Here are some practice problems in number theory. They are, very roughly, in increasing order of difficulty.

1. (a) Show that \( n^7 - n \) is divisible by 42 for every positive integer \( n \).

(b) Show that every prime not equal to 2 or 5 divides infinitely many of the numbers 1, 11, 111, 1111, etc.

2. Show that if \( p > 3 \) is a prime, then \( p^2 \equiv 1 \pmod{24} \).

3. How many zeros are at the end of 1000!?

4. If \( p \) and \( p^2 + 2 \) are primes, show that \( p^3 + 2 \) is prime.

5. Show that \( \gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1 \) for positive integers \( a, b \).

6. Suppose that \( a, b, c \) are distinct integers and that \( p(x) \) is a polynomial with integer coefficients. Show that it is not possible to have \( p(a) = b, p(b) = c, p(c) = a \).

7. A triangular number is a positive integer of the form \( n(n + 1)/2 \). Show that \( m \) is a sum of two triangular numbers iff \( 4m + 1 \) is a sum of two squares. \cite{A-1, Putnam 1975}

8. For positive integers \( n \) define \( d(n) = n - m^2 \), where \( m \) is the greatest integer with \( m^2 \leq n \). Given a positive integer \( b_0 \), define a sequence \( b_i \) by taking \( b_{k+1} = b_k + d(b_k) \). For what \( b_0 \) do we have \( b_i \) constant for sufficiently large \( i \)? \cite{B-1, Putnam 1991}

9. \( d, e \) and \( f \) each have nine digits when written in base 10. Each of the nine numbers formed from \( d \) by replacing one of its digits by the corresponding digit of \( e \) is divisible by 7. Similarly, each of the nine numbers formed from \( e \) by replacing one of its digits by the corresponding digit of \( f \) is divisible by 7. Show that each of the nine differences between corresponding digits of \( d \) and \( f \) is divisible by 7. \cite{A-3, Putnam 1993}

10. Define the sequence of decimal integers \( a_n \) as follows: \( a_1 = 0; a_2 = 1; a_{n+2} \) is obtained by writing the digits of \( a_{n+1} \) immediately followed by those of \( a_n \). When is \( a_n \) a multiple of 11? \cite{A-4, Putnam 1998}

11. Suppose \( n > 1 \) is an integer. Show that \( n^4 + 4^n \) is not prime.

12. (a) Let \( \alpha \) and \( \beta \) be irrational numbers such that \( 1/\alpha + 1/\beta = 1 \). Then the sequences \( f(n) = \lfloor \alpha n \rfloor \) and \( g(n) = \lfloor \beta n \rfloor \), \( n = 1, 2, 3, \ldots \) are disjoint and their union is the set of positive integers. \cite{A classic due to Beatty; variations of this appear again and again.}

(b) Show the following converse: if \( \alpha, \beta \) are positive reals such that the sequences \( f(n) = \lfloor \alpha n \rfloor \) and \( g(n) = \lfloor \beta n \rfloor \), \( n = 1, 2, 3, \ldots \) are disjoint and their union is the set of positive integers, then \( \alpha, \beta \) are irrational and \( 1/\alpha + 1/\beta = 1 \).
13. If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum
\[
\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}
\]
of binomial coefficients is divisible by $p^2$. (A-5, Putnam 1996)

14. Find all positive integers $a$, $b$, $m$, $n$ with $m$ relatively prime to $n$ such that
\[
(a^2 + b^2)^m = (ab)^n.
\]
(A-3, Putnam 1992)

15. Suppose the positive integers $x$, $y$ satisfy $2x^2 + x = 3y^2 + y$. Show that
$x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

16. Find all solutions of $x^{n+1} - (x + 1)^n = 2001$ in positive integers $x$ and $n$.
(A-5, Putnam 2001)