = \mu_I$.

Note that $X_*(T)^+ = \{\mu \in X_*(T) : \mu_1 \geq \cdots \geq \mu_n\}$. If $\mu \in X_*(T)^+$, then $\mu(\varpi) \in Z^+$ and hence $E_{\mu(\varpi)} = T_{\mu(\varpi)}$ and formula (6.3.5) holds. In general, choose $\mu' \in X_*(T)^+$ such that $\mu + \mu' \in X_*(T)^+$. Then formula (6.3.5) follows easily from the following three assertions: (1) $T^Z_{\mu(\varpi)} T^Z_{\mu'(\varpi)} = T^Z_{\mu(\varpi)\mu'(\varpi)}$; (2) $E_{\mu(\varpi)} E_{\mu'(\varpi)} = (q_{\mu(\varpi)} q_{\mu'(\varpi)} q^{-1}_{\mu(\varpi)\mu'(\varpi)})^{1/2} E_{\mu(\varpi)\mu'(\varpi)}$

in the notation of [Vig16, §4.4], where we take the positive square root; and (3) Lemma 6.3.1. Assertion (1) is clear and assertion (2) is [Vig16, Cor. 5.28]. □

The following simple and presumably well-known lemma will be used below.

Lemma 6.3.6. *Suppose that $\rho : W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a smooth representation. Then for any $\gamma \in W_F$ the valuations of the eigenvalues of $\rho(\gamma)$ depend only on the image of γ in $W_F/I_F \cong \mathbb{Z}$.*

Proof. Fix a geometric Frobenius element $\mathrm{Frob}_F \in W_F$, and let $v_1 \le \cdots \le v_n$ denote the valuations of the eigenvalues of $\rho(\mathrm{Frob}_F)$. We need to show that the eigenvalues of $\rho(\mathrm{Frob}_F^r\, g)$ have valuations $rv_1 \le \cdots \le rv_n$ for any $g \in I_F$. As $\rho(I_F)$ is finite and normalized by $\rho(\mathrm{Frob}_F)$, we see that $\rho(\mathrm{Frob}_F)^{\ell r}$ and $\rho(I_F)$ commute for some $\ell \ge 1$, so $\rho(I_F)$ preserves the generalized eigenspaces of $\rho(\mathrm{Frob}_F^{\ell r})$. Hence the valuations of the eigenvalues of $\rho(\mathrm{Frob}_F^{\ell r}\, g)$ are independent of $g \in I_F$, and the claim follows by passing to ℓ-th powers. □

We now fix an isomorphism $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

Proposition 6.3.7. *Suppose that Π is an irreducible generic smooth representation of $\mathrm{GL}_{nd}(F)$ over $\mathbb{C}$ that is essentially unitarizable and such that the representation $\pi := |\mathrm{LJ}_{\mathrm{GL}_n(D)}|(\Pi)$ of $\mathrm{GL}_n(D)$ is non-zero. Suppose that $\imath^{-1}(\pi^{I(1)})$ is a non-zero integral $\mathcal{H}_{\overline{\mathbb{Q}}_p}$-module with non-supersingular reduction, and let $v_1 \le \cdots \le v_{nd}$ denote the valuations of the eigenvalues of a geometric Frobenius on $\imath^{-1}(\mathrm{rec}_F(\Pi))$. Then there exists $1 \le j \le n-1$ such that*

$$\sum_{i=1}^{jd} v_i = -\frac{d^2 j(n-j)}{2}\,\mathrm{val}(q).$$

Proof. Step 1: We compute the action of $\mathscr{Z}_1, \ldots, \mathscr{Z}_n$ on the Hecke module $\imath^{-1}(\pi^{I(1)})$ and show in particular that it is scalar.

Note by Lemma 6.2.2 that $\pi^{I(1)}$ is a subquotient of $(\rho'_1 \times \cdots \times \rho'_n)^{I(1)}$ for some irreducible representations ρ'_i of $D^\times/D(1)$, and $\rho'_1 \times \cdots \times \rho'_n \cong \mathrm{Ind}_B^{\mathrm{GL}_n(D)}(\rho'_1\nu^{d(n-1)/2} \otimes \cdots \otimes \rho'_n\nu^{-d(n-1)/2})$ (unnormalized induction). By [OV18, Prop. 4.4] we have

(6.3.8)
$$\imath^{-1}(\rho'_1 \times \cdots \times \rho'_n)^{I(1)} \cong \imath^{-1}(\rho'_1\nu^{d(n-1)/2} \otimes \cdots \otimes \rho'_n\nu^{-d(n-1)/2})^{Z\cap I(1)} \otimes_{\mathcal{H}_{Z,\overline{\mathbb{Q}}_p},\widetilde{\theta}} \mathcal{H}_{\overline{\mathbb{Q}}_p},$$

where we used the homomorphism $\widetilde{\theta}$ of Lemma 6.3.3.

By Proposition 6.2.3(i) we can write $\rho'_i \cong \pi_D(\zeta_i)$ for some admissible tame pair $(F_i/F, \zeta_i)$. We let $f_i := [F_i : F]$ and $e_i := d/f_i$. Let $\zeta'_i := \imath^{-1}(\zeta_i)$. From equations (6.3.4), (6.3.8) and Proposition 6.2.3(ii) we deduce that $\mathscr{Z}_j$ acts on $\imath^{-1}(\pi^{I(1)})$ as the scalar

$$\begin{aligned} \lambda_j &:= \sum_{|I|=j} \left(q^{d^2(\sum_{i\in I} i - \binom{j+1}{2})} q^{d^2((n+1)j/2 - \sum_{i\in I} i)} \prod_{i\in I} \zeta'_i(\varpi)^{-e_i} \right) \\ &= q^{-d^2\binom{j}{2}} \sum_{|I|=j} \left(q^{d^2(n-1)j/2} \prod_{i\in I} \zeta'_i(\varpi)^{-e_i} \right). \end{aligned} \tag{6.3.9}$$

Step 2: We complete the proof. By assumption, the Hecke module $\imath^{-1}(\pi^{I(1)})$ is integral, so $\lambda_i \in \overline{\mathbb{Z}}_p$ for all i and $\lambda_n \in \overline{\mathbb{Z}}_p^{\times}$. Moreover, as the reduction of $\imath^{-1}(\pi^{I(1)})$ is non-supersingular we deduce by Lemma 6.3.2 that $\lambda_{n-j} \in \overline{\mathbb{Z}}_p^{\times}$ for some $1 \le j \le n-1$.

From now on assume for convenience that the ζ_i' are ordered such that the sequence $\mathrm{val}(\zeta_i'(\varpi)^{-e_i})$ is non-increasing. Consider the polynomial $\prod_{i=1}^n (1-q^{d^2(n-1)/2}\zeta_i'(\varpi)^{-e_i}X)$. By (6.3.9) its Newton polygon is defined by the points $(i, \mathrm{val}(\lambda_i) + d^2\binom{i}{2}\mathrm{val}(q))$ for $0 \le i \le n$. From $\lambda_{n-j} \in \overline{\mathbb{Z}}_p^{\times}$, $\lambda_i \in \overline{\mathbb{Z}}_p$, and the convexity of the quadratic function $x(x-1)/2$ we deduce that $(n-j, d^2\binom{n-j}{2}\mathrm{val}(q))$ is a vertex of the Newton polygon. It follows for the sum of the largest j root valuations that

$$\sum_{i=1}^{j} \mathrm{val}(q^{d^2(n-1)/2}\zeta_i'(\varpi)^{-e_i}) = d^2\left(\binom{n}{2} - \binom{n-j}{2}\right)\mathrm{val}(q). \tag{6.3.10}$$

Again by convexity we obtain the root valuation bounds

$$\mathrm{val}(q^{d^2(n-1)/2}\zeta_i'(\varpi)^{-e_i}) \ge d^2(n-j)\,\mathrm{val}(q) \quad \forall i \le j, \tag{6.3.11}$$

$$\mathrm{val}(q^{d^2(n-1)/2}\zeta_i'(\varpi)^{-e_i}) \le d^2(n-j-1)\,\mathrm{val}(q) \quad \forall i > j. \tag{6.3.12}$$

From Lemma 6.2.2 and Proposition 6.2.3(iii) we see that

$$\mathrm{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{i=1}^{n}\bigoplus_{k=0}^{e_i-1} \mathrm{Ind}_{W_{F_i}}^{W_F}(\eta_{F_i}^{e_i(f_i-1)}\zeta_i)|\cdot|_F^{(e_i-1)/2-k}.$$

If Frob_F denotes a geometric Frobenius of W_F, then $\mathrm{Frob}_F^{f_i}$ is a geometric Frobenius of W_{F_i}. We see that all eigenvalues of Frob_F on $\mathrm{Ind}_{W_{F_i}}^{W_F}(\eta_{F_i}^{e_i(f_i-1)}\zeta_i')$ have valuation $\frac{1}{f_i}\mathrm{val}(\zeta_i'(\mathrm{Frob}_F^{f_i})) = \frac{1}{f_i}\mathrm{val}(\zeta_i'(\varpi))$. Hence, for $i \le j$ and $0 \le k \le e_i - 1$ all eigenvalues of Frob_F on $\mathrm{Ind}_{W_{F_i}}^{W_F}(\eta_{F_i}^{e_i(f_i-1)}\zeta_i')|\cdot|_F^{(e_i-1)/2-k}$ have valuation

$$\begin{aligned}\frac{1}{f_i}\mathrm{val}(\zeta_i'(\varpi)) - \Big(\frac{e_i-1}{2} - k\Big)\mathrm{val}(q) &\le \frac{1}{d}\mathrm{val}(\zeta_i'(\varpi)^{e_i}) + \Big(\frac{e_i-1}{2}\Big)\mathrm{val}(q)\\ &< d\Big(\frac{n-1}{2} - (n-j)\Big)\mathrm{val}(q) + \frac{d}{2}\mathrm{val}(q) = \frac{d(2j-n)}{2}\mathrm{val}(q),\end{aligned}$$

where we used (6.3.11) and that $e_i - 1 < d$. Similarly, for $i > j$ and $0 \le k \le e_i - 1$ we find that the eigenvalues of Frob_F on $\mathrm{Ind}_{W_{F_i}}^{W_F}(\eta_{F_i}^{e_i(f_i-1)}\zeta_i')|\cdot|_F^{(e_i-1)/2-k}$ have valuation greater than $\frac{d(2j-n)}{2}\mathrm{val}(q)$. Therefore, from (6.3.10) we deduce that

$$\begin{aligned}\sum_{i=1}^{jd} v_i = \sum_{i=1}^{j}\sum_{k=0}^{e_i-1} f_i\left(\frac{1}{f_i}\mathrm{val}(\zeta_i'(\varpi)) - \Big(\frac{e_i-1}{2} - k\Big)\mathrm{val}(q)\right) &= \sum_{i=1}^{j}\mathrm{val}(\zeta_i'(\varpi)^{e_i})\\ = d^2\left(\binom{n-j}{2} - \binom{n}{2} + \frac{j(n-1)}{2}\right)\mathrm{val}(q) &= -\frac{d^2 j(n-j)}{2}\mathrm{val}(q).\end{aligned}$$

$\square$

6.4. **A reducibility lemma.** Let F_0 denote the maximal absolutely unramified intermediate field of $F/\mathbb{Q}_p$. The following lemma generalizes [EGH13, Prop. 4.5.2], which dealt with regular crystalline Galois representations.

Lemma 6.4.1. *Suppose that $\rho : \Gamma_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a de Rham Galois representation. Let $v_1 \le \cdots \le v_n$ denote the valuations of the eigenvalues of a geometric Frobenius element acting on $\mathrm{WD}(\rho)$, and for each embedding $\tau : F \to \overline{\mathbb{Q}}_p$ let $h_{\tau,1} \le \cdots \le h_{\tau,n}$ denote the τ-Hodge–Tate weights of ρ. Then $\sum_{i=1}^j v_i \ge [F : F_0]^{-1} \sum_{i=1}^j \sum_{\tau:F\to\overline{\mathbb{Q}}_p} h_{\tau,i}$ for any $0 \le j \le n$.*

Suppose that $h_{\tau,1} < h_{\tau,n}$ for some τ and that for some $1 \le j \le n-1$ we have $\sum_{i=1}^j v_i = [F : F_0]^{-1} \sum_{i=1}^j \sum_{\tau:F\to\overline{\mathbb{Q}}_p} h_{\tau,i}$. Then ρ is reducible.

Proof. We first choose $E/\mathbb{Q}_p$ a sufficiently large finite subextension of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$, so that in particular ρ can be defined over E and all embeddings τ have image contained in E. Choose F'/F a finite Galois extension over which ρ becomes semistable. Let $D := \mathrm{D}_{\mathrm{st}}(\rho|_{\Gamma_{F'}})$ be the covariantly associated free $F_0' \otimes_{\mathbb{Q}_p} E$-module, equipped with actions of φ, N, $\mathrm{Gal}(F'/F)$, where F_0' denotes the maximal absolutely unramified intermediate field of $F'/\mathbb{Q}_p$. As usual, we write $D \cong \bigoplus_{\sigma:F_0'\to E} D_\sigma$. Fix any embedding $\sigma_0 : F_0' \to E$ and let $f' := [F_0' : \mathbb{Q}_p]$. Note that $\varphi^{f'}$ acts linearly on D and stabilizes each D_σ.

By construction of $\mathrm{WD}(\rho)$ and Lemma 6.3.6, the eigenvalues of $\varphi^{f'}$ on D_{σ_0} have valuations $rv_1 \le \cdots \le rv_n$, where $r := [F_0' : F_0]$. For any $0 \le j \le n$, choose a $\varphi^{f'}$-stable E-subspace $D'_{\sigma_0} \subset D_{\sigma_0}$ of dimension j such that the eigenvalues of $\varphi^{f'}$ on D'_{σ_0} have valuations $rv_1 \le \cdots \le rv_j$. Then D'_{σ_0} is also N-stable, since $N\varphi = p\varphi N$. Now for each $\sigma : F_0' \to E$ choose the unique E-subspace $D'_\sigma \subset D_\sigma$ that agrees with our choice of D'_{σ_0} when $\sigma = \sigma_0$ and such that $D' := \bigoplus_{\sigma:F_0'\to E} D'_\sigma$ is φ-stable. Then D' is stable under the actions of $F_0' \otimes_{\mathbb{Q}_p} E$, φ, N. As in the proof of [EGH13, Prop. 4.5.2] (see also the proof of [BS07, Prop. 5.1]) we now compute that $t_N(D') = \frac{[E:\mathbb{Q}_p]}{[F_0:\mathbb{Q}_p]} \sum_{i=1}^j v_i$ and that $t_H(D') \ge \frac{[E:\mathbb{Q}_p]}{[F:\mathbb{Q}_p]} \sum_{i=1}^j \sum_{\tau:F\to E} h_{\tau,i}$. By weak admissibility of D we have

$$(*_j) \qquad \sum_{i=1}^j v_i \ge [F : F_0]^{-1} \sum_{i=1}^j \sum_{\tau:F\to E} h_{\tau,i},$$

proving the first claim (with equality when $j = 0$ or $j = n$).

Now suppose that equality holds in $(*_j)$ for some $1 \le j \le n-1$. If $v_j = v_{j+1}$ then monotonicity of the $h_{\tau,i}$ and $(*_{j+1})$ imply that equality holds in both $(*_{j-1})$, $(*_{j+1})$ and that $h_{\tau,j} = h_{\tau,j+1}$ for all τ. Thus, by modifying j, we may assume without loss of generality that equality holds in $(*_j)$ and that $v_j < v_{j+1}$ (as $h_{\tau,1} < h_{\tau,n}$ for some τ, by assumption).

Let D' be the sum of all generalized $\varphi^{f'}$-eigenspaces in the E-vector space D whose corresponding eigenvalues have valuation at most rv_j. As $v_j < v_{j+1}$ we see that D' is a free $F_0' \otimes_{\mathbb{Q}_p} E$-module of rank j, stable under the actions of φ, N, and $\mathrm{Gal}(F'/F)$. Equality in $(*_j)$ gives that $t_H(D') = t_N(D')$, so ρ admits a j-dimensional subrepresentation. □

Remark 6.4.2. The lemma can fail when $h_{\tau,1} = h_{\tau,n}$ for all τ. For example, let $F/\mathbb{Q}_p$ be a quadratic extension and $\chi : \Gamma_F \to \overline{\mathbb{Q}}_p^\times$ a potentially unramified character that does not extend to $\Gamma_{\mathbb{Q}_p}$. Then $\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}_p}} \chi$ is irreducible and de Rham with all Hodge–Tate weights equal to 0 (since it is potentially unramified). In particular, $v_1 = v_2 = 0$. Concretely, via local class field theory, we can take $F = \mathbb{Q}_{p^2}$ and $\chi : \widehat{\mathbb{Q}_{p^2}^\times} \to \overline{\mathbb{Q}}_p^\times$ tame and non-trivial on $\mu_{p+1}(\mathbb{Q}_{p^2})$.

6.5. **On base change and descent for compact unitary groups.** The purpose of this section is to discuss base change and descent results for compact unitary groups that go slightly beyond those in [Lab11], namely allowing that the unitary group is non-quasisplit at some finite places. The proofs will be provided by Sug Woo Shin in Appendix A.

Suppose that $\widetilde{F}/\widetilde{F}^+$ is a CM extension of number fields with $\widetilde{F}^+ \neq \mathbb{Q}$ and G a unitary group over $\widetilde{F}^+$ such that

(i) $G_{/\widetilde{F}}$ is an inner form of GL_{nd};
(ii) $G(\widetilde{F}_u^+)$ is compact for any place $u \mid \infty$ of $\widetilde{F}^+$;
(iii) G is quasi-split at all finite places that are inert in $\widetilde{F}/\widetilde{F}^+$.

Let c denote the complex conjugation of $\widetilde{F}/\widetilde{F}^+$. Let $\Delta^+(G)$ denote the set of finite places of $\widetilde{F}^+$ where G is not quasi-split. This is a finite set of places that split or ramify in $\widetilde{F}$. Let $\Delta(G)$ denote the set of places of $\widetilde{F}$ lying over a place of $\Delta^+(G)$.

Proposition 6.5.1. *Suppose that π is a (cuspidal) automorphic representation of $G(\mathbb{A}_{\widetilde{F}^+})$. Then there exists a partition $n = n_1 + \cdots + n_r$ and discrete automorphic representations Π_i of $\mathrm{GL}_{n_i d}(\mathbb{A}_{\widetilde{F}})$ satisfying $\Pi_i^\vee \cong \Pi_i^c$ such that $\Pi := \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is a weak base change of π. More precisely, at every finite split place $v = ww^c$ of $\widetilde{F}^+$ we have $|\mathrm{LJ}_{G(\widetilde{F}_w)}|(\Pi_w) \cong \pi_v$ as representations of $G(\widetilde{F}_w) \cong G(\widetilde{F}_v^+)$, and at infinity the compatibility is as in* [Lab11, Cor. 5.3].

Proposition 6.5.2. *Suppose that $\widetilde{F}/\widetilde{F}^+$ is unramified at all finite places and that Π is a cuspidal automorphic representation of $\mathrm{GL}_{nd}(\mathbb{A}_{\widetilde{F}})$ such that $\Pi^\vee \cong \Pi^c$, Π_∞ is cohomological, and Π_w is supercuspidal for all $w \in \Delta(G)$ (in particular $|\mathrm{LJ}_{G(\widetilde{F}_w)}|(\Pi_w) \neq 0$). Then there exists a (cuspidal) automorphic representation π of $G(\mathbb{A}_{\widetilde{F}^+})$ such that at every finite split place $v = ww^c$ of $\widetilde{F}^+$ we have $|\mathrm{LJ}_{G(\widetilde{F}_w)}|(\Pi_w) \cong \pi_v$ as representations of $G(\widetilde{F}_w) \cong G(\widetilde{F}_v^+)$.*

6.6. **Supersingular representations of $\mathrm{GL}_n(D)$.** We now prove the existence of supersingular representations of $\mathrm{GL}_n(D)$ and $\mathrm{PGL}_n(D)$.

Theorem 6.6.1. *Suppose that C is algebraically closed of characteristic p. For any smooth character $\zeta : F^\times \to C^\times$ there exists an irreducible admissible supercuspidal C-representation of $\mathrm{GL}_n(D)$ with central character ζ. In particular, there exists an irreducible admissible supercuspidal C-representation of $\mathrm{PGL}_n(D)$.*

Corollary 6.6.2. *If C is any field of characteristic p, then $\mathrm{PGL}_n(D)$ admits an irreducible admissible supercuspidal representation over C.*

The proof uses Galois representations associated to automorphic representations on certain unitary groups. We now make a few relevant definitions in preparation for the proof.

As in §6.3 we fix an isomorphism $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Recall that if $\widetilde{F}/\widetilde{F}^+$ is a CM extension of number fields and Π is a regular algebraic cuspidal polarizable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\widetilde{F}})$ (in the sense of [BLGGT14b, §2.1]) we have an associated semisimple potentially semistable p-adic Galois representation $r_{p,\imath}(\Pi) : \Gamma_{\widetilde{F}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ that satisfies and is determined by local-global compatibility with Π at all finite places [BLGGT14b, Thm. 2.1.1], [BLGGT14a].

Suppose that $\widetilde{F}^+ \neq \mathbb{Q}$ and that G is a unitary group over $\widetilde{F}^+$ as in §6.5. If π is an automorphic representation of $G(\mathbb{A}_{\widetilde{F}^+})$, then its weak base change $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ of Proposition 6.5.1 is regular algebraic and each Π_i is polarizable. By the Moeglin–Waldspurger classification of the discrete spectrum and the previous paragraph it follows that Π has an associated semisimple potentially semistable p-adic Galois representation $r_{p,\imath}(\pi) = r_{p,\imath}(\Pi) : \Gamma_{\widetilde{F}} \to \mathrm{GL}_{nd}(\overline{\mathbb{Q}}_p)$ that satisfies and is determined by local-global compatibility with π at all finite places of $\widetilde{F}$ that split over $\widetilde{F}^+$ and are not contained in $\Delta(G)$. (We note that the Chebotarev density theorem shows that the set of Frobenius elements at places w of $\widetilde{F}$ that split over $\widetilde{F}^+$ is dense in $\Gamma_{\widetilde{F}}$.) In particular, if Π is not cuspidal, then $r_{p,\imath}(\pi)$ is reducible.

Proof of Theorem 6.6.1.

Step 0: We show that it suffices to prove the theorem when $C = \overline{\mathbb{F}}_p$.

Given a smooth character $\zeta : F^\times \to C^\times$ we can define $\zeta' : F^\times \to \overline{\mathbb{F}}_p^\times$ by extending $\zeta|_{\mathcal{O}_F^\times}$ (which is of finite order and hence takes values in $\overline{\mathbb{F}}_p^\times$) arbitrarily. If Theorem 6.6.1 holds over $\overline{\mathbb{F}}_p$, there exists an irreducible admissible supercuspidal $\overline{\mathbb{F}}_p$-representation π of $\mathrm{GL}_n(D)$ with central character ζ'. Then by Step 3 of the proof of Proposition 3.2.1 there exists an irreducible admissible supercuspidal C-representation π' of $\mathrm{GL}_n(D)$ with central character ζ'. As C is algebraically closed, a suitable unramified twist of π' has central character ζ.

We will assume from now on that $C = \overline{\mathbb{F}}_p$.

Step 1: We find a CM field $\widetilde{F}$ with maximal totally real subfield $\widetilde{F}^+ \neq \mathbb{Q}$ and a place $v \mid p$ of $\widetilde{F}^+$ such that

(i) $\widetilde{F}/\widetilde{F}^+$ is unramified at all finite places;
(ii) any place of $\widetilde{F}^+$ that divides p splits in $\widetilde{F}$;
(iii) $\widetilde{F}^+_v \cong F$;

and a cyclic totally real extension $L^+/\widetilde{F}^+$ of degree nd in which v is inert.

By Krasner's lemma we can find a totally real number field H, a place u of H, and an isomorphism $H_u \xrightarrow{\sim} F$. Now we apply [Hen83, Lemma 3.6] and its proof to find finite totally real extensions $L^+/\widetilde{F}^+/H$ and a place v of $\widetilde{F}^+$ above u such that $L^+/\widetilde{F}^+$ is cyclic of degree nd, $\widetilde{F}^+_v = H_u$, and v is inert in L^+. (We briefly recall the proof: pick a monic polynomial Q of degree nd over F whose splitting field is the unramified extension of degree nd. Then let L^+ be the splitting field of a monic polynomial P over H that is u-adically very close to Q and let $\widetilde{F}^+$ be the decomposition field of some place above u. We can use sign changes of P at real places to ensure that L^+ is totally real.)

Now pick any totally imaginary quadratic extension $\widetilde{F}/\widetilde{F}^+$ in which any place dividing p splits. By [CHT08, Lemma 4.1.2] we can find a finite solvable Galois totally real extension $K^+/\widetilde{F}^+$ that is linearly disjoint from $L^+/\widetilde{F}^+$, such that v splits in K^+, and such that for any prime v' of $\widetilde{F}^+$ that ramifies in $\widetilde{F}$ and any prime w' of K^+ above v' the extension $K^+_{w'}/\widetilde{F}^+_{v'}$ is isomorphic to the extension $\widetilde{F}_{v'}/\widetilde{F}^+_{v'}$. Then we can replace $\widetilde{F}/\widetilde{F}^+$ by $K^+\widetilde{F}/K^+$, L^+ by K^+L^+, and v by any place of K^+ lying above v to ensure that, without loss of generality, $\widetilde{F}/\widetilde{F}^+$ is unramified at all finite places. (In particular, we can always achieve $\widetilde{F}^+ \neq \mathbb{Q}$ in this way.)

We let w denote a place of $\widetilde{F}$ lying over v and fix an isomorphism of topological fields $\widetilde{F}_w \xrightarrow{\sim} F$. We let $L := L^+\widetilde{F}$ and let c denote the unique complex conjugation of L.

Step 2: Letting $v_1 \nmid p$ denote any place of $\widetilde{F}^+$ that is inert in L^+ and splits in $\widetilde{F}$, we now find a unitary group G over $\widetilde{F}^+$ such that

(i) $G_{/\widetilde{F}}$ is an inner form of GL_{nd};
(ii) $G(\widetilde{F}^+_u)$ is compact for any place $u \mid \infty$ of $\widetilde{F}^+$;
(iii) $G(\widetilde{F}_w) \cong \mathrm{GL}_n(D)$;
(iv) G is quasi-split at all finite places not contained in $\{v, v_1\}$.

Let G^* denote the unique quasi-split outer form of GL_{nd} over $\widetilde{F}^+$ that splits over $\widetilde{F}$. By [Clo91, §2] we can find an inner form G of G^* that satisfies all the above conditions. (If nd is odd we do not need the auxiliary place v_1. If nd is even we use v_1 to ensure our local conditions can be globally realized.)

The set $\Delta^+(G)$ (defined in §6.5) contains v if $d > 1$ and is contained in $\{v, v_1\}$. Any place of $\Delta(G)$ is inert in L and splits over $\widetilde{F}^+$, and the set $\Delta_L(G)$ of places of L lying above $\Delta(G)$ is in canonical bijection with $\Delta(G)$.

For any finite place $v' \notin \Delta^+(G)$ of $\widetilde{F}^+$ that splits as $v' = w'w'^c$ in $\widetilde{F}$ we obtain an isomorphism $\iota_{w'} : G(\widetilde{F}^+_{v'}) = G(\widetilde{F}_{w'}) \xrightarrow{\sim} \mathrm{GL}_{nd}(\widetilde{F}_{w'})$ that is unique up to conjugacy. Moreover, $c \circ \iota_{w'}$ and $\iota_{w'^c}$ differ by an outer automorphism of $\mathrm{GL}_{nd}(\widetilde{F}_{w'^c})$. We also fix an isomorphism $\iota_w : G(\widetilde{F}^+_v) = G(\widetilde{F}_w) \xrightarrow{\sim} \mathrm{GL}_n(D)$. (It is canonical, up to conjugacy, by condition (i).)

Step 3: We find an algebraic Hecke character $\chi : \mathbb{A}_L^\times/L^\times \to \mathbb{C}^\times$ with associated potentially crystalline p-adic Galois representation $\psi = r_{p,\imath}(\chi) : \Gamma_L \to \overline{\mathbb{Q}}_p^\times$ (cf. [CHT08, Lemma 4.1.3]) such that

(i) $\psi\psi^c = \varepsilon^{-(nd-1)}$;
(ii) for any place $w' \in \Delta_L(G)$ the induced representation $\mathrm{Ind}_{W_{L_{w'}}}^{W_{\widetilde{F}_{w'}}} \chi_{w'}$ is irreducible;
(iii) the representation $r := \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi$ has regular Hodge–Tate weights, i.e., for each $\kappa' : \widetilde{F} \to \overline{\mathbb{Q}}_p$ the nd integers $\mathrm{HT}_{\kappa'}(\mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi)$ are pairwise distinct;
(iv) the restriction $\overline{r}|_{\Gamma_{\widetilde{F}_w}}$ to $\Gamma_{\widetilde{F}_w}$ of the reduction $\overline{r} \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \overline{\psi}$ is irreducible.

We first introduce some notation. Let Δ_p denote the places w' of L that divide p. Note that, by construction, any place $w' \in \Delta_L(G) \cup \Delta_p$ splits over L^+, i.e. $w' \neq w'^c$. Let $S_K := \mathrm{Hom}_{\mathrm{cts}}(K, \overline{\mathbb{Q}}_p)$ for any topological field K of characteristic zero and $S_k := \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ for any field k of characteristic p.

Our strategy is to carefully choose continuous characters $\theta_{w'} : \Gamma_{L_{w'}} \to \overline{\mathbb{Q}}_p^\times$ for any $w' \in \Delta_L(G) \cup \Delta_p$ that satisfy $(\theta_{w'}\theta^c_{w'^c})|_{I_{L_{w'}}} = \varepsilon^{-(nd-1)}|_{I_{L_{w'}}}$ and are potentially crystalline when $w' \in \Delta_p$. We then deduce by [BLGGT14b, Lemma A.2.5(1)] that there exists a character $\psi : \Gamma_L \to \overline{\mathbb{Q}}_p^\times$ such that $\psi\psi^c = \varepsilon^{-(nd-1)}$ and $\psi|_{I_{L_{w'}}} = \theta_{w'}|_{I_{L_{w'}}}$ for all $w' \in \Delta_L(G) \cup \Delta_p$. In particular, ψ is potentially crystalline, and we let χ be the associated algebraic Hecke character. It follows that condition (i) holds.

For any $w' \in \Delta_L(G)$ we can choose a smooth character $\zeta_{w'} : \Gamma^{\mathrm{ab}}_{L_{w'}} \cong \widehat{L^\times_{w'}} \to \overline{\mathbb{Q}}_p^\times$ such that the $\mathrm{Gal}(L_{w'}/\widetilde{F}_{w'})$-conjugates of $\zeta_{w'}|_{\mathcal{O}^\times_{L_{w'}}}$ are pairwise distinct. (For example, we can take a faithful character of $k^\times_{L_{w'}}$ and inflate it to $\mathcal{O}^\times_{L_{w'}}$.) We may assume without loss of generality that $\zeta_{w'}\zeta^c_{w'^c} = 1$.

Now suppose that $w' \in \Delta_p$. Suppose that we are given any integers λ_κ ($\kappa \in S_L$) satisfying $\lambda_\kappa + \lambda_{\kappa c} = nd - 1$ for all $\kappa \in S_L$. Let $\theta^{\mathrm{cr}}_{w'} : \Gamma_{L_{w'}} \to \overline{\mathbb{Q}}_p^\times$ be any crystalline character with $\mathrm{HT}_\kappa(\theta^{\mathrm{cr}}_{w'}) = \lambda_\kappa$ for all $\kappa \in S_{L_{w'}} \subset S_L$. Without loss of generality, by our constraint on the λ_κ, we may assume that $\theta^{\mathrm{cr}}_{w'}(\theta^{\mathrm{cr}}_{w'^c})^c = \varepsilon^{-(nd-1)}$.

For $w' \in \Delta_L(G) \cup \Delta_p$ define

$$\theta_{w'} := \begin{cases} \zeta_{w'} & \text{if } w' \in \Delta_L(G) - \Delta_p; \\ \theta^{\mathrm{cr}}_{w'}\zeta_{w'} & \text{if } w' \in \Delta_L(G) \cap \Delta_p; \\ \theta^{\mathrm{cr}}_{w'} & \text{if } w' \in \Delta_p - \Delta_L(G). \end{cases}$$

This completes the construction of a potentially crystalline character ψ and its associated algebraic Hecke character χ. By construction, for any $w' \in \Delta_L(G)$ the character $\imath\zeta_{w'}|_{I_{L_{w'}}}$ corresponds to $\chi_{w'}|_{\mathcal{O}^\times_{L_{w'}}}$ under the local Artin map. Therefore, since the $\mathrm{Gal}(L_{w'}/\widetilde{F}_{w'})$-conjugates of $\zeta_{w'}|_{\mathcal{O}^\times_{L_{w'}}}$ are pairwise distinct, we deduce that condition (ii) holds.

Finally, we will choose the integers λ_κ ($\kappa \in S_L$) so that conditions (iii) and (iv) hold. Note that condition (iii) is equivalent to the condition

(iii′) for any $\kappa' \in S_{\widetilde{F}}$ the nd integers $\{\lambda_\kappa : \kappa \in S_L, \kappa|_{\widetilde{F}} = \kappa'\}$ are pairwise distinct.

First choose the λ_κ for those $\kappa \in S_L$ that do not induce either of the places w, w^c on L so that condition (iii′) holds for any $\kappa' \in S_{\widetilde{F}}$ not inducing either of the places w, w^c on $\widetilde{F}$. It remains to choose the λ_κ for those κ that induce the place w on L (since the remaining λ_κ are determined by the condition $\lambda_\kappa + \lambda_{\kappa c} = nd - 1$ for all κ), i.e. for $\kappa \in S_{L_w}$.

To choose the λ_κ for $\kappa \in S_{L_w}$, we note that $\overline{r}|_{\Gamma_{\widetilde{F}_w}} \cong \mathrm{Ind}_{\Gamma_{L_w}}^{\Gamma_{\widetilde{F}_w}}(\overline{\psi}|_{\Gamma_{L_w}})$ is irreducible if and only if the $\mathrm{Gal}(L_w/\widetilde{F}_w)$-conjugates of $\overline{\psi}|_{\Gamma_{L_w}}$ are pairwise distinct, or equivalently if the characters $\overline{\psi}|^{q^i}_{I_{L_w}}$ ($0 \leq i \leq nd-1$) are pairwise distinct. (Recall that $q = \#k_F$.) We have $\overline{\psi}|_{I_{L_w}} \cong \overline{\theta^{\mathrm{cr}}_w\zeta_w}|_{I_{L_w}}$ if $d > 1$ or $\overline{\psi}|_{I_{L_w}} \cong \overline{\theta^{\mathrm{cr}}_w}|_{I_{L_w}}$ otherwise. By [GHS18, Cor. 7.1.2], noting our opposite conventions concerning Hodge–Tate weights, we have $\overline{\theta_w^{\mathrm{cr}}}|_{I_{L_w}} = \prod_{\sigma \in S_{k_{L_w}}} \omega_\sigma^{-b_\sigma}$, where ω_σ corresponds to the character $\mathcal{O}_{L_w}^\times \twoheadrightarrow k_{L_w}^\times \xrightarrow{\sigma} \overline{\mathbb{F}}_p^\times$ under local class field theory and $b_\sigma := \sum_{\kappa \in S_{L_w} : \overline{\kappa} = \sigma} \lambda_\kappa$. Choose the λ_κ for $\kappa \in S_{L_w}$ so that $\overline{\psi}|_{I_{L_w}} = \omega_\sigma$. As the $\omega_\sigma^{q^i}$ $(i = 0, 1, \dots, nd-1)$ are pairwise distinct, condition (iv) holds. Finally, we can ensure that condition (iii′) holds for all $\kappa' \in S_{\widetilde{F}_w}$ while keeping $\overline{r}|_{I_{\widetilde{F}_w}}$ unchanged by varying the λ_κ (for $\kappa \in S_{L_w}$) modulo $q^{nd} - 1$. This completes Step 3.

Step 4: Using automorphic induction and descent we define an automorphic representation π' of $G(\mathbb{A}_{\widetilde{F}^+})$ with associated Galois representation $r = \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi$.

Let Π'' denote the automorphic induction of χ with respect to the cyclic extension $L/\widetilde{F}$. It is an automorphic representation of $\mathrm{GL}_{nd}(\mathbb{A}_{\widetilde{F}})$ that is parabolically induced from a cuspidal representation. (For the functoriality of automorphic induction in cyclic extensions we refer to [Hen12], which shows in particular that it is compatible with local automorphic induction at all places. Note the results of [Hen12] apply to unitary representations, but by twisting they continue to hold for twists of unitary representations.)

We claim that Π'' is cuspidal. This follows from [Hen12], Theorems 2, 3, and Proposition 2.5, provided that the Hecke characters $\{\chi^\sigma : \sigma \in \mathrm{Gal}(L/\widetilde{F})\}$ are pairwise distinct. Equivalently, the Galois characters $\{\psi^\sigma : \sigma \in \mathrm{Gal}(L/\widetilde{F})\}$ are pairwise distinct, which in turn is equivalent to the condition that $\mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi$ is irreducible. This is a consequence of condition (iv) in Step 3, so Π'' is cuspidal.

Let $\Pi' := \Pi'' \otimes |\det|_{\widetilde{F}}^{(nd-1)/2}$. By condition (i) in Step 3 we have $\chi\chi^c = |\cdot|_L^{-(nd-1)}$, hence $(\Pi')^\vee \cong \Pi'^c$. On the other hand, Π'_∞ is cohomological by [Clo90, Lemma 3.14], as it is regular by condition (iii′) in Step 3. It follows that Π' is regular algebraic and polarizable in the sense of [BLGGT14b, §2.1], so we have an associated Galois representation $r_{p,\imath}(\Pi')$. By local-global compatibility at unramified places and Chebotarev we deduce that $r_{p,\imath}(\Pi') \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi$.

For $w' \in \Delta(G)$ the local factor $\Pi'_{w'}$ is supercuspidal, as $\mathrm{rec}_{\widetilde{F}_{w'}}(\Pi'_{w'})$ is irreducible by condition (ii) in Step 3. It follows from what we recalled in §6.2 that $|\,\mathrm{LJ}_{G(\widetilde{F}_{w'})}|(\Pi'_{w'}) \neq 0$.

By Proposition 6.5.2 we deduce that Π' descends to a (cuspidal) automorphic representation π' of $G(\mathbb{A}_{\widetilde{F}^+})$, such that for all finite places $v' \notin \Delta^+(G)$ of $\widetilde{F}^+$ that split as $v' = w'w'^c$ in $\widetilde{F}$ we have $\pi'_{v'} \cong \Pi'_{w'}$ as representations of $G(\widetilde{F}^+_{v'}) \cong \mathrm{GL}_{nd}(\widetilde{F}_{w'})$. We deduce that $r_{p,\imath}(\pi') \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\widetilde{F}}} \psi$.

Step 5: We use the automorphic representation π' to define an irreducible admissible $\overline{\mathbb{F}}_p$-representation σ of $G(\widetilde{F}^+_v) \cong \mathrm{GL}_n(D)$.

Fix a maximal compact open subgroup K_p of $\prod_{v'|p} G(\widetilde{F}^+_{v'})$. If U is any compact open subgroup of $K_p G(\mathbb{A}_{\widetilde{F}^+}^{\infty,p})$ and $\mathcal{W}$ is any $\overline{\mathbb{Z}}_p[K_p]$-module, we let $S(U, \mathcal{W})$ be the $\overline{\mathbb{Z}}_p$-module of functions $f : G(\widetilde{F}^+)\backslash G(\mathbb{A}_{\widetilde{F}^+}^\infty) \to \mathcal{W}$ such that $f(gu) = u_p^{-1} f(g)$ for all $g \in G(\mathbb{A}_{\widetilde{F}^+}^\infty)$ and $u \in U$ (where u_p denotes the projection of u to K_p).

Using the compactness of G at infinity, we see as in [EGH13, Lemma 7.1.6] that there exists a $\mathbb{Q}_p$-algebraic representation $\mathcal{W}_{\mathrm{alg}}$ of $\prod_{v'|p} G(\widetilde{F}^+_{v'})$ over $\overline{\mathbb{Q}}_p$ such that $\varinjlim_U S(U, \mathcal{W}_{\mathrm{alg}})$ contains $\imath^{-1}\pi'^{\infty}$ as $G(\mathbb{A}^{\infty}_{\widetilde{F}^+})$-representation. Choose a K_p-invariant $\overline{\mathbb{Z}}_p$-lattice $\mathcal{W}^{\circ}_{\mathrm{alg}}$ in $\mathcal{W}_{\mathrm{alg}}$ and let $\overline{\mathcal{W}_{\mathrm{alg}}} := \mathcal{W}^{\circ}_{\mathrm{alg}} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$.

Pick a compact open subgroup $U = \prod_{v'\nmid\infty} U_{v'}$ of $G(\mathbb{A}^{\infty}_{\widetilde{F}^+})$ such that

(i) $(\pi'^{\infty})^U \neq 0$;
(ii) there exists a place $v' \nmid p\infty$ of $\widetilde{F}^+$ such that $U_{v'}$ contains no element of finite order other than the identity;
(iii) the group $\prod_{v'|p} U_{v'}$ is contained in K_p and acts trivially on $\overline{\mathcal{W}_{\mathrm{alg}}}$.

Note that condition (ii) implies that for any compact open subgroup $U' = U'_p \prod_{v'\nmid p\infty} U_{v'}$ with $U'_p \leq K_p$ we have $S(U', \mathcal{W}) \cong \mathcal{W}^{\oplus s}$ as $\overline{\mathbb{Z}}_p$-modules for some $s \geq 1$ depending only on U'. In particular,

$$S(U', \mathcal{W}) \otimes_{\overline{\mathbb{Z}}_p} R \xrightarrow{\sim} S(U', \mathcal{W} \otimes_{\overline{\mathbb{Z}}_p} R) \tag{6.6.3}$$

for any $\overline{\mathbb{Z}}_p$-algebra R (see e.g. [EGH13, §7.1.2]). We will apply this with $R = \overline{\mathbb{Q}}_p$ and $R = \overline{\mathbb{F}}_p$.

Let $\mathcal{P}$ denote the set of places $w' \nmid p$ of $\widetilde{F}$ that split over a place v' of $\widetilde{F}^+$ not contained in $\Delta^+(G)$, and are such that $U_{v'}$ is a maximal compact subgroup of $G(\widetilde{F}^+_{v'})$. For each such w' we conjugate the isomorphism $\iota_{w'}$ of Step 2 so that $\iota_{w'}(U_{v'}) = \mathrm{GL}_{nd}(\mathcal{O}_{\widetilde{F}_{w'}})$. Note that the set $\mathcal{P}$ has finite complement in the set of places of $\widetilde{F}$ that split over $\widetilde{F}^+$. Let $\mathbb{T}^{\mathcal{P}}$ denote the commutative polynomial $\overline{\mathbb{Z}}_p$-algebra in the variables $T^{(i)}_{w'}$ for $w' \in \mathcal{P}$ and $0 \leq i \leq nd$, acting on any $S(U, \mathcal{W})$ as double coset operators as in [EGH13, §7.1.2]. Note that the ring $\mathbb{T}^{\mathcal{P}}$ acts by scalars on $(\imath^{-1}\pi'^{\infty})^U$ inside $S(U, \mathcal{W}_{\mathrm{alg}})$ and stabilizes the $\overline{\mathbb{Z}}_p$-lattice $S(U, \mathcal{W}^{\circ}_{\mathrm{alg}})$. Therefore there exists a unique maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{\mathcal{P}}$ with residue field $\overline{\mathbb{F}}_p$ such that $(\imath^{-1}\pi'^{\infty})^U \subset S(U, \mathcal{W}_{\mathrm{alg}})_{\mathfrak{m}}$.

Applying (6.6.3) and localizing at $\mathfrak{m}$ we obtain that $S(U, \overline{\mathcal{W}_{\mathrm{alg}}})_{\mathfrak{m}} \neq 0$. Then

$$S(U, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \otimes_{\overline{\mathbb{F}}_p} \overline{\mathcal{W}_{\mathrm{alg}}} \cong S(U, \overline{\mathcal{W}_{\mathrm{alg}}})_{\mathfrak{m}} \neq 0,$$

where the isomorphism uses condition (iii) on U. Writing $U^v := \prod_{v'\neq v} U_{v'}$ and $S(U^v, \overline{\mathbb{F}}_p) := \varinjlim_{U_v} S(U^v U_v, \overline{\mathbb{F}}_p)$, we have $S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$. This is a non-zero admissible smooth representation of $G(\widetilde{F}^+_v) \cong \mathrm{GL}_n(D)$, using the isomorphism ι_w of Step 2. Let σ be an irreducible (admissible) $\mathrm{GL}_n(D)$-subrepresentation of $S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, which exists by the proof of Lemma 9.9 in [Her11] or [HV12, Lemma 7.10].

Step 6: We show that σ is supersingular, or equivalently, supercuspidal.

By [OV18, Thm. 3] it suffices to show that the $\mathcal{H}_{\overline{\mathbb{F}}_p}$-module $\sigma^{I(1)}$ is supersingular, where $I(1)$ denotes the pro-p Iwahori subgroup of $\mathrm{GL}_n(D) \cong G(\widetilde{F}^+_v)$ defined in §6.3. In fact, we will even show that $(S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{I(1)} \cong S(U^v \cdot I(1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is supersingular. Assume by contradiction that this is false, so one of the operators $\mathscr{Z}_j$ for $1 \leq j \leq n-1$ has a nonzero eigenvalue λ_j on $S(U^v \cdot I(1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$.

Again from (6.6.3) we know that $S(U^v \cdot I(1), \overline{\mathbb{Z}}_p) \otimes_{\overline{\mathbb{Z}}_p} R \cong S(U^v \cdot I(1), R)$ for $R = \overline{\mathbb{Q}}_p$ and $R = \overline{\mathbb{F}}_p$. By applying [EGH13, Lemma 4.5.1] (a version of the Deligne–Serre lemma) with $A = \mathbb{T}^{\mathcal{P}}[\mathscr{Z}_j]$, $M = S(U^v \cdot I(1), \overline{\mathbb{Z}}_p)$, $\mathfrak{n}$ the maximal ideal of A generated by $\mathfrak{m}$ and $\mathscr{X}_j - \lambda_j$, we deduce that there exists a homomorphism $\theta : A \to \overline{\mathbb{Z}}_p$ such that the θ-eigenspace of $S(U^v \cdot I(1), \overline{\mathbb{Q}}_p)$ is non-zero, $\ker(\overline{\theta}|_{\mathbb{T}^{\mathcal{P}}}) = \mathfrak{m}$, and $\theta(\mathscr{X}_j) \in \overline{\mathbb{Z}}_p^\times$. By [EGH13, Lemma 7.1.6], there exists an automorphic representation π of $G(\mathbb{A}_{\widetilde{F}^+})$ satisfying

(i) $(\imath^{-1}\pi^\infty)^{U^v \cdot I(1)}$ has a non-trivial θ-eigenspace;
(ii) π_∞ is trivial.

It follows from (i) that $\imath^{-1}\pi_v^{I(1)} \neq 0$ is an integral $\mathcal{H}_{\overline{\mathbb{Q}}_p}$-module whose reduction is non-supersingular. (A priori, we get that $(\imath^{-1}\pi_v^{I(1)})^{\oplus s}$ is integral for some $s \geq 1$, but then we can project to any factor. Note that any finitely generated submodule of a finite free $\overline{\mathbb{Z}}_p$-module is free.)

By local-global compatibility and [CHT08, Cor. 3.1.2], for any $w' \in \mathcal{P}$ the characteristic polynomial of $\overline{r}(\mathrm{Frob}_{w'})$ equals $\sum_{i=0}^{nd} (-1)^i (\mathbf{N}w')^{i(i-1)/2} T_{w'}^{(i)} X^{nd-i}$ modulo $\mathfrak{m}$, where $\mathrm{Frob}_{w'}$ denotes a geometric Frobenius element at w'. The same is true for $\overline{r}_{p,\imath}(\pi)$, as $\ker(\overline{\theta}|_{\mathbb{T}^{\mathcal{P}}}) = \mathfrak{m}$, and hence we deduce by the Chebotarev density theorem that $\overline{r_{p,\imath}(\pi)} \cong \overline{r}$.

By Proposition 6.5.1 we obtain an automorphic representation Π of $\mathrm{GL}_{nd}(\mathbb{A}_{\widetilde{F}})$ with associated Galois representation $r_{p,\imath}(\Pi)$ lifting $\overline{r}$ such that $|\,\mathrm{LJ}_{G(\widetilde{F}_{w'})}|(\Pi_{w'}) \cong \pi_{v'}$ for all finite places v' of $\widetilde{F}^+$ that split as $v' = w'w'^c$ in $\widetilde{F}$. As $\overline{r}$ is irreducible by construction we know that Π is cuspidal. In particular, $\Pi_{w'}$ is essentially unitarizable and generic for each finite place w' of $\widetilde{F}$. Let $v_1 \leq \cdots \leq v_{nd}$ denote the valuations of the eigenvalues of a geometric Frobenius on $\imath^{-1}(\mathrm{rec}_F(\Pi_w))$. From Proposition 6.3.7 (applied to Π_w) we deduce that there exists $1 \leq j \leq n-1$ such that

$$\sum_{i=1}^{jd} v_i = -\frac{d^2 j(n-j)}{2}\,\mathrm{val}(q). \tag{6.6.4}$$

Note that the infinitesimal character of Π is the same as that of the trivial representation. By [BLGGT14b, Thm. 2.1.1] we deduce that $\mathrm{HT}_\tau(r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}}) = \{0, 1, \dots, nd-1\}$ for all $\tau : \widetilde{F}_w \to \overline{\mathbb{Q}}_p$ and that $\imath\mathrm{WD}(r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}_F(\Pi_w \otimes |\det|_F^{(1-nd)/2})$. Together with (6.6.4) it follows that

$$\sum_{i=1}^{jd} v_i' = -\frac{d^2 j(n-j)}{2}\,\mathrm{val}(q) + jd\,\mathrm{val}(q^{(nd-1)/2}) = \binom{jd}{2}\mathrm{val}(q),$$

where $v_1' \leq \cdots \leq v_{nd}'$ denote the valuations of the eigenvalues of a geometric Frobenius on $\mathrm{WD}(r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}})$. By Lemma 6.4.1, noting that $\mathrm{val}(q) = [F_0 : \mathbb{Q}_p]$, it follows that $r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}}$ is reducible, which contradicts that its reduction $\overline{r}|_{\Gamma_{\widetilde{F}_w}}$ is irreducible by Step 3.

Step 7: We fix the central character.

Suppose we are given a smooth character $\zeta : F^\times \to \overline{\mathbb{F}}_p^\times$. As in Step 0 it is enough to construct an irreducible admissible supercuspidal representation such that $\mathcal{O}_F^\times$ acts via $\zeta|_{\mathcal{O}_F^\times}$.

Note that σ has a central character χ_σ, as it is irreducible and admissible. We claim that $\chi_\sigma|_{\mathcal{O}_F^\times} = \det(\overline{r}|_{I_{\widetilde{F}_w}}) \cdot \overline{\varepsilon}^{nd(nd-1)/2}$ under the local Artin map. The central character of the $\mathrm{GL}_n(D)$-representation $\imath^{-1}\pi_v$ in Step 6 lifts χ_σ and is equal to the central character of $\imath^{-1}\Pi_w$. (This equality follows from the definition of LJ in [Bad08, §2.7], noting that Π_w is generic and hence fully induced from an essentially square-integrable representation.) By local-global compatibility at p (cf. Step 6) the latter character equals $\mathrm{WD}(\det r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}})|_{I_{\widetilde{F}_w}}$ on $\mathcal{O}_F^\times$, under the local Artin map. As $r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}}$ has parallel Hodge–Tate weights $0, 1, \dots, nd-1$, we have $\det r_{p,\imath}(\Pi)|_{I_{\widetilde{F}_w}} = \varepsilon^{-nd(nd-1)/2} \cdot \mathrm{WD}(\det r_{p,\imath}(\Pi)|_{\Gamma_{\widetilde{F}_w}})|_{I_{\widetilde{F}_w}}$ and hence deduce the claim.

It thus suffices to show that in Step 3 above we can choose r such that $\det(\overline{r}|_{I_{\widetilde{F}_w}})$ is any prescribed character that is extendable to $\Gamma_{\widetilde{F}_w}$. Let us fix any $\overline{\kappa} \in S_{k_{L_w}}$ and write $\overline{\psi}|_{I_{L_w}} = \omega_{\overline{\kappa}}^s$ for some $s \in \mathbb{Z}$. Then the condition that the $\overline{\psi}|_{I_{L_w}}^{q^i}$ $(i = 0, 1, \dots, nd-1)$ are pairwise distinct means:

$$s \not\equiv 0 \pmod{\tfrac{q^{nd}-1}{q^\ell-1}} \quad \forall \ell \mid nd,\ \ell < nd. \tag{6.6.5}$$

On the other hand, $\det(\overline{r}|_{I_{\widetilde{F}_w}}) = \prod_{i=0}^{nd-1} \overline{\psi}|_{I_{L_w}}^{q^i} = \omega_{\overline{\kappa}'}^s$, where $\overline{\kappa}' \in S_{k_{\widetilde{F}_w}}$ is the restriction of $\overline{\kappa}$ to $k_{\widetilde{F}_w}$. As any character $\Gamma_{\widetilde{F}_w} \to \overline{\mathbb{F}}_p^\times$ restricts to a power of $\omega_{\overline{\kappa}'}$ on inertia, we can prescribe $\det(\overline{r}|_{I_{\widetilde{F}_w}})$ if and only if we can choose s in any residue class modulo $q-1$. Since $\frac{q^{nd}-1}{q^\ell-1} \geq q+1$ for any $\ell \mid nd$, $\ell < nd$, it follows that we can pick any s in the interval $[1, q-1]$, completing the proof. □

Proof of Corollary 6.6.2. Going back to Step 5 of the proof of Theorem 6.6.1, it is clear that the representation $S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$ is defined over a finite field (as $\overline{r}$ is), and hence so is its irreducible subrepresentation σ. This proves the corollary when C is a sufficiently large finite field of characteristic p. We conclude by Proposition 3.2.1. □

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Appendix A. Base change

Sug Woo Shin[1]

In this appendix we will prove Propositions 6.5.1 and 6.5.2.

We need a character identity for the Jacquet–Langlands correspondence. We fix compatible Haar measures on $\mathrm{GL}_{nd}(F)$ and $\mathrm{GL}_n(D)$ in the sense of [Kot88, p.631]. We say that $f \in C_c^\infty(\mathrm{GL}_n(D))$ and $f^* \in C_c^\infty(\mathrm{GL}_{nd}(F))$ are associated, or that f^* is a transfer of f, if the orbital integral identity $O_\delta(f) = O_{\delta^*}(f^*)$ holds for every regular

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semisimple elements $\delta \in \mathrm{GL}_n(D)$ and $\delta^* \in \mathrm{GL}_{nd}(F)$ with the same characteristic polynomial. (We use the same Haar measures on the centralizers of δ and δ^* in $\mathrm{GL}_n(D)$ and $\mathrm{GL}_{nd}(F)$, respectively, to compute the orbital integrals.) A well-known fact, proven in [DKV84], is that every $f \in C_c^\infty(\mathrm{GL}_n(D))$ admits a transfer in $C_c^\infty(\mathrm{GL}_{nd}(F))$. (This is a special case of the Langlands–Shelstad transfer.) Let $e(\mathrm{GL}_n(D)) \in \{\pm 1\}$ denote the Kottwitz sign [Kot83]. Explicitly $e(\mathrm{GL}_n(D)) = (-1)^{nd-n}$.

Proposition A.0.1. *Let π^* be an irreducible unitarizable representation of $\mathrm{GL}_{nd}(F)$. For every associated pair $f \in C_c^\infty(\mathrm{GL}_n(D))$ and $f^* \in C_c^\infty(\mathrm{GL}_{nd}(F))$, we have*

$$\operatorname{tr}\pi^*(f^*) = e(\mathrm{GL}_n(D)) \cdot \operatorname{tr}\left(|\operatorname{LJ}_{\mathrm{GL}_n(D)}|(\pi^*)\right)(f).$$

Proof. This follows from [Bad07, Prop 3.3] and the Weyl integration formula [DKV84, A.3.f] for $\mathrm{GL}_n(D)$ and $\mathrm{GL}_{nd}(F)$. □

We assume that the CM extension $\widetilde{F}/\widetilde{F}^+$ and the unitary group G over $\widetilde{F}^+$ are as in Section 6.5.

Write G^* for a quasi-split inner twist of G over $\widetilde{F}^+$ (with an isomorphism between G^* and G over an algebraic closure of $\widetilde{F}^+$). By convention, every trace considered on p-adic or adelic points of G^* over $\widetilde{F}$ (as opposed to over $\widetilde{F}^+$) will mean the *twisted* trace relative to the action of $\mathrm{Gal}(\widetilde{F}/\widetilde{F}^+)$ on $\mathrm{Res}_{\widetilde{F}/\widetilde{F}^+} G^*$ (with the Whittaker normalization), unless specified otherwise.

Proof of Proposition 6.5.1. This proposition is implied by [Lab11, Cor. 5.3] except possibly the relation $|\operatorname{LJ}_{G(\widetilde{F}_w)}|(\Pi_w) \cong \pi_v$.[2] We elaborate on this point. Thus we assume $v = ww^c$ as in the proposition. We will omit the subscript for $|\operatorname{LJ}|$ when there is little danger of confusion.

Let S be a finite set of places of $\widetilde{F}^+$ containing all infinite places as well as all finite place where either π or G is ramified. Denote by S_{f} the subset of finite places in S. In particular $S_{\mathrm{f}} \supset \Delta^+(G)$. For an irreducible admissible representation σ of $G(\mathbb{A}_{\widetilde{F}^+})$ unramified outside S, we write $\mathrm{BC}(\sigma^S) = \Pi^S$ to mean that the local unramified base change of σ_u is Π_u at all places $u \notin S$. (The unramified base change is defined via the Satake transform.) Using the Langlands parametrization at archimedean places, we write $\mathrm{BC}(\sigma_\infty) = \Pi_\infty$ to mean that the local base change of σ_∞ is Π_∞.

For each finite place u and $f_u \in C_c^\infty(G(\widetilde{F}_u^+))$ let $f_u^* \in C_c^\infty(G^*(\widetilde{F}_u^+))$ denote a transfer. There exists $\phi_u \in C_c^\infty(G^*(\widetilde{F} \otimes_{\widetilde{F}^+} \widetilde{F}_u^+))$ whose base change transfer is f_u^* by [Lab11, Lem. 4.1]. Write $f_{S_{\mathrm{f}}} := \prod_{u \in S_{\mathrm{f}}} f_u$ and $\phi_{S_{\mathrm{f}}} := \prod_{u \in S_{\mathrm{f}}} \phi_u$.

Let $\Pi_v := \Pi_w \otimes \Pi_{w^c}$ be the v-component of Π, which is a representation of $G^*(\widetilde{F} \otimes_{\widetilde{F}^+} \widetilde{F}_v^+)$. Let $\pi_v^* := \Pi_w$ via the isomorphism $G^*(\widetilde{F}_v^+) \cong G^*(\widetilde{F}_w)$. Then we have the following character identities, where $\operatorname{tr}\Pi_v(\phi_v)$ means the twisted trace by abuse of notation:

$$\operatorname{tr}\Pi_v(\phi_v) = \operatorname{tr}\pi_v^*(f_v^*) = e(G(\widetilde{F}_v^+)) \cdot \operatorname{tr}\left(|\operatorname{LJ}|(\pi_v^*)\right)(f_v). \tag{A.0.2}$$

[2] In fact this assertion is implicit in [Lab11, Cor. 5.3] where it reads "Aux places non ramifiées ou décomposées la correspondance $\sigma_v \mapsto \pi_v$ est donnée par le changement de base local." However when $v = ww^c$ the author introduced the notion of local base change (§4.10 of *op. cit.*) only when $\mathbf{U}$ is a general linear group at v (in his notation). We need the case when $\mathbf{U}$ is a nontrivial inner form of a general linear group at v.

The first equality holds by the same computation as for [Rog90, Prop. 4.13.2 (a)]. The second equality is Proposition A.0.1. On the other hand, the trace formula argument of [Lab11, Thm. 5.1] shows

$$\sum_{\sigma} m(\sigma)\,\mathrm{tr}\,\sigma_{S_{\mathrm{f}}}(f_{S_{\mathrm{f}}}) = c\cdot \mathrm{tr}\,\Pi_{S_{\mathrm{f}}}(\phi_{S_{\mathrm{f}}}), \tag{A.0.3}$$

with a constant c and the automorphic multiplicity $m(\sigma)\in\mathbb{Z}_{\ge 0}$, where the sum runs over σ such that $\mathrm{BC}(\sigma^S)=\Pi^S$ and $\mathrm{BC}(\sigma_\infty)=\Pi_\infty$. Again the trace on the right-hand side is the twisted trace. Since (A.0.3) holds for each $f^\infty=\prod_{u\nmid\infty} f_u$ (and f_u^* and ϕ_u constructed from f_u at each u as above), we choose f_u to be the characteristic function on a sufficiently small compact open subgroup of $G(\widetilde{F}_u^+)$ at $u\in S_{\mathrm{f}}\backslash\{v\}$. Then $\mathrm{tr}\,\sigma_u(f_u)\ge 0$, so we obtain

$$\sum_{\sigma} n(\Pi_v,\sigma)\,\mathrm{tr}\,\sigma_v(f_v) = c'\cdot \mathrm{tr}\,\Pi_v(\phi_v), \quad \text{with } n(\Pi_v,\sigma)\ge 0,$$

where c' is a new constant and the sum runs over σ such that $\mathrm{BC}(\sigma^S)=\Pi^S$, $\mathrm{BC}(\sigma_\infty)=\Pi_\infty$, and $\mathrm{tr}\,\sigma_u(f_u)\neq 0$ at every $u\in S_{\mathrm{f}}\backslash\{v\}$. Notice that $\sigma=\pi$ contributes to the sum with $n(\Pi_v,\pi)>0$, by choice of f_u at $u\in S_{\mathrm{f}}\backslash\{v\}$. By choosing a suitable f_v we deduce that $c'\neq 0$. Substituting (A.0.2) we obtain

$$\sum_{\sigma} n(\Pi_v,\sigma)\,\mathrm{tr}\,\sigma_v(f_v) = c'\cdot e(G(\widetilde{F}_v^+))\cdot \mathrm{tr}\left(|\,\mathrm{LJ}\,|(\pi_v^*)\right)(f_v),$$

with the sum running over the same set of σ. Since the sum is not identically zero, $|\,\mathrm{LJ}\,|(\pi_v^*)$ is irreducible (rather than 0). By linear independence of characters of $G(\widetilde{F}_v^+)$, we deduce that the coefficients on the left-hand side are zero unless $\sigma_v\cong|\,\mathrm{LJ}\,|(\pi_v^*)$. Since $n(\Pi_v,\pi)>0$, we must have $\pi_v\cong|\,\mathrm{LJ}\,|(\pi_v^*)$, noting that no cancellation takes place in the sum as the coefficients are non-negative. □

Proof of Proposition 6.5.2. The proposition would follow from [Lab11, Thm. 5.4] but we need some care since our G is not quasi-split[3]; we also need some more information at split places. Thus we sketch the trace formula argument. Again we drop the subscript from $|\,\mathrm{LJ}\,|$.

The argument of [Lab11, Thm. 5.4] shows the identity (adapted to our notation)

$$\sum_{\sigma} m(\sigma)\,\mathrm{tr}\,\sigma(f) = \mathrm{tr}\,\Pi(\phi) \tag{A.0.4}$$

with the functions $\phi=\prod_u \phi_u$ on $G^*(\mathbb{A}_{\widetilde{F}})$ and $f=\prod_u f_u$ on $G(\mathbb{A}_{\widetilde{F}^+})$ as in the proof there, where the sum runs over automorphic representations σ of $G(\mathbb{A}_{\widetilde{F}^+})$ with multiplicity $m(\sigma)$ whose weak base change is Π. The right-hand side is interpreted as the twisted trace by the convention mentioned earlier.

The key point to show is that the right-hand side does not always vanish. There is a subtlety when G is not quasi-split, because not every test function ϕ may be allowed in

[3]Contrary to the assumption on $\mathbf{U}$ above [Lab11, Thm. 5.4] that U is quasi-split at all *inert* places, it seems the assumption ought to be that $\mathbf{U}$ is quasi-split at *all* finite places. We believe that "Le second membre étant non identiquement nul" (in the proof of [Lab11, Thm. 5.4], between the second and third displays) is not always true, e.g. if Π_w is a principal series representation at a non-quasi-split place that splits in $\widetilde{F}$. (See the third paragraph of the current proof.) If it were true, we could deduce Proposition 6.5.2 directly from [Lab11, Thm. 5.4].

(A.0.4). The potential problem is that a base change transfer of ϕ_u at u from $G^*(\widetilde{F}_u)$ to $G^*(\widetilde{F}_u^+)$ is not in the image of endoscopic transfer from $G(\widetilde{F}_u^+)$ to $G^*(\widetilde{F}_u^+)$. We make a choice of test functions avoiding this problem.

At ∞ one does the same as in Labesse's proof so that $\operatorname{tr}\Pi_\infty(\phi_\infty) \neq 0$. At finite places u, we recall that f_u and ϕ_u are related as follows: writing f_u^* for a transfer of f_u from $G(\widetilde{F}_u^+)$ to $G^*(\widetilde{F}_u^+)$, the functions f_u^* and ϕ_u are associated in the sense of [Lab11, 4.5]. There is no problem when $u \notin \Delta^+(G)$ as G and G^* are isomorphic outside $\Delta^+(G)$; more precisely we choose ϕ_u on $G(\widetilde{F} \otimes_{\widetilde{F}^+} \widetilde{F}_v^+)$ such that

$$\operatorname{tr}\Pi_u(\phi_u) \neq 0$$

and choose f_u to be a base change transfer to $G(\widetilde{F}_v^+)$ (which exists by [Lab11, Lem. 4.1], where it is called an "associated" function). At each $v = ww^c \in \Delta^+(G)$, choose f_v and let f_v^* be a transfer. Write $\pi_v^* := \Pi_w$ via the chosen isomorphism $G^*(\widetilde{F}_w) \cong G^*(\widetilde{F}_v^+)$. Then by Proposition A.0.1,

$$\operatorname{tr}\pi_v^*(f_v^*) = e(G(\widetilde{F}_v^+)) \cdot \operatorname{tr}\left(|\operatorname{LJ}|(\pi_v^*)\right)(f_v).$$

Note that $|\operatorname{LJ}|(\pi_v^*)$ is irreducible (i.e. nonzero) since π_v^* is supercuspidal by assumption. If we choose f_v such that $\operatorname{tr}\left(|\operatorname{LJ}|(\pi_v^*)\right)(f_v) \neq 0$ then the above identity tells us that $\operatorname{tr}\pi_v^*(f_v^*) \neq 0$. Choosing ϕ_v to be a function associated with f_v^* (such a ϕ_v exists by either [Lab11, Lem. 4.1]), we have as in (A.0.2),

$$\operatorname{tr}\Pi_v(\phi_v) = \operatorname{tr}\pi_v^*(f_v^*) \neq 0.$$

We have exhibited a choice of f and ϕ above such that (A.0.4) is valid with the right-hand side non-vanishing. Therefore there exists some π on the left-hand side contributing with positive multiplicity. Let S be the set of places of $\widetilde{F}^+$ containing all infinite places and the finite places where G and Π are ramified. Write S_{f} for the subset of finite places in S. As we are free to choose ϕ_u in the unramified Hecke algebra at each $u \in S_{\mathrm{f}}$, we may assume that π^S is unramified with $\operatorname{BC}(\pi^S) = \Pi^S$. The nonvanishing of $\operatorname{tr}\pi_\infty(f_\infty)$ tells us that $\operatorname{BC}(\pi_\infty) = \Pi_\infty$. Thus (A.0.4) is reduced to a formula of the form (A.0.3), with π contributing nontrivially to the sum. Arguing as in the proof of the preceding proposition, we deduce that $|\operatorname{LJ}|(\pi_v^*) \cong \pi_v$. □

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