THE MOD p REPRESENTATION THEORY OF p-ADIC GROUPS

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Introduction and Motivation

1. p-adic groups

1.1. The p-adic numbers. A rational number $x \in \mathbb{Q}^\times$ may be uniquely written as $x = \frac{a}{b}p^n$ with a, b and n nonzero integers such that $p \nmid ab$. We define $\mathrm{ord}_p(x) = n$, $|x|_p = p^{-n}$, $|0|_p = 0$. $|\cdot|_p$ defines an absolute value on $\mathbb{Q}$, satisfying the stronger ultrametric triangle equality $|x+y|_p \leq \max(|x|_p, |y|_p)$. We define $\mathbb{Q}_p$ to be the completion $\mathbb{Q}$ with respect to this metric and we use the same notation $|\cdot|_p$ for the extension of $|\cdot|_p$ to $\mathbb{Q}_p$; $(\mathbb{Q}_p, |\cdot|_p)$ is a complete valued field. Put $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$, it is a local discrete valuation ring with maximal ideal $p\mathbb{Z}_p$. The collection $(p^n\mathbb{Z}_p)_{n\geq 0}$ of compact open subgroups forms a fundamental system of neighbourhoods of 0.

1.2. p-adic reductive groups. There is a general theory p-adic reductive groups and their points over some extension $F/\mathbb{Q}_p$ but for simplicity we will stick to $G = \mathrm{GL}_n(\mathbb{Q}_p)$. We give G the subspace topology from the inclusion $G \subseteq M_n(\mathbb{Q}_p) \cong \mathbb{Q}_p^{n^2}$. With respect to this topology the maps $g \mapsto g_{ij}$ and $g \mapsto \det(g)^{-1}$ are continuous and G is a topological group.

Remark. Let H be any topological group.

i) Left and right translations $H \to H$ are homeomorphisms

ii) Any open subgroup of H is also closed (the complement is a union of cosets, hence open)

iii) Any closed subgroup of finite index of H is open (the complement is a finite union of cosets, hence closed)

iv) If H is also compact then any open subgroup has finite index (the cosets form a disjoint open cover)

In G, $K = \mathrm{GL}_n(\mathbb{Z}_p)$ is a maximal compact open subgroup. We define $K(r) = 1 + p^r M_n(\mathbb{Z}_p)$ for $r \geq 1$, these are compact open subgroups of G that forms a fundamental system of neighbourhoods of 1 (hence G is totally disconnected). The quotient $K/K(r)$ is $\mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})$.

Next we define the some special subgroups of G. Let $n_1, \ldots, n_r \geq 1$ be integers such that $\sum n_i = n$. Let $P_{n_1,\ldots,n_r}$ be the subgroup of block upper-triangular matrices in G with blocks along the diagonal of size $n_1 \times n_1, \ldots, n_r \times n_r$. $P_{n_1,\ldots,n_r}$ has two distinguished subgroups: $M_{n_1,\ldots,n_r}$, consisting of the block diagonal matrices (again with diagonal blocks of $n_1 \times n_1, \ldots, n_r \times n_r$), and $N_{n_1,\ldots,n_r}$, which consists of those matrices in $P_{n_1,\ldots,n_r}$ with the zero-matrix (of the appropriate size) in each diagonal block. $N_{n_1,\ldots,n_r}$is called the unipotent radical of $P_{n_1,\ldots,n_r}$; the $P_{n_1,\ldots,n_r}$ (for varying $n_1, \ldots, n_r$) are called the standard parabolic subgroups of G and the $M_{n_1,\ldots,n_r}$ are the standard Levi subgroups.

[1] Thanks to Re'em Waxman for some helpful comments.

$N_{n_1,\ldots,n_r}$ is normal in $P_{n_1,\ldots,n_r}$ and we have $P_{n_1,\ldots,n_r} = M_{n_1,\ldots,n_r} \ltimes N_{n_1,\ldots,n_r}$. We will often write $P = MN$ to mean that P is a standard parabolic subgroup with standard Levi subgroup M and unipotent radical N.

Remark. 1) The subgroup $P_{1,\ldots,1}$ of upper-triangular matrices is called the Borel subgroup and will be denoted B. Its standard Levi is the maximal torus of diagonal matrices and will be denote T, and its unipotent radical is the subgroup of unipotent matrices of G and will be denoted U.

2) A parabolic subgroup is any conjugate of a standard parabolic subgroup.

3) In fact, any subgroup containing B is a standard parabolic.

4) For any standard parabolic P, we will denote its transpose (i.e. the corresponding subgroup of block-lower triangular matrices) by $\overline{P}$. Similarly we denote the transpose of N by $\overline{N}$.

There are several useful decompositions of G in terms of various of the above subgroups that are useful. We start with:

Proposition 1. *(Iwasawa decomposition) $G = \overline{B}K$. Hence $G = \overline{P}K$ for any standard parabolic P, and $\overline{P}\backslash G$ is compact.*

Proof. (sketch) We will use integral column operations to reduce any matrix to lower triangular form. The column permutation matrices are integral, so without loss of generality the $(1,1)$-entry has the smallest valuation in the first row. Using this, we may add suitable integral multiples of the first column to the others to reduce to the case where the $(1,i)$-entry is 0 for $2 \leq i \leq n$. An induction on n finishes the proof. □

Before moving on to the next, we recall, in our setting, a theorem from the theory of principal ideal domains. Before we state it, we define a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^n$ to be a finitely generated $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$ that spans $\mathbb{Q}_p^n$ (such a submodule is necessarily free of rank n over $\mathbb{Z}_p$).

Theorem. *(Theorem on elementary divisors) Given a lattice $\Lambda \subseteq \mathbb{Q}_p^n$ there exists a basis $e'_1,\ldots,e'_n$ of $\mathbb{Z}_p^n$ and unique integers $a_1 \leq \cdots \leq a_n$ such that $p^{a_1}e'_1,\ldots,p^{a_n}e'_n$ is a basis for Λ.*

Remark. The theorem is usually stated for $\Lambda \subseteq \mathbb{Z}_p^n$ but we may reduce to this case by scaling.

Using this, we may now prove

Proposition 2. *(Cartan decomposition)* $G = \coprod_{a_1 \leq \cdots \leq a_n} K \cdot \mathrm{diag}\,(p^{a_1},\ldots,p^{a_n}) \cdot K$, *where* $\mathrm{diag}\,(p^{a_1},\ldots,p^{a_n})$ *is the diagonal matrix with entries $p^{a_1},\ldots,p^{a_n}$ along the diagonal.*

Proof. Given $g \in G$, let $\Lambda = g\mathbb{Z}_p^n$. If $e_1,\ldots,e_n$ is the standard basis of $\mathbb{Z}_p^n$ then $ge_1,\ldots,ge_n$ is a basis for Λ. By the theorem of elementary divisors there is a basis $e'_1,\ldots,e'_n$ of $\mathbb{Z}_p^n$ and integers $a_1 \leq \cdots \leq a_n$ such that $p^{a_1}e'_1,\ldots,p^{a_n}e'_n$ is a basis for Λ. Thus, as bases for $\mathbb{Z}_p^n$ resp. Λ, $e_1,\ldots,e_n$ and $e'_1,\ldots,e'_n$ are related by some $k_1 \in K$ and $ge_1,\ldots,ge_n$ and $p^{a_1}e'_1,\ldots,p^{a_n}e'_n$ are related by some $k_2 \in K$ (say $e'_i = k_1 e_i$, $p^{a_i}e'_i = k_2 g e_i$). Then $g = k_2^{-1}\mathrm{diag}\,(p^{a_1},\ldots,p^{a_n})\,k_1$. Uniqueness of the a_i imply the disjointness of the decomposition. □

Proposition 3. *("big cell lemma") Let $P = MN$ be a standard parabolic subgroup of G. The multiplication map $\overline{P} \times N \to G$ is injective and has open image (it is not a group homomorphism).*

Proof. (sketch) We assume $P = B$, the general case is analogous, using blocks. $\overline{B} \times U \to G$ is injective as $\overline{B} \cap U = 1$, so it remains to show that the image is open. We claim that the image S consists of those matrices g for which the upper left $i \times i$ minor $\det_i(g)$ is nonzero for all $i = 1, \dots, n$. This set is open as the map $G \to \mathbb{Q}_p^n$, $g \mapsto (\det_i(g))_i$ is continuous and S is the preimage of the open set $\mathbb{Q}_p^{\times n}$ under this map. To show that the image is S, note that $\det_i(bu) = \det_i(b)\det_i(u) \neq 0$ for all i and $b \in \overline{B}$, $u \in U$ as b resp. u are lower resp. upper triangular. To see that the image is all of S, pick $g \in S$ and write down the linear equations that the entries of some $u \in U$ would have to satisfy in order for gu to be lower triangular. These turn out to be solvable exactly because $g \in S$. □

2. Smooth Representations of p-adic Groups

In this section, G is any Hausdorff topological group that has a fundamental system of neighbourhoods of 1 consisting of compact open subgroups, E is a field and G-representation is an E-vector space π with a linear action of G. If U is a subgroup of G we let $\pi^U = \{x \in \pi \mid ux = x \,\forall u \in U\}$denote the invariants of U.

Lemma 4. *The following are equivalent:*

1) The stabilizer of any $x \in \pi$ is open.

2) For any $x \in \pi$ there exists an open subgroup U of G such that $x \in \pi^U$.

3) $\pi = \bigcup \pi^U$ where the union is taken over all open subgroups of G.

4) The action map $G \times \pi \to \pi$ is continuous with respect to the discrete topology.

Definition. A G-representation π is smooth if the equivalent conditions of the previous lemma hold.

Example. Let $G = \mathbb{Q}_p^\times$. A character $\chi : \mathbb{Q}_p^\times \to E^\times$ is smooth if and only if its kernel is open. We have $\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times p^{\mathbb{Z}}$ where $p^{\mathbb{Z}}$ is equipped with the discrete topology. To give a smooth character we need:

1) An element $\chi(p) \in E^\times$

2) A character $\mathbb{Z}_p^\times/(1 + p^r\mathbb{Z}_p) \to E^\times$

Any smooth irreducible representation of $\mathbb{Q}_p^\times$ is a character (this follows from the commutativity of $\mathbb{Q}_p^\times$).

Remark. 1) Any subquotient of a smooth representation is smooth.

2) To form the category of smooth representations we take as morphisms G-linear maps (smoothness is equivalent to continuity with respect to the discrete topology on the vector spaces, so there is no topological condition required on the G-linear maps).

Definition. We say a smooth G-representation π is irreducible if it has no subrepresentations apart from 0 and π.

Induced representations. Let $H \leq G$ be a closed subgroup (note that this implies that $H\backslash G$ is Hausdorff) and let σ be a smooth H-representation.

Definition. We define the induction $\mathrm{Ind}_H^G\sigma$ of σ from H to G to be the vector space of functions $f : G \to \sigma$ such that $f(hg) = h.f(g)$ for all $h \in H$ and $g \in G$, and such that there is an open subgroup $U \leq G$ such that $f(gu) = f(g)$ for all $u \in U$ and $g \in G$. We let G act on $\mathrm{Ind}_H^G\sigma$ by $(g.f)(x) = f(xg)$; the last condition in the definition ensures that this is a smooth representation of G.

Example. If $\sigma = 1_H$ then $\mathrm{Ind}_H^G 1_H$ is the space of uniformly locally constant functions $f : H\backslash G \to E$.

In general, if $f \in \mathrm{Ind}_H^G\sigma$ we define the support of f as $\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$; it is a union of right cosets of H. By the smoothness condition in the definition of $\mathrm{Ind}_H^G\sigma$, f is locally constant and hence $\mathrm{supp}(f)$ is open and closed, which implies that the image of $\mathrm{supp}(f)$ in $H\backslash G$ is also open and closed.

Definition. We define the compact induction $\mathrm{ind}_H^G\sigma$ of σ from H to G to be the subspace of $\mathrm{Ind}_H^G\sigma$ of functions f such that the image of $\mathrm{supp}(f)$ in $H\backslash G$ is compact. This is a G-subrepresentation of $\mathrm{Ind}_H^G\sigma$ since the action of G translates the support of functions.

Remark. 1) If $H\backslash G$ is compact, then $\mathrm{Ind}_H^G\sigma = \mathrm{ind}_H^G\sigma$.

2) Ind_H^G- and ind_H^G- are left exact functors.

Proposition. *(Frobenius Reciprocity)*

1) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) = \mathrm{Hom}_H(\pi|_H, \sigma)$

2) If U is an open subgroup, then $\mathrm{Hom}_G(\mathrm{ind}_U^G\sigma, \pi) = \mathrm{Hom}_H(\sigma, \pi|_H)$ *and* ind_H^G- *is exact.*

Proof. (sketch) 1) Given $\phi : \pi \to \mathrm{Ind}_H^G\sigma$ define $\overline{\varphi} : \pi|_H \to \sigma$ by $x \mapsto \varphi(x)(1)$. Conversely, given $\psi : \pi|_H \to \sigma$ define $\overline{\psi} : \pi \to \mathrm{Ind}_H^G\sigma$ by sending $x \in \pi$ to the function $(g \mapsto \psi(gx)) \in \mathrm{Ind}_H^G\sigma$. One checks that these are well-defined and give the adjunction in 1).

2) $U\backslash G$ is discrete since U is open, so the support of an element in $\mathrm{ind}_U^G\sigma$ is finite in $U\backslash G$. To give an element f of $\mathrm{ind}_U^G\sigma$ we need a finite number of distinct cosets Ug_i and elements $y_i \in \sigma$; the function is given by $f(ug_i) = uy_i$ on the Ug_i and zero outside them (using this description, one checks that ind_U^G- is exact). Write $[g, y]$ for the function with support Ug^{-1} sending g^{-1} to y. Then $\gamma.[g, y] = [\gamma g, y]$ for all $g \in G$ and $[gu, y] = [g, uy]$ for all $u \in U$. Now given $\varphi : \mathrm{ind}_U^G\sigma \to \pi$ define $\overline{\varphi} : \sigma \to \pi|_U$ by sending $y \in \sigma$ to $\varphi([1, y])$, and conversely, given $\psi : \sigma \to \pi|_U$ define $\overline{\psi} : \mathrm{ind}_U^G\sigma \to \pi$ by $[g, y] \mapsto g.\psi(y)$ and extending linearly. One checks that these are well-defined and give the adjunction in 2). ☐

Remark. Alternatively, one may prove 2) by noting that $\mathrm{ind}_U^G\sigma$ is isomorphic $E[G] \otimes_{E[U]} \sigma$, the isomorphism being given by $g \otimes y \mapsto [g, y]$. The adjunction in 2) is then just the standard extension/restriction of scalars-adjunction and the exactness follows from the fact that $E[G]$ is free over $E[U]$ (generated as a left module by a set of representatives of $U\backslash G$).

Proposition 5. *1) Assume that the projection map $\pi : G \to H\backslash G$ has a continuous section s and that $H\backslash G$ is compact. Then* Ind_H^G- *is exact.*

2) If $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and P is a standard parabolic, then Ind_P^G- *is exact.*

Proof. 1) Compactness of $H\backslash G$ ensures that "locally constant" is equivalent to "uniformly locally constant". Define a map $\mathrm{Ind}_H^G\sigma \to \mathcal{C}^\infty(H\backslash G,\sigma)$ by $f \mapsto (Hg \mapsto f(s(Hg)))$ and a map the other way by

$$\varphi \mapsto \overline{\varphi} = (g = h.s(Hg) \mapsto h.\varphi(Hg))$$

The first map is well-defined as f is locally constant. To check that the second map is well-defined, write $h.\varphi(Hg) = g.s(\pi(g))^{-1}.\varphi(\pi(g))$ and note this is locally constant as $g.s(\pi(g))^{-1}$ is continuous in g, σ is smooth and φ is locally constant. One then checks that these maps are inverses to one another and defines an natural isomorphism of functors $\mathrm{Ind}_H^G- \cong \mathcal{C}^\infty(H\backslash G,-)$ (the spaces $\mathcal{C}^\infty(H\backslash G,\sigma)$ have a left action of G coming from the right action on $H\backslash G$). Exactness of Ind_H^G- now follows from exactness of $\mathcal{C}^\infty(H\backslash G,-)$ (left exactness is straightforward; to prove that it preserves surjections $\sigma \twoheadrightarrow \tau$ choose any vector space section $\tau \to \sigma$ and use that this is continuous with respect to the discrete topology).

2) Write $P = MN$. We want to deduce 2) from 1) so we need to construct a continuous section $s : \overline{P}\backslash G \to G$. By the "big cell" lemma the image of $N \hookrightarrow \overline{P}\backslash G$ is open, hence the image Ω of the compact open subgroup $N(\mathbb{Z}_p) = N \cap K$ is open and closed, so we can define a continuous section on Ω. By translation we may define a continuous section on Ωg for any $g \in G$; these sets form an open cover of $\overline{P}\backslash G$. Since $\overline{P}\backslash G$ is compact we can take a finite subcover $\{\Omega g_i\}_{1\le i\le r}$. Now chop these up into 2^r disjoint open and closed subsets $\left(\bigcap_{i\in I}\Omega g_i\right) \cap \left(\bigcap_{i\in I^c}(\Omega g_i)^c\right)$ (for $I \subseteq \{1,\dots,r\}$) on which a continuous section exists, and then glue to get a continuous section on all of $\overline{P}\backslash G$. ☐

3. Smooth Representations in Characteristic p

We now to back to the case $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and furthermore assume, from now, that the characteristic of E is p.

Definition. A profinite group is a compact Hausdorff topological group with a fundamental system of neighbourhoods of 1 consisting of normal subgroups. A profinite group is called pro-p if furthermore the index of each of these normal subgroups is a power of p.

Any closed subgroup or quotient of a profinite (resp. pro-p) group is profinite (resp. pro-p).

Example. $\mathbb{Z}_p$ is a pro-p group. $K = \mathrm{GL}_n(\mathbb{Z}_p)$ is profinite.

Lemma 6. *$K(1)$ is a pro-p group.*

Proof. The $K(r)$ form a fundamental system of neighbourhoods of 1 so it suffices to show that $(K(1) : K(r))$ is a power of p for all $r \ge 2$. By multiplicativity of indices this reduces to showing that $(K(r) : K(r+1))$ is a power of p for $r \ge 1$. But as groups, $K(r)/K(r+1)$ is isomorphic to $M_n(\mathbb{F}_p)$ via the map sending $1 + p^r A \in K(r)/K(r+1)$ to the reduction of A modulo p, and therefore $(K(r) : K(r+1)) = p^{n^2}$. ☐

Similarly, the subgroup $I(1) \subseteq K$ consisting of matrices that are unipotent modulo p is pro-p, as $I(1)/K(1) \cong U(\mathbb{F}_p)$, where $U(\mathbb{F}_p) \subseteq \mathrm{GL}_n(\mathbb{F}_p)$ is the subgroup of unipotent matrices, which has order $p^{n(n-1)/2}$ (we remark that $I(1)$ is a Sylow pro-p subgroup of K, i.e. a maximal pro-p subgroup).

Lemma 7. *("p-groups lemma") Any smooth representation $\tau \neq 0$ in characteristic p of a pro-p group H has invariant vectors, i.e. $\tau^H \neq 0$.*

Proof. Without loss of generality we may assume $E = \mathbb{F}_p$ (forget the E-action). Pick any nonzero $x \in \tau$. Since τ is smooth and H is profinite there exists an open normal subgroup U of H such that $x \in \tau^U$. Then H/U is a finite p-group acting on $\tau^U \neq 0$, so we may reduce to the case where H is finite.

Then, the $\mathbb{F}_p$-span of $Hx = \{hx \mid h \in H\}$ is finite dimensional (as Hx is finite), hence this span is finite as a set, so without loss of generality τ is a finite set, of p-power order. Decompose τ into H-orbits; by the orbit-stabilizer theorem all orbits have size a power of p and there is at least one orbit of size 1, namely $\{0\}$, so looking modulo p there must be at least p orbits of size 1. In particular, $\tau^H \neq 0$. □

Remark. The coinvariants of τ as in the lemma may be 0. Because of this, duals do not work well in characteristic p.

Corollary. *A pro-p group H has a unique irreducible representation, which is the trivial representation.*

For us, the following corollaries are important:

Corollary. *If π is a smooth representation of G, then $\pi^{K(1)} \supseteq \pi^{I(1)} \neq 0$.*

We note that this fails in characteristic 0, even if $n = 1$.

Corollary. *Any smooth irreducible representation of K factors through $K/K(1) = \mathrm{GL}_n(\mathbb{F}_p)$, so there is a natural bijection between smooth irreducible representations of K and $\mathrm{GL}_n(\mathbb{F}_p)$.*

Proof. Let V be a smooth irreducible representation of K. By above, $V^{K(1)} \neq 0$ and since $K(1)$ is normal in K, $V^{K(1)} \subseteq V$ is K-stable, so we must have $V^{K(1)} = V$ since V is irreducible, i.e. $K(1)$ acts trivially on V. □

From now on we will assume that E is algebraically closed.

Definition. A smooth irreducible representation of K (or equivalently, a smooth irreducible representation of $\mathrm{GL}_n(\mathbb{F}_p)$) is called a weight.

Corollary. *Any smooth representation π of G contains a weight, i.e. there is a weight V such that V is a subrepresentation of $\pi|_K$.*

Proof. Pick a nonzero $x \in \pi^{K(1)}$. Then the E-span of Kx is a finite dimensional K-subrepresentation of $\pi|_K$ (Kx is finite since $K(1)$ acts trivially on x and $K/K(1)$ is finite), and therefore contains an irreducible subrepresentation. □

Example. ($n = 1$) Any smooth character $\chi : \mathbb{Q}_p^\times \to E^\times$ is trivial on $K(1) = 1 + p\mathbb{Z}_p$. To define χ we need an element $\chi(p) \in E^\times$ and a character $\chi|_{\mathbb{Z}_p^\times} : \mathbb{F}_p^\times \to E^\times$ (a weight for GL_1). There are $p - 1$ weights, all one-dimensional.

Next we determine the weights of GL_2:

Proposition 8. *The weights of $\mathrm{GL}_2(\mathbb{F}_p)$ are $F(a, b) = \mathrm{Sym}^{a-b} E^2 \otimes \det^b$, where $0 \leq a - b \leq p - 1$ and $0 \leq b < p - 1$, E^2 is the standard representation $\mathrm{GL}_2(\mathbb{F}_p) \hookrightarrow \mathrm{GL}(E)$ on column vectors and* det *is the determinant representation.*

Before proving the proposition, we make a few remarks. First, we note that we may take b to be arbitrary if we take into account that $F(a+p-1, b+p-1) \cong F(a,b)$. Second, the action on $\mathrm{Sym}^d E^2$ is by $g(v_1 \cdots v_d) = (gv_1) \cdots (gv_d)$. More concretely, $\mathrm{Sym}^d E^2 \cong E[X,Y]_{(d)}$, the space of homogeneous polynomials of degree d in X and Y. The isomorphism takes the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^i \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{d-i}$ to the basis $X^i Y^{d-i}$, and the action of a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p)$ takes a degree d homogeneous polynomial $f(X,Y)$ to $f(\alpha X + \gamma Y, \beta X + \delta Y)$. We now prove the proposition:

Proof. Step 1: Prove that the $F(a,b)$ are irreducible

We may twist and with loss of generality assume that $b = 0$. First we prove that $(\mathrm{Sym}^a E^2)^{U(\mathbb{F}_p)} = E.X^a$. The inclusion $\supseteq$ is clear. Let $f \in (\mathrm{Sym}^a E^2)^{U(\mathbb{F}_p)}$. We have, for all $u \in \mathbb{F}_p$,

$$\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} f \right)(X,Y) = f(X, uX+Y) = f(X,Y)$$

Consider $g(Y) = f(X,Y) - f(X,0) \in E(X)[Y]$. The degree (in Y) of g is $\leq a < p$. By the above equation we have

$$g(-uX) = f(X,-uX) - f(X,0) = 0$$

for all $u \in \mathbb{F}_p$, so g has $p > \deg(g)$ distinct roots, hence $g = 0$. Thus $f(X,Y) = f(X,0)$ for all X, Y, so f is a monomial in X, hence in $E.X^a$.

Next, we want to show that X^a generates $\mathrm{Sym}^a E^2$. For $u \in \mathbb{F}_p$, we have

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^a = (X+uY)^a = \sum_{i=0}^{a} \begin{pmatrix} a \\ i \end{pmatrix} u^i X^{a-i} Y^i$$

$\mathrm{Sym}^a E^2$ has a basis $\begin{pmatrix} a \\ i \end{pmatrix} X^{a-i} Y^i$ for $0 \leq i \leq a$. Consider the set $\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^a$ for $0 \leq u \leq a$. It follows from the equation above that the determinant for passing from $\begin{pmatrix} a \\ i \end{pmatrix} X^{a-i} Y^i$ to $\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^a$ is a Vandermonde determinant and hence nonzero, so $\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^a$, $0 \leq u \leq a$ forms a basis for $\mathrm{Sym}^a E^2$, and hence X^a generates it.

Finally, we may finish the proof of Step 1. Let $V \neq 0$ be a subrepresentation of $\mathrm{Sym}^a E^2$. By the p-groups lemma $V^{U(\mathbb{F}_p)} \neq 0$,so it must be $E.X^a$. Since this subspace generates $\mathrm{Sym}^a E^2$, $V = \mathrm{Sym}^a E^2$ and $\mathrm{Sym}^a E^2$ is irreducible.

Step 2: The $F(a,b)$ are distinct.

Since $T(\mathbb{F}_p)$ normalises $U(\mathbb{F}_p)$ it acts on $F(a,b)^{U(\mathbb{F}_p)} = E.X^{a-b}$. We compute this action:

$$\begin{pmatrix} x & \\ & y \end{pmatrix} X^{a-b} = xX^{a-b}(xy)^b = x^a y^b X^{a-b}$$

i.e. $T(\mathbb{F}_p)$ acts by the character $\chi_{a,b}$ sending $\mathrm{diag}(x,y)$ to $x^a y^b$. If $F(a,b) \cong F(a',b')$, then first we must have $a - b = a' - b'$ by looking at the dimensions. Second, we must also have $\chi_{a,b} = \chi_{a',b'}$ by above, so $a \equiv a'$, $b \equiv b' \bmod p-1$. By our constraints on b this forces $b = b'$, and hence $a = a'$ as desired.

Step 3: These are all irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$.

By modular representation theory of finite groups, the number of irreducible representations is equal to the number of conjugacy classes of prime-to-p order. We may use Jordan normal form and the fact that matrices are conjugate over $\overline{\mathbb{F}}_p$ if and only if they are conjugate over $\mathbb{F}_p$ to find the conjugacy classes (the latter follows e.g. from the rational canonical form; one may also use this directly). We get the following four types of conjugacy classes:

(i) Central elements $\begin{pmatrix} x & \\ & x \end{pmatrix}$, $x \in \mathbb{F}_p^\times$

(ii) Diagonal non-central elements $\begin{pmatrix} x & \\ & y \end{pmatrix}$, $x \neq y$, $x, y \in \mathbb{F}_p^\times$

(iii) Elements that may be diagonalised over $\mathbb{F}_{p^2}$ $\begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix}$, $\alpha \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$

(iv) Non-diagonalisable elements $\begin{pmatrix} x & 1 \\ & x \end{pmatrix}$, $x \in \mathbb{F}_p^\times$

The first three have order prime to p, the fourth does not. There are $p-1$ conjugacy classes in (i), $(p-1)(p-2)/2$ in (ii) and $p(p-1)/2$ in (iii). Summing up, we get in total $p(p-1)$ conjugacy classes of order prime to p, which is equal to the number of pairs (a, b) in the parametrisation of the $F(a, b)$. This finishes the proof of the proposition. $\square$

There is an alternative way to prove the Step 3 without resorting to general results in modular representation theory. Assume that V is any irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Then we get a nonzero $T(\mathbb{F}_p)$-representation $V^{U(\mathbb{F}_p)}$. As $T(\mathbb{F}_p)$ is commutative of order prime to p, this representation splits as a direct sum of characters. Let χbe one of these characters, we have $\chi \hookrightarrow V^{U(\mathbb{F}_p)}$ as $T(\mathbb{F}_p)$-representations and hence $\chi \hookrightarrow V$ as $B(\mathbb{F}_p)$-representations (where we extend $\chi$to a character of $B(\mathbb{F}_p)$ by letting $U(\mathbb{F}_p)$ act trivially). Frobenius reciprocity now gives us a nonzero map

$$\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \chi \twoheadrightarrow V$$

which has to surject since V is irreducible. Thus, to show that V is one of the $F(a, b)$ it suffices to classify the quotients of the representations $\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \chi$, as χ ranges over the characters of $T(\mathbb{F}_p)$. First we need a lemma:

Lemma 9. *$F(a, b)_{\overline{U}(\mathbb{F}_p)} \cong \chi_{a,b}$ as $T(\mathbb{F}_p)$-representations, and the $T(\mathbb{F}_p)$-linear map*

$$F(a, b)^{U(\mathbb{F}_p)} \hookrightarrow F(a, b) \twoheadrightarrow F(a, b)_{\overline{U}(\mathbb{F}_p)}$$

is an isomorphism.

Proof. Without loss of generality $b = 0$, so $F(a, b) = F(a, 0) = \mathrm{Sym}^a E^2$. As we saw in the previous proof we have

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^a - X^a = \sum_{i=1}^{a} \begin{pmatrix} a \\ i \end{pmatrix} u^i X^{a-i} Y^i$$

and they have the same span as $X^{a-1}Y,\dots,Y^a$, moreover they get mapped to 0 in the coinvariants. For $1 \le i \le a$ we have

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} X^{a-i}Y^i - X^{a-i}Y^i = \sum_{k=1}^{a} \begin{pmatrix} a-i \\ k \end{pmatrix} u^k X^{a-i-k}Y^{i+k}$$

which shows that the kernel of $F(a,b) \twoheadrightarrow F(a,b)_{\overline{U}(\mathbb{F}_p)}$ is spanned by $X^{a-1}Y,\dots,Y^a$ and hence that as a $T(\mathbb{F}_p)$-representation we have

$$F(a,0) = F(a,0)^{U(\mathbb{F}_p)} \oplus \mathrm{Ker}\left(F(a,b) \twoheadrightarrow F(a,b)_{\overline{U}(\mathbb{F}_p)}\right)$$

This gives both statements of the lemma. □

We can now describe the irreducible subquotients of the $\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\chi$ and hence finish the alternative proof of Step 3:

Lemma 10. *Let $a, b \in \mathbb{Z}$ with $0 \le a-b < p-1$. Then the irreducible quotients of $\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\chi_{a,b}$ are $F(a,b)$, and also $F(a+p-1,b)$ if $a=b$. The irreducible subrepresentations are $F(b+p-1,a)$, and also $F(b,a)$ if $a=b$. In any case, the irreducible constituents are $F(a,b)$ and $F(b+p-1,a)$.*

Proof. (sketch) By Frobenius reciprocity $\mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\chi_{a,b} \twoheadrightarrow F(a',b')$ if and only if $\chi_{a,b} \hookrightarrow F(a',b')^{U(\mathbb{F}_p)} = \chi_{a',b'}$ and similarly $F(a',b') \hookrightarrow \mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\chi_{a,b}$ if and only if $F(a',b')_{U(\mathbb{F}_p)} \twoheadrightarrow \chi_{a,b}$, which is seen to be equivalent to $\chi_{a',b'} = F(a',b')_{\overline{U}(\mathbb{F}_p)} \cong \chi_{b,a}$ by conjugating with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence we know that there do exist subrepresentations and quotients as in the statement of the lemma. To see that these are the only ones, note that

$$\dim F(a,b) + \dim F(b+p-1,a) = (a-b+1) + (b+p-a) = p+1$$

and

$$\dim \mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)}\chi_{a,b} = (G(\mathbb{F}_p) : B(\mathbb{F}_p)) = \#\mathbb{P}^1(\mathbb{F}_p) = p+1$$

□

Weights of principal series ($n=2$). Let $\chi : T \to E^\times$ be a smooth character (remember that $\chi|_{T(\mathbb{Z}_p)}$ factors through $T(\mathbb{F}_p)$). Consider χ as a $\overline{B}$-representation (via $\overline{B} \twoheadrightarrow T$). Then χ defines a so-called principal series representation $\mathrm{Ind}_{\overline{B}}^{G}\chi$.

Proposition 11. *The weights of $\mathrm{Ind}_{\overline{B}}^{G}\chi$ are all V such that $V_{\overline{U}(\mathbb{F}_p)} \cong \chi|_{T(\mathbb{Z}_p)}$ as a $T(\mathbb{F}_p)$-representation (Lemma 9 therefore implies that $\mathrm{Ind}_{\overline{B}}^{G}\chi$ has one or two weights). Each V occurs with multiplicity one.*

Proof. Let V be a weight. By the Iwasawa decomposition (Proposition 1) the restriction map $\mathrm{Ind}_{\overline{B}}^{G}\chi \to \mathrm{Ind}_{\overline{B}\cap K}^{K}\chi$ is an isomorphism of K-representations. Thus

$$\mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^{G}\chi) = \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}\cap K}^{K}\chi)$$

which by Frobenius reciprocity is $\mathrm{Hom}_{\overline{B}\cap K}(V,\chi)$, and

$$\mathrm{Hom}_{\overline{B}\cap K}(V,\chi) = \mathrm{Hom}_{\overline{B}(\mathbb{F}_p)}(V,\chi) = \mathrm{Hom}_{T(\mathbb{F}_p)}(V_{\overline{U}(\mathbb{F}_p)},\chi)$$

which is one-dimensional or 0 depending on whether $V_{\overline{U}(\mathbb{F}_p)} \cong \chi|_{T(\mathbb{Z}_p)}$ or not, which is what we wanted. $\square$

Remark. Usually a principal series representation has a unique weight V occurring with multiplicity one. In this case, the G-subrepresentation generated by V is irreducible (because any G-subrepresentation of this contains a weight). Later we will see that in fact V generates the whole principal series representation.

4. Hecke Algebras – Generalities

Let π be a smooth irreducible representation of G. We know that there is a weight $V \hookrightarrow \pi|_K$; Frobenius reciprocity gives us a G-linear map $\mathrm{ind}_K^G V \twoheadrightarrow \pi$. We would therefore wish to understand the representations $\mathrm{ind}_K^G V$.

Definition. The Hecke algebra of the weight V is $\mathcal{H}_G(V) = \mathrm{End}_G(\mathrm{ind}_K^G V)$.

Remark. More generally, one could study such Hecke algebras for $\mathrm{ind}_H^G W$ with W a finite dimensional smooth representation of any arbitrary open compact subgroup H of G.

Proposition 12. *$\mathcal{H}_G(V)$ is isomorphic to the algebra of functions $\varphi : G \to \mathrm{End}_E(V)$ such that $\mathrm{supp}(\varphi)$ is compact and $\varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2$ for all $k_1, k_2 \in K$, $g \in G$. The product on this algebra is convolution*

$$(\varphi_1 \star \varphi_2)(g) = \sum_{\gamma \in K\backslash G} \varphi_1(g\gamma^{-1})\varphi_2(\gamma)$$

where we use the notation "$\gamma \in K\backslash G$" to mean that the γ runs through a set of coset representatives of $K\backslash G$, and that the sum is independent of this choice.

Proof. As vector spaces

$$\mathcal{H}_G(V) = \mathrm{End}_G(\mathrm{ind}_K^G V) = \mathrm{Hom}_K(V, \mathrm{ind}_K^G V) \subseteq \mathrm{Map}(V, \mathrm{Map}(G, V)) = \mathrm{Map}(G, \mathrm{Map}(V, V))$$

where we write $\mathrm{Map}(R, S)$ for the set of functions from one set R to another set S. It is naturally a vector space when S is a vector space. To see that the image of $\mathrm{Hom}_K(V, \mathrm{ind}_K^G V)$ inside $\mathrm{Map}(G, \mathrm{Map}(V, V))$ is as claimed, take $(v \mapsto f_v) \in \mathrm{Hom}_K(V, \mathrm{ind}_K^G V)$, then its image is defined by $\varphi(g)(v) = f_v(g)$. The K-linearity of $(v \mapsto f_v)$ means that $f_v(gk) = f_{kv}(g)$ and by the definition of the (compact) induction $f_v(kg) = k.f_v(g)$. This translates into $\varphi(k_1 g k_2)(v) = f_v(k_1 g k_2) = k_1.f_{k_2 v}(g) = k_1(\varphi(g)(k_2 v))$ which is the desired identity, and furthermore the compact support condition on φ translates into the (uniformly) compact support condition on the f_v (using linearity in v of f_v and the finite dimensionality of V).

Next we check the product. Take φ_1, φ_2 corresponding to $\psi'_1, \psi'_2 \in \mathcal{H}_G(V)$ and to $\psi_1, \psi_2 \in \mathrm{Hom}_K(V, \mathrm{ind}_K^G V)$; we have $\psi_i(x) = \psi'_i([1, x])$ by definition. Then

$$\psi'_2([1, x])(\gamma_2) = \psi_2(x)(\gamma_2) = \varphi_2(\gamma_2)(x)$$

and hence by K-equivariance

$$\psi'_2([1, x]) = \sum_{\gamma_2 \in K\backslash G} [\gamma_2^{-1}, \varphi_2(\gamma_2)(x)] = \sum_{\gamma_2 \in K\backslash G} \gamma_2^{-1}[1, \varphi_2(\gamma_2)(x)]$$

which implies that

$$\psi_1'(\psi_2'([1,x])) = \psi_1'\left(\sum_{\gamma_2 \in K\backslash G} \gamma_2^{-1}[1, \varphi_2(\gamma_2)(x)]\right) =$$

$$= \sum_{\gamma_2 \in K\backslash G} \gamma_2^{-1}. \sum_{\gamma_1 \in K\backslash G} [\gamma_1^{-1}, (\varphi_1(\gamma_1) \circ \varphi_2(\gamma_2))(x)]$$

which is equal to

$$\sum_{\gamma_2 \in K\backslash G} \sum_{\gamma_1 \in K\backslash G} [\gamma_2^{-1}\gamma_1^{-1}, (\varphi_1(\gamma_1) \circ \varphi_2(\gamma_2))(x)]$$

We need to make a change of variables in order to be able to interchange the sums. Put $\gamma = \gamma_1\gamma_2$ and interchange γ for γ_1, then we get

$$\sum_{\gamma_2 \in K\backslash G} \sum_{\gamma \in K\backslash G} [\gamma^{-1}, (\varphi_1(\gamma\gamma_2^{-1}) \circ \varphi_2(\gamma_2))(x)] = \sum_{\gamma \in K\backslash G} \sum_{\gamma_2 \in K\backslash G} [\gamma^{-1}, (\varphi_1(\gamma\gamma_2^{-1}) \circ \varphi_2(\gamma_2))(x)]$$

which is equal to

$$\sum_{\gamma \in K\backslash G} \left[\gamma^{-1}, \sum_{\gamma_2 \in K\backslash G} (\varphi_1(\gamma\gamma_2^{-1}) \circ \varphi_2(\gamma_2))(x)\right] = \sum_{\gamma \in K\backslash G} [\gamma^{-1}, (\varphi_1 \star \varphi_2)(\gamma)(x)]$$

as desired. $\square$

Let π be a smooth G-representation. Then $\mathcal{H}_G(V)$ acts on the right on $\mathrm{Hom}_K(V, \pi|_K) \cong \mathrm{Hom}_G(\mathrm{ind}_K^G V, \pi)$. This action is given explicitly by

$$(f \star \varphi)(x) = \sum_{\gamma \in K\backslash G} \gamma^{-1} f(\varphi(\gamma)(x))$$

for $f \in \mathrm{Hom}_K(V, \pi|_K)$ and φ in the explicit description of $\mathcal{H}_G(V)$. To see this, let ψ' and ψ correspond to φ as in the above proof and let $\bar{f} \in \mathrm{Hom}_G(\mathrm{ind}_K^G V, \pi)$ correspond to f. Then, by the above proof

$$\bar{f}(\psi(x)) = \bar{f}\left(\sum_{\gamma \in K\backslash G} \gamma^{-1}.[1, \varphi(\gamma)(x)]\right) = \sum_{\gamma \in K\backslash G} \gamma^{-1}.f(\varphi(\gamma)(x))$$

Example. Let V be the trivial representation, then $\mathcal{H}_G(V) = \mathcal{C}_c(K\backslash G/K, E)$, the algebra of bi-$K$-invariant functions on G with compact support, under convolution. Let 1_{KgK} denote the characteristic function of the double coset KgK. Then, for a smooth G-representation π it acts on $\pi^K = \mathrm{Hom}_K(V, \pi)$ in the usual way; if

$$KgK = \coprod_i g_i K$$

then for $x \in \pi^K$,

$$1_{KgK}(x) = \sum_i g_i^{-1} x.$$

5. Hecke Algebras for GL_2

Recall the Cartan decomposition $G = \coprod_{r \leq s} K \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} K$ (Proposition 2).

Theorem 13. *Fix a weight V.*

1) For any pair of integers $r \leq s$ there is a unique Hecke operator $T_{r,s} \in \mathcal{H}_G(V)$ such that $\mathrm{supp}(T_{r,s}) = K.\mathrm{diag}(p^r, p^s).K$ *and the endomorphism*

$$T_{r,s}\left[\begin{pmatrix} p^r & \\ & p^s \end{pmatrix}\right] \in \mathrm{End}_E(V)$$

is a linear projection. The $T_{r,s}$ form a basis of $\mathcal{H}_G(V)$.

2) $\mathcal{H}_G(V)$ is isomorphic to $E[T_1, T_2, T_2^{-1}]$ with $T_{0,1}$ going to T_1 and $T_{1,1}$ going to T_2. In particular $\mathcal{H}_G(V)$ is commutative.

Proof. 1) Suppose that $\varphi \in \mathcal{H}_G(V)$ is such that $A = \varphi \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} \neq 0$. Whenever $k_1, k_2 \in K$ are such that

$$k_1 \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} = \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} k_2$$

we must have $k_1 \circ A = A \circ k_2$. It is easy to check that conversely, such an A determines a Hecke operator supported on $K.\mathrm{diag}(p^r, p^s).K$ (just define φ to be $k_1 \circ A \circ k_2$ for elements $k_1.\mathrm{diag}(p^r, p^s).k_2$ and 0 everywhere else, the condition on A then assures that this is well defined). For

$$k_1 \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} = \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} k_2$$

to hold, we need

$$k_1 \in K \cap \begin{pmatrix} p^r & \\ & p^s \end{pmatrix} K \begin{pmatrix} p^r & \\ & p^s \end{pmatrix}^{-1}$$

If $r = s$ this is just K, and as $\begin{pmatrix} p^r & \\ & p^r \end{pmatrix}$ is central we have

$$k \begin{pmatrix} p^r & \\ & p^r \end{pmatrix} = \begin{pmatrix} p^r & \\ & p^r \end{pmatrix} k$$

for all $k \in K$, hence $k \circ A = A \circ k$ for all $k \in K$. By the irreducibility of V and Schur's Lemma, A is a scalar. Since A is a nonzero projection, A must be the identity. For $r < s$, the intersection above is

$$\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^{s-r}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

Write $k_1 = \begin{pmatrix} \alpha & \beta \\ p^{s-r}\gamma & \delta \end{pmatrix}$, then $k_2 = \begin{pmatrix} \alpha & p^{s-r}\beta \\ \gamma & \delta \end{pmatrix}$. As the action of $k \in K$ on V only depends the reduction $\bar{k} \in G(\mathbb{F}_p)$, the condition $k_1 \circ A = A \circ k_2$ may be written as

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\delta} \end{pmatrix} \circ A = A \circ \begin{pmatrix} \bar{\alpha} & 0 \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$

which is equivalent to the three conditions

$$\begin{pmatrix} 1 & \bar{\beta} \\ 0 & 1 \end{pmatrix} \circ A = A$$

$$A = A \circ \begin{pmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \bar{\delta} \end{pmatrix} \circ A = A \circ \begin{pmatrix} \bar{\alpha} & 0 \\ 0 & \bar{\delta} \end{pmatrix}$$

The first two implies the following factorization of A

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d] & V \\ V_{\overline{U}(\mathbb{F}_p)} \arrow[r, "\overline{A}"] & V^{U(\mathbb{F}_p)} \arrow[u] \end{tikzcd}$$

and the third implies that $\overline{A}$ is $T(\mathbb{F}_p)$-linear. We know that $V_{\overline{U}(\mathbb{F}_p)} \cong V^{U(\mathbb{F}_p)}$ as $T(\mathbb{F}_p)$-representations and that these are one-dimensional. Thus the space of Hecke operators supported on $K.\mathrm{diag}(p^r, p^s).K$ is one-dimensional and we may take $T_{r,s}$ to be the one corresponding to the projection

$$P = (V \twoheadrightarrow V_{\overline{U}(\mathbb{F}_p)} \leftarrow V^{U(\mathbb{F}_p)} \hookrightarrow V)$$

This finishes the proof of 1).

2) We first claim that $T_{i,i}T_{r,s} = T_{r+i,s+i} = T_{r,s}T_{i,i}$ for all i, r, s. In particular this implies that $T_{0,0}$ is the identity and that $T_2 = T_{1,1}$ is invertible, with $T_2^r = T_{r,r}$. To prove the claim, first note that

$$K \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix} K = K \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix}$$

which implies that

$$(T_{r,s}T_{i,i})(g) = \sum_{\gamma \in K \backslash G} T_{r,s}(g\gamma^{-1})T_{i,i}(\gamma) = T_{r,s}\left(g \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix}^{-1}\right) =$$

$$= \begin{cases} 1 & \text{if } r = s \text{ and } g \in K \begin{pmatrix} p^{r+i} & 0 \\ 0 & p^{s+i} \end{pmatrix} K \\ k_1 P k_2 & \text{if } r < s \text{ and } g = k_1 \begin{pmatrix} p^{r+i} & 0 \\ 0 & p^{s+i} \end{pmatrix} k_2 \in K \begin{pmatrix} p^{r+i} & 0 \\ 0 & p^{s+i} \end{pmatrix} K \\ 0 & \text{otherwise} \end{cases}$$

which is equal to $T_{r+i,s+i}(g)$ as desired. A similar calculation shows that $T_{r+i,s+i} = T_{r,s}T_{i,i}$, which proves the claim. Note also that this implies that T_2 is central.

Next, we claim that $T_{r,s}T_1 = T_{r,s+1} + \sum_{i>0} a_i T_{r+i,s+1-i}$ for some $a_i \in E$ (note that implicitly $r + i \leq s + 1 - i$, i.e. $i \leq (s + 1 - r)/2$). To show this we may first multiply by T_2^{-r} and hence without loss of generality assume that $r = 0$. If $s = 0$ as well we already know the result from above, so we may assume $s > 0$. By looking at the convolution formula, we see that

$$\operatorname{supp}(T_{0,s}T_1) \subseteq K \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix} K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K$$

and hence that

$$\operatorname{supp}(T_{0,s}T_1) \subseteq \coprod_{0 \leq i \leq (s+1)/2} K \begin{pmatrix} p^i & 0 \\ 0 & p^{s+1-i} \end{pmatrix} K$$

by looking at determinants in the Cartan decomposition. Thus we have an equation

$$T_{0,s}T_1 = \sum_{0 \leq i \leq (s+1)/2} a_i T_{i,s+1-i}$$

and it remains to show that $a_0 = 1$. We have

$$T_{0,s}T_1 \begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} = \sum_{\gamma \in K \backslash G} T_{0,s} \left(\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \gamma^{-1} \right) T_1(\gamma) =$$

$$= \sum_{\gamma \in K \backslash K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K} T_{0,s} \left(\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \gamma^{-1} \right) P$$

The double coset $K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K$ has a right coset decomposition

$$K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K = \left(\coprod_{0 \leq u \leq p-1} K \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix} \right) \sqcup K \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$$

so the sum becomes

$$\left(\sum_{0 \leq u \leq p-1} T_{0,s} \left(\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix}^{-1} \right) T_1 \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix} \right) +$$

$$+ T_{0,s} \left(\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}^{-1} \right) T_1 \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$$

Since $T_{0,s}$ is supported inside $M_2(\mathbb{Z}_p)$, we need the arguments for $T_{0,s}$ in the equation above to lie in $M_2(\mathbb{Z}_p)$ for those terms not to vanish. But

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -up^{-1} \\ 0 & p^s \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & p^{-1} \\ p^{s+1} & 0 \end{pmatrix}$$

so only term corresponding to $u = 0$ survives, so we get

$$T_{0,s} \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix} T_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = P \circ P = P = T_{0,s+1} \begin{pmatrix} 1 & 0 \\ 0 & p^{s+1} \end{pmatrix}$$

so $a_0 = 1$ as desired. Finally, combining these two results we have, for all $r \leq s$,

$$T_1^{s-r}T_2^r = T_2^r T_1^{s-r} = T_{r,s} + \sum_{1\leq i\leq (s-r)/2} a_i' T_{r+i,s-i}$$

for some $a_i' \in E$ (this follows by induction on $s-r$). Thus for all $r \leq s$ fixed there is a unipotent (hence invertible) matrix expressing $(T_1^{s-r-2i}T_2^{r+i})_{0\leq i\leq (s-r)/2}$ in terms of $(T_{r+i,s-i})_{0\leq i\leq (s-r)/2}$. Therefore, the set $(T_1^{s-r}T_2^r)_{r\leq s}$ forms another basis of $\mathcal{H}_G(V)$. This gives us the desired algebra structure (as T_2 is central). □

Corollary. *Let V be a weight, π a smooth representation of G and $f : V \hookrightarrow \pi|_K$ a K-linear injection. Then $f \star T_2 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} \circ f$ ($\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1}$ acts on π).*

Proof. Let $x \in V$. Then

$$(f \star T_2)(x) = \sum_{g\in K\backslash G} g^{-1} f(T_2(g)x) =$$

$$= \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} f\left(T_2\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} x\right) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} f(x)$$

as $T_2\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ is the identity. □

Proposition 14. *Suppose that $\chi : T \to E^\times$ is a smooth character and $f : V \hookrightarrow \mathrm{Ind}_B^G\chi$ is a K-linear injection. Then f is an eigenvector for $\mathcal{H}_G(V)$ and*

$$f \star T_1 = \chi\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} f$$

$$f \star T_2 = \chi\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} f$$

Proof. We know from Proposition 11that $\dim \mathrm{Hom}_K(V, \mathrm{Ind}_B^G\chi) = 1$ so f has to be an eigenvector. For T_2, the formula for the eigenvalue follows from the previous Corollary upon noting that the center of G acts as χ on $\mathrm{Ind}_B^G\chi$ (this follows directly from the definition of inductions).

For T_1, since we already know that f is an eigenvector it suffices to compute the eigenvalue by evaluating $f \star T_1$ and f suitably. First, we claim that $f(x)(1) \neq 0$ if $0 \neq x \in V^{U(\mathbb{F}_p)}$. To prove this, note that $f : V \hookrightarrow \mathrm{Ind}_B^G\chi \xrightarrow{\sim} \mathrm{Ind}_{B\cap K}^K\chi$ corresponds to the $T(\mathbb{F}_p)$-linear map $\theta : V_{\overline{U}(\mathbb{F}_p)} \xrightarrow{\sim} \chi|_{T(\mathbb{Z}_p)}$ given by $\theta(kv) = f(v)(k)$ for $k \in K$, $v \in V$. Thus $f(x)(1) = \theta(x) \neq 0$ since θ is an isomorphism. Therefore the T_1-eigenvalue will be the ratio between $(f \star T_1)(x)(1)$ and $f(x)(1)$. We have

$$(f \star T_1)(x) = \sum_{\gamma\in K\backslash G} \gamma^{-1} f(T_1(\gamma)(x))$$

hence

$$(f \star T_1)(x)(1) = \sum_{\gamma\in K\backslash G} f(T_1(\gamma)(x))(\gamma^{-1})$$

$T_1(\gamma)$ is 0 unless $\gamma \in K\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}K$, so we can use the coset decomposition $K\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}K = \left(\coprod_{u\in\mathbb{F}_p} K\begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix}\right) \sqcup K\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ to write the above as

$$\left(\sum_{u\in\mathbb{F}_p} f\left(T_1\begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix}x\right)\left(\begin{pmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{pmatrix}\right)\right) + f\left(T_1\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}x\right)\left(\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}\right)$$

Write $\begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, remembering that $T_1\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = P$ and that $x \in V^{U(\mathbb{F}_p)}$ we get

$$\left(\sum_{u\in\mathbb{F}_p} f(x)\left(\begin{pmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{pmatrix}\right)\right) + f\left(P\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}x\right)\left(\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}\right)$$

(using $P(x) = x$ because $x \in V^{U(\mathbb{F}_p)}$). $\begin{pmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{pmatrix}$ has Iwasawa decomposition

$$\begin{pmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} up^{-1} & 0 \\ -p^{-1} & u^{-1} \end{pmatrix}\begin{pmatrix} 1 & pu^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where the last two matrices are in K. We get

$$f(x)\left(\begin{pmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{pmatrix}\right) = \begin{cases} \chi\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} f(x)(1) \\ \chi\begin{pmatrix} up^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix} f(x)\left[\begin{pmatrix} 1 & pu^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \end{cases}$$

(the first option if $u = 0$, the second if $u \neq 0$) and we have (θ corresponding to f via Frobenius reciprocity as in Proposition 12)

$$f(x)\left[\begin{pmatrix} 1 & pu^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \theta\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}x\right)$$

Now note that $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}x = 0$ in $V_{\overline{U}(\mathbb{F}_p)}$ unless $\dim V = 1$, in which case it is x (use the explicit description of the weights to see this), i.e. corresponding to $a = b$ in our parametrisation of weights. Thus if $\dim V > 1$ we are done. If $\dim V = 1$ then $V = \det^b = \chi|_{T(\mathbb{Z}_p)}$ and we get

$$\chi\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} f(x)(1) + \left(\sum_{u\neq 0}\chi\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\theta(x)\right) + f(x)\left(\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}\right)$$

and

$$f(x)\left(\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}\right) = f(x)\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

so using this and simplifying we get

$$\chi\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} f(x)(1) + p\left(\chi\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\theta(x)\right).$$

As $p = 0$ in E we get the desired equality

$$(f \star T_1)(x)(1) = \chi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} f(x)(1)$$

□

6. Comparison Between Compact and Parabolic Induction

Let V be a weight, χ a character of T assume that we have a K-linear embedding $f : V \hookrightarrow \mathrm{Ind}_B^G \chi$. By Frobenius reciprocity we have a nonzero G-linear map $\tilde{f} : \mathrm{ind}_K^G V \to \mathrm{Ind}_B^G \chi$. Proposition 14 implies that $\tilde{f}$ is a Hecke eigenvector and we know the eigenvalues. Let χ' is the character of $\mathcal{H}_G(V)$ defined by $\chi'(T_1) = \chi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}$ and $\chi'(T_2) = \chi \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1}$. Then $\tilde{f}$ factors as (think of χ' as a quotient of $\mathcal{H}_G(V)$)

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \to \mathrm{Ind}_B^G \chi$$

Theorem 15. *This map is an isomorphism if* $\dim V > 1$.

We will prove a stronger, universal version of this statement. Since $V_{\overline{U}(\mathbb{F}_p)} \cong \chi|_{T(\mathbb{Z}_p)}$ we get a T-linear surjection $\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)} \twoheadrightarrow \chi$, and the converse is also true, so $\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$ is universal for characters χ such that $V_{\overline{U}(\mathbb{F}_p)} \cong \chi|_{T(\mathbb{Z}_p)}$. $\tilde{f}$ then factors as

$$\mathrm{ind}_K^G V \xrightarrow{F} \mathrm{Ind}_B^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \twoheadrightarrow \mathrm{Ind}_B^G \chi$$

where F is obtained from the canonical map $V_{\overline{U}(\mathbb{F}_p)} \to \mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$ (sending y to $[1, y]$) by the following series of equalities:

$$\mathrm{Hom}_{T(\mathbb{Z}_p)}(V_{\overline{U}(\mathbb{F}_p)}, \mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) = \mathrm{Hom}_{\overline{B}(\mathbb{Z}_p) = \overline{B} \cap K}(V, \mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) =$$
$$= \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B} \cap K}^K(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})) = \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}))$$

noting that, as before $\mathrm{Ind}_{\overline{B} \cap K}^K(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \cong \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})$ as K-representations. Our theorem is then

Theorem 16. *Let F be as above. Then F is also $\mathcal{H}_G(V)$-linear, where T_1 acts as* $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}$ *and T_2 acts as* $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1}$ *on* $\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$ *(these matrices are in T so act on* $\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$*). The induced map*

$$(\mathrm{ind}_K^G V)[T_1^{-1}] \to \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})$$

is injective, and is an isomorphism if $\dim V > 1$.

Here, $(\mathrm{ind}_K^G V)[T_1^{-1}] \cong \mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \mathcal{H}_G(V)[T_1^{-1}]$ is the localisation of the $\mathcal{H}_G(V)$-module $\mathrm{ind}_K^G V$ at T_1.

This theorem implies Theorem 15 in the following way. Let $\dim V > 1$. Apply $- \otimes_{\mathcal{H}_G(V)[T_1^{-1}]} \chi'$ to the isomorphism in Theorem 16 to get an isomorphism

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \longrightarrow \mathrm{Ind}_{\overline{B}}^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \otimes_{\mathcal{H}_G(V)[T_1^{-1}]} \chi'$$

We wish to show that $\mathrm{Ind}_B^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \otimes_{\mathcal{H}_G(V)[T_1^{-1}]} \chi' \cong \mathrm{Ind}_B^G \chi$. Note that we have a surjection

$$\mathcal{H}_G(V)[T_1^{-1}] \cong E[T_1^{\pm 1}, T_2^{\pm 1}] \xrightarrow{\chi'} E = \frac{E[T_1^{\pm 1}, T_2^{\pm 1}]}{(T_1 - \lambda_1, T_2 - \lambda_2)}$$

where $\lambda_1^{-1} = \chi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\lambda_2^{-1} = \chi \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. By exactness of Ind_B^G (Proposition 5) we deduce

$$\mathrm{Ind}_B^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \otimes_{\mathcal{H}_G(V)[T_1^{-1}]} \chi' \cong \mathrm{Ind}_B^G \left(\frac{\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}}{(T_1 - \lambda_1, T_2 - \lambda_2)} \right)$$

and we know from before that the quotient in the right-hand side is χ.

Let us now prove the surjectivity in Theorem 16; the injectivity will be proven later.

Proof. First we prove that F is $\mathcal{H}_G(V)$-linear, proceeding as in Proposition 14. Fix a nonzero $x \in V^{U(\mathbb{F}_p)}$. We have

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_B^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})) \cong \mathrm{Hom}_{T(\mathbb{Z}_p)}(V_{\overline{U}(\mathbb{F}_p)}, \mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}) \cong$$

$$\cong \mathrm{End}_T(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})$$

(where the first isomorphism was explained above) and thus $\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_B^G(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}))$ is a free $\mathrm{End}_T(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)})$-module of rank 1. The same calculation as in Proposition 14, transposed to our setting, shows that

$$(F \circ T_1)(x)(1) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} F(x)(1)$$

$$(F \circ T_2)(x)(1) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} F(x)(1)$$

As $F(x)(1) = [1, \bar{x}]$ generates $\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$ as a T-representation (since $V_{\overline{U}(\mathbb{F}_p)}$ is one-dimensional; in general, for any set (s_i) of σ as a U-representation, the $[1, s_i]$ generate $\mathrm{ind}_U^G \sigma$ as a G-representation) we have that $F \circ T_1 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} F$ and $F \circ T_2 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} F$.

Let us now get down to proving surjectivity. It suffices to show that $f_0 = F(x)$ (x as above) generates the target under the G- and $\mathcal{H}_G(V)$-actions. f_0 is a function $G \to \mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}$, sending an element $t\bar{u}k \in G$ ($t \in T$, $\bar{u} \in \overline{U}$, $k \in K$) to $[t, \overline{kx}]$. We want to work out when $\overline{kx} \neq 0$ in $V_{\overline{U}(\mathbb{F}_p)}$. The Bruhat decomposition says that

$$\mathrm{GL}_2(\mathbb{F}_p) = B(\mathbb{F}_p) \sqcup B(\mathbb{F}_p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B(\mathbb{F}_p)$$

Multiplying by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the left we get

$$\mathrm{GL}_2(\mathbb{F}_p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B(\mathbb{F}_p) \sqcup \overline{B}(\mathbb{F}_p) B(\mathbb{F}_p)$$

Elements in the first summand on the right-hand side kills x because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$ is in the kernel of $V \twoheadrightarrow V_{\overline{U}(\mathbb{F}_p)}$ if $\dim V > 1$. Thus k needs to map to the second summand, which we may simplify to $\overline{B}(\mathbb{F}_p)U(\mathbb{F}_p)$. Thus k lies in

$$\begin{pmatrix} \mathbb{Z}_p^\times & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} =$$

$$= \left(\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \times \begin{pmatrix} 1 & p\mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} = \overline{B}(\mathbb{Z}_p) \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

and hence the support of f_0 lies in $\overline{B} \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$. Moreover, for any $a \in \mathbb{Z}_p$, $f_0 \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) = [1, \bar{x}]$, so we have equality. The same technique that was used in Proposition 5 to construct the section we deduce

$$\left\{ f \in \mathrm{Ind}_{\overline{B}}^{G}(\mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)}) \mid \mathrm{supp}(f) \subseteq \overline{B}U \right\} \cong \mathcal{C}_c^\infty(\mathbb{Q}_p, \mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$$

$$f \longmapsto \left(f' : a \mapsto f \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right)$$

(the inverse of this map is extension by 0, this works because of the compact support condition) This is compatible with the $\mathcal{H}_G(V)$-action. Since $\mathrm{supp}(g.f) = \mathrm{supp}(f).g^{-1}$, the subspace of $\mathrm{Ind}_{\overline{B}}^{G}(\mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$ above is B-stable. Transfer of structure to $\mathcal{C}_c^\infty(\mathbb{Q}_p, \mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$ gives us a B-representation, with the following actions:

$$\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f' \right)(a) = f'(a+u)$$

$$\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} f' \right)(a) = f'(ay/x)$$

We have

$$f_0'(a) = \begin{cases} [1, \bar{x}] & \text{if } a \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

Now acting by $\mathcal{H}_G(V)[T_1^{-1}]$ gives all functions supported and constant on $\mathbb{Z}_p$. The T-action scales, so we get any function supported and constant on some $p^n\mathbb{Z}_p$, for any $n \in \mathbb{Z}$. Finally the U-action translates, and so we get functions supported and constant on some $a + p^n\mathbb{Z}_p$, $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$. These functions span $\mathcal{C}_c^\infty(\mathbb{Q}_p, \mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$. Translating this back to $\mathrm{Ind}_{\overline{B}}^{G}(\mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$ we know have all functions supported on $\overline{B}U$. Translating by G, we have any function supported on $\overline{B}Ug$, for any $g \in G$. As these sets cover G, we now argue as in Proposition 5 to get all of $\mathrm{Ind}_{\overline{B}}^{G}(\mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$, which finishes the proof that $f_0 = F(x)$ generates $\mathrm{Ind}_{\overline{B}}^{G}(\mathrm{ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)})$ and hence that F is surjective. □

Remark. If $\dim V = 1$, then F is not surjective.

Corollary. *If V is a weight of $\mathrm{Ind}_{\overline{B}}^{G}\chi$ such that $\dim V > 1$, then V generates $\mathrm{Ind}_{\overline{B}}^{G}\chi$ as a G-representation. In particular, if $\chi = \chi_1 \otimes \chi_2$ (χ_1, χ_2 characters of $\mathbb{Q}_p^\times$, $\chi = \chi_1 \otimes \chi_2$ defined by $\chi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \chi_1(x)\chi_2(y)$; we will use this notation in the future without further comment) with $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$, then $\mathrm{Ind}_{\overline{B}}^{G}\chi$ is irreducible.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
V \arrow[d, hook] \arrow[dr, hook] & \\
\mathrm{ind}_K^G V \arrow[r] \arrow[d] & \mathrm{Ind}_{\overline{B}}^G \chi \\
\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \arrow[ur] &
\end{tikzcd}$$

The map $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \to \mathrm{Ind}_{\overline{B}}^G \chi$ is an isomorphism, so the horizontal map $\mathrm{ind}_K^G V \to \mathrm{Ind}_{\overline{B}}^G \chi$ is surjective. As $V \subseteq \mathrm{ind}_K^G V$ generates $\mathrm{ind}_K^G V$, $V \subseteq \mathrm{Ind}_{\overline{B}}^G \chi$ generates $\mathrm{Ind}_{\overline{B}}^G \chi$. If furthermore V is the unique weight (necessarily of multiplicity one), then $\mathrm{Ind}_{\overline{B}}^G \chi$ is irreducible (as any subrepresentation has to contain a weight).

By Proposition 11 V is a weight of $\mathrm{Ind}_{\overline{B}}^G \chi$ if and only if $V_{\overline{U}(\mathbb{F}_p)} \cong \chi|_{T(\mathbb{Z}_p)} = \chi_{a,b}$ for unique $0 \le a-b < p-1$, $0 \le b < p-1$, and

$$V = \begin{cases} F(a,b) & \text{if } a \neq b \\ F(b,b),\ F(b+p-1,b) & \text{if } a = b \end{cases}$$

where in the second case $F(b+p-1,b)$ generates $\mathrm{Ind}_{\overline{B}}^G \chi$ and $F(b,b) = \det^b$ is one-dimensional. Thus, in the first case we see that $\mathrm{Ind}_{\overline{B}}^G \chi$ is irreducible, and the condition $a \neq b$ is equivalent to $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$. □

7. Steinberg Representation for GL_2

The principal series $\mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}}) = \mathcal{C}^\infty(\overline{B}\backslash G, E) = \mathcal{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)$ has a one-dimensional trivial subrepresentation 1_G consisting of the constant functions.

Definition. The quotient of $\mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}})$ by 1_G is called the Steinberg representation and will be denoted by St.

If $\chi : \mathbb{Q}_p^\times \to E^\times$ is a smooth character we may tensor the exact sequence

$$0 \to 1_G \to \mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}}) \to \mathrm{St} \to 0$$

by $\chi \circ \det$ to get an exact sequence

$$0 \to \chi \circ \det \to \mathrm{Ind}_{\overline{B}}^G(\chi \circ \det) \to \mathrm{St} \otimes (\chi \circ \det) \to 0$$

Theorem 17. St *is irreducible.*

Proof. $\mathrm{St}^{I(1)}$ is one-dimensional (exercise: use the description of $\mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}})$ as $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)$ and show that any function which is $I(1)$-invariant in the Steinberg quotient is actually $I(1)$-invariant in $\mathcal{C}^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)$). This shows that St must have a unique weight, since each weight gives a positive-dimensional contribution to the $I(1)$-invariants. St is therefore irreducible, proving the theorem. □

Remark. 1) Let $\overline{\mathrm{St}}$ denote the weight $F(p-1,0)$ ("Steinberg weight"); it is a weight of $\mathrm{Ind}_{\overline{B}}^{G}(1_{\overline{B}})$ and is disjoint from 1_G, hence a weight of St and therefore the unique weight of St.

2) The Hecke eigenvalues of $\overline{\mathrm{St}}$ in St (and $1_K \subseteq 1_G$) are the same as in $\mathrm{Ind}_{\overline{B}}^{G}(1_{\overline{B}})$, namely

$$T_1, T_2 \mapsto 1$$

Similarly, for $\mathrm{St} \otimes (\chi \circ \det)$, we get $T_1 \mapsto \chi(p)^{-1}$, $T_2 \mapsto \chi(p)^{-2}$.

3) The sequence $0 \to 1_G \to \mathrm{Ind}_{\overline{B}}^{G}(1_{\overline{B}}) \to \mathrm{St} \to 0$ is nonsplit.

8. "Change of Weight" for GL_2

We wish to show that the remaining principal series representations are irreducible. To this, we will study G-linear maps $\mathrm{ind}_K^G V \to \mathrm{ind}_K^G V'$ for (distinct) weights V,V'.

Let $\mathcal{H}_G(V,V') = \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{ind}_K^G V')$, it is a $(\mathcal{H}_G(V'), \mathcal{H}_G(V))$-bimodule, with the extra structure of composition maps (or three weights V, V', V'')

$$\mathcal{H}_G(V',V'') \times \mathcal{H}_G(V,V') \to \mathcal{H}_G(V,V'')$$

We have the following generalisation of Proposition 12:

Proposition 18. *(i) $\mathcal{H}_G(V,V')$ is isomorphic to the space of functions $\varphi : G \to \mathrm{Hom}_E(V,V')$ such that φ has compact support and $\varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2$ for any $k_1, k_2 \in K$, $g \in G$, with the composition maps given by convolution as in Proposition 12.*

(ii) $\mathcal{H}_G(V,V') \neq 0$ if and only if $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$ as $T(\mathbb{F}_p)$-representations.

(iii) If $V \not\cong V'$ and $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$, there is a Hecke operator $\varphi : G \to \mathrm{Hom}_E(V,V')$ supported on $K \begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix} K$ if and only if $r < s$, and it is unique up to scalar.

Proof. This is proved in exactly the same way as Proposition 12. For parts (ii) and (iii) we trace the explicit computation of the Hecke operators in Proposition 12 (changing one V for V') and note that to find a Hecke operator supported on $K \begin{pmatrix} p^r & 0 \\ 0 & p^r \end{pmatrix} K$ we need a K-linear map $\varphi \begin{pmatrix} p^r & 0 \\ 0 & p^r \end{pmatrix} V \to V'$, which is zero unless $V \cong V'$, and to find a Hecke operator supported on $K \begin{pmatrix} p^r & 0 \\ 0 & p^s \end{pmatrix} K$ for $r < s$ we needed a $T(\mathbb{F}_p)$-linear map $V_{\overline{U}(\mathbb{F}_p)} \to (V')^{U(\mathbb{F}_p)}$ (denoted by $\overline{A}$ in the proof of Proposition 12), which is zero unless $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$, in which case it is unique to scalar. $\square$

From this proposition we conclude that $\mathcal{H}_G(V,V')$ gives us something new only in the case $V = F(b,b)$, $V' = F(b+p-1,b)$ (or vice versa), and that there are nonzero G-linear maps

$$\mathrm{ind}_K^G V \underset{\varphi^+}{\overset{\varphi^-}{\leftrightarrows}} \mathrm{ind}_K^G V'$$

such that $\mathrm{supp}(\varphi^{\pm}) = K \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K$; they are furthermore unique up to scalar. Identify $\mathcal{H}_G(V)$ and $\mathcal{H}_G(V')$ by identifying the T_i from Theorem 13 (we will see later that this identification is in fact natural) and write $\mathcal{H}_G$ for this algebra.

Proposition 19. *1) $\varphi^{\pm}$ are $\mathcal{H}_G$-linear.*

2) $\varphi^- \circ \varphi^+ = \varphi^+ \circ \varphi^- = T_1^2 - T_2$ up to a nonzero scalar.

Proof. 1) and the first equality in 2) will be formal consequences later of the Satake isomorphism. The second equality in 2) is obtained by explicit calculation (again, see later for the generalisation to GL_n). □

Remark. Analogues of the $\varphi^{\pm}$ were used by Kisin in his paper "The Fontaine-Mazur conjecture for GL_2".

Corollary. *If $\chi' : \mathcal{H}_G \to E$ is an algebra homomorphism with $\chi'(T_1^2 - T_2) \neq 0$, then $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi' \cong \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G} \chi'$.*

Proof. $\varphi^{\pm}$ induce G-linear maps between the these two representations, and their compositions either way are $T_1^2 - T_2$, which acts invertibly (by the nonzero scalar $\chi'(T_1^2 - T_2)$) on $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi'$ and $\mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G} \chi'$, so $\varphi^{\pm}$ induce isomorphisms. □

Proposition 20. *If $\chi = \chi_1 \otimes \chi_2 : T \to E^\times$ is a smooth character and $\chi_1 \neq \chi_2$, then $\mathrm{Ind}_B^G \chi$ is irreducible.*

Proof. If $\chi_1|_{T(\mathbb{Z}_p)} \neq \chi_2|_{T(\mathbb{Z}_p)}$ then this is the Corollary to Theorem 15. Thus we may assume $\chi_1|_{T(\mathbb{Z}_p)} = \chi_2|_{T(\mathbb{Z}_p)}$; it follows that $\chi_1(p) \neq \chi_2(p)$. Suppose the character $\chi_1|_{T(\mathbb{Z}_p)} = \chi_2|_{T(\mathbb{Z}_p)}$ is given by $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times \to E^\times$ where the latter map is $x \mapsto x^b$ for some $b \in \mathbb{Z}$. Then $\mathrm{Ind}_B^G \chi$ has weights $V = F(b, b)$ and $V' = F(b + p - 1, b)$ with Hecke eigenvalues

$$T_1 \mapsto \chi_2(p)^{-1} \,,\; T_2 \mapsto \chi_1(p)^{-1}\chi_2(p)^{-1}$$

Suppose that $\sigma \subseteq \mathrm{Ind}_B^G \chi$ is a nonzero G-subrepresentation. σ has to contains one of the two weights above. If it contains V' then $\sigma = \mathrm{Ind}_B^G \chi$ by the Corollary to Theorem 15. Suppose that a priori we only know that σ contains V. We have

$$\chi'(T_1^2 - T_2) = \chi_2(p)^{-2} - \chi_1(p)^{-1}\chi_2(p)^{-1} \neq 0$$

as $\chi_1(p) \neq \chi_2(p)$. Therefore by the Corollary above,

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi' \cong \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G} \chi'$$

and the former has a nonzero G-linear map to σ as V is a weight of σ, hence we get a nonzero G-linear map $\mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G} \chi' \to \sigma$ and so a K-linear embedding $V' \hookrightarrow \sigma$, so V' is a weight of σ, and hence $\sigma = \mathrm{Ind}_B^G \chi$ as above. We conclude that $\mathrm{Ind}_B^G \chi$ is irreducible. □

9. Admissible Representations

In this section we will let G be any Hausdorff topological group that has a fundamental system of open neighbourhoods of 1 consisting of pro-p subgroups, unless stated otherwise.

Definition. A smooth G-representation π is admissible if $\dim \pi^U$ is finite for any open subgroup $U \subseteq G$.

Remark. 1) Any subrepresentation of an admissible representation is admissible (clear from the definition).

2) In fact, for certain groups (e.g. $\mathrm{GL}_n(\mathbb{Q}_p)$) quotients of admissible representations are admissible.

Lemma 21. *π is admissible if and only if there is an open pro-p subgroup U of G such that $\dim \pi^U$ is finite.*

Proof. If π is admissible then by definition $\mathrm{di},\mathrm{m}\,\pi^U$ is finite for any open subgroup U. Conversely, let U be as in the statement of the lemma and let U' be any other open subgroup of G, we wish to show that $\pi^{U'}$ is finite-dimensional. As $\pi^{U'} \subseteq \pi^{U \cap U'}$, we may assume that $U' \subseteq U$. We have

$$\pi^{U'} = \mathrm{Hom}_{U'}(1_{U'}, \pi) = \mathrm{Hom}_U(\mathrm{ind}_{U'}^U 1_{U'}, \pi)$$

with $\mathrm{ind}_{U'}^U 1_{U'}$ finite-dimensional as the index $(U : U')$ is finite. To finish we show, by induction on $\dim \sigma$, that $\mathrm{Hom}_U(\sigma, \pi)$ is finite-dimensional for any finite-dimensional smooth representation σ of U. If σ is one-dimensional, then $\sigma = 1_U$ (by Lemma 7) and this reduces to the hypothesis of the lemma. For the induction step, by Lemma 7 we have $1_U \subseteq \sigma$, and so applying $\mathrm{Hom}_U(-, \pi)$ to

$$0 \to 1_U \to \sigma \to \sigma/1_U \to 0$$

we get an exact sequence

$$0 \to \mathrm{Hom}_U(\sigma/1_U, \pi) \to \mathrm{Hom}_U(\sigma, \pi) \to \mathrm{Hom}_U(1_U, \pi)$$

and hence $\mathrm{Hom}_U(\sigma, \pi)$ is finite-dimensional as $\mathrm{Hom}_U(\sigma/1_U, \pi)$ and $\mathrm{Hom}_U(1_U, \pi)$ are finite-dimensional. □

Remark. This is false if the characteristic of E is $\neq p$ ($G = \mathbb{Q}_p^\times$ gives some easy examples).

Lemma 22. *If $\pi \neq 0$ is an admissible G-representation, then π contains an irreducible subrepresentation.*

Proof. Fix an open pro-p subgroup U of G. Then, for any nonzero G-subrepresentation $\tau \subseteq \pi$ we have $0 \neq \tau^U \subseteq \pi^U$. As π^U is finite-dimensional we may pick τ such that τ^U has minimal (nonzero) dimension. Then the G-subrepresentation τ' generated by τ^U is irreducible, as any nonzero subrepresentation $\sigma \subseteq \tau'$ must have $\sigma^U \subseteq \tau'^U = \tau^U$ but also $\dim \sigma^U \geq \dim \tau^U$, so $\sigma \supseteq \sigma^U = \tau^U$ and hence $\sigma = \tau'$. □

Lemma 23. *Here we take $G = \mathrm{GL}_n(\mathbb{Q}_p)$. Suppose that π is any smooth G-representation. Then π is admissible if and only if π has finitely many weights (counted with multiplicity).*

Proof. Assume π is admissible. For any weight V its multiplicity in π is the dimension of $\mathrm{Hom}_K(V, \pi) = \mathrm{Hom}_K(V, \pi^{K(1)})$ which is finite-dimensional as $\pi^{K(1)}$ is. As there are only finitely many weights, π only has finitely many weights counting with multiplicity.

For the converse, by Lemma 21 it is enough to show that $\pi^{K(1)}$ is finite-dimensional. We have

$$\pi^{K(1)} = \mathrm{Hom}_{K(1)}(1_{K(1)}, \pi) = \mathrm{Hom}_K(\mathrm{ind}_{K(1)}^K 1_{K(1)}, \pi)$$

with $\mathrm{ind}_{K(1)}^K 1_{K(1)}$ finite-dimensional. As in Lemma 21, $\mathrm{Hom}_K(\sigma, \pi)$ is finite-dimensional for any finite-dimensional K-representation σ by induction on the number of irreducible subquotients in σ where the base case $\mathrm{Hom}_K(V, \pi)$ finite-dimensional for irreducible V is the hypothesis of the lemma. □

Remark. By this lemma and what we have proven earlier, all principal series representations are admissible, as well as $\chi \circ \det$ and $\mathrm{St} \otimes (\chi \circ \det)$ for any smooth character $\chi : \mathbb{Q}_p^\times \to E^\times$.

Lemma 24. *Take $G = \mathrm{GL}_n(\mathbb{Q}_p)$.*

1) (Schur's Lemma) If π is an irreducible admissible representation, then $\mathrm{End}_G(\pi) = E$.

2) Any irreducible admissible representation π of G has a central character $\chi_\pi : \mathbb{Q}_p^\times \to E^\times$.

Proof. 1) Let $\phi \in \mathrm{End}_G(\pi)$. Then $\pi^{K(1)} \neq 0$ is finite-dimensional and $\phi(\pi^{K(1)}) \subseteq \pi^{K(1)}$, so ϕ has an eigenvector $0 \neq v \in \pi^{K(1)}$ with eigenvalue $\lambda \in E$ say, so $\mathrm{Ker}(\phi - \lambda.\mathrm{id}_\pi) \neq 0$. But it is also a G-subrepresentation of π, and hence must be π itself as π is irreducible, so $\phi = \lambda.id_\pi$.

2) If $z \in Z(G)$, then $z \in \mathrm{Aut}_G(\pi)$ (thinking of z as an automorphism of the underlying vector space; since z is central it intertwines the G-action). Moreover this map $Z(G) \to \mathrm{Aut}_G(\pi)$ is a group homomorphism. But $\mathrm{Aut}_G(\pi) = E^\times$ by 1) and $Z(G) = \mathbb{Q}_p^\times$, so we obtain a character $\mathbb{Q}_p^\times \to E^\times$. □

Remark. This result is not know if we merely assume π to be smooth and irreducible. Neither is it known if any smooth irreducible π with a central character is admissible, except for $n = 1, 2$.

10. Classification of Irreducible Admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

In this section we are back to be the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let π be an irreducible admissible representation of G and let V be a weight of π, then $\mathrm{Hom}_K(V, \pi)$ is nonzero and finite-dimensional (i.e. the multiplicity of V in π is finite, by admissibility of π) and moreover a module for the commutative algebra $\mathcal{H}_G(V)$. Hence we may find $0 \neq f \in \mathrm{Hom}_K(V, \pi)$ which is a simultaneous eigenvector for all elements of $\mathcal{H}_G(V)$, i.e. there is an algebra homomorphism $\chi' : \mathcal{H}_G(V) \to E$ such that

$$f \star \varphi = \chi'(p) f$$

for all $\varphi \in \mathcal{H}_G(V)$. So we get a G-linear map surjection $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \twoheadrightarrow \pi$. Recall that for principal series representations we found that $\chi'(T_1) \neq 0$.

Definition. An irreducible admissible G-representation π is called supersingular if, for any weight V, the following two equivalent conditions hold:

1) T_1 is nilpotent on $\mathrm{Hom}_K(V, \pi)$ (i.e. the only eigenvalue is 0)

2) $\mathrm{Hom}_K(V, \pi)[T_1^{-1}] = 0$.

Theorem 25. *(Barthel-Livné) The irreducible admissible G-representations π fall into the following 4 disjoint categories:*

1) Irreducible principal series $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, $\chi_1 \neq \chi_2$.

2) One-dimensional representations $\chi \circ \det$, $\chi : \mathbb{Q}_p^\times \to E^\times$ smooth character.

3) $\mathrm{St} \otimes (\chi \circ \det)$, $\chi : \mathbb{Q}_p^\times \to E^\times$ smooth character.

4) Supersingular representations.

Remark. 1) Barthel and Livné proved this under the (a priori) weaker assumption that π is smooth irreducible and has a central character.

2) Any supersingular representation is by definition a quotient of $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi'$ with $\chi'(T_1) = 0$. Breuil showed that all these representations are irreducible (and hence supersingular). This gives a classification of supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. A nice proof can be found in Emerton's

paper "On a class of coherent rings, with applications to the smooth representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ in characteristic p". Work of Breuil and Paskunas shows that, for $\mathrm{GL}_2(F)$ with F a proper finite extension of $\mathbb{Q}_p$, these compact inductions typically (always?) have infinitely many non-isomorphic irreducible admissible quotients.

Proof. We know that the four cases are disjoint and that all representations in them are irreducible (in the cases 1) – 3) we proved this, for 4) this is in the definition). Moreover we have also proved that the Hecke eigenvalues and the weights determine the characters χ_1 and χ_2 in 1) and χ in 2) and 3). Let π be any irreducible admissible representation of G. We need to show that π is in one of the four categories above, so assume that π is not supersingular and let V be a weight of π and $\chi' : \mathcal{H}_G(V) \to E$ an algebra homomorphism with $\chi'(T_1) \neq 0$ such that $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \twoheadrightarrow \pi$. If $\dim V > 1$, then $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' = \mathrm{Ind}_{\overline{B}}^G \chi$ by Theorem 15, where χ is determined by

$$\chi|_{T(\mathbb{Z}_p)} \cong V_{\overline{U}(\mathbb{F}_p)}$$

$$\chi\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} = \chi'(T_1) \neq 0$$

$$\chi\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} = \chi'(T_2)$$

and hence π is in one of 1) – 3). If $\dim V = 1$ and $\chi'(T_1^2 - T_2) \neq 0$, then by the Corollary to Proposition 19, $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V)} \chi' \cong \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G(V)} \chi'$ where V' is the unique weight such that $\dim V' > 1$ and $V'_{\overline{U}(\mathbb{F}_p)} \cong V_{\overline{U}(\mathbb{F}_p)}$. This then brings us into the previous case.

Now assume $\dim V = 1$ and $\chi'(T_1^2 - T_2) = 0$. By replacing π with $\pi \otimes (\eta \circ \det)$ where $(\eta \circ \det)^{-1}|_K \cong V$ and $\eta(p) = \chi'(T_1)$, we may assume that $V = 1_K$ and $\chi'(T_1) = \chi'(T_2) = 1$. We claim that $\mathrm{ind}_K^G 1_K \otimes_{\mathcal{H}_G(V)} \chi'$ has finite length and that $(\mathrm{ind}_K^G 1_K \otimes_{\mathcal{H}_G(V)} \chi')^{ss} \cong (\mathrm{Ind}_{\overline{B}}^G 1_{\overline{B}})^{ss}$; if this holds then we are done. We will need the following two facts:

1) $(\mathrm{ind}_K^G 1_K)[T_1^{-1}]$ is a free $\mathcal{H}_G(V)[T_1^{-1}]$-module.

2) $(\mathrm{ind}_K^G \overline{\mathrm{St}})[T_1^{-1}]$ is a free $\mathcal{H}_G(V)[T_1^{-1}]$-module.

The second fact follows from Theorem 16. 1) follows from comparison of Theorem 16 for 1_K and $\overline{\mathrm{St}}$.

Next, let us write $\mathcal{H}_G$ for $\mathcal{H}_G(1_K) \cong \mathcal{H}_G(\overline{\mathrm{St}})$ (identified as in Section 8). By Proposition 19 there are $\mathcal{H}_G$- and G-linear maps

$$\mathrm{ind}_K^G 1_K \underset{\varphi^+}{\overset{\varphi^-}{\leftrightarrows}} \mathrm{ind}_K^G \overline{\mathrm{St}}$$

such that $\varphi^- \circ \varphi^+ = \varphi^+ \circ \varphi^- = T_2 - T_1^2$ (scale so the second equality holds on the nose). Let

$$\sigma = \frac{\mathrm{ind}_K^G 1_K}{(T_1 - 1)\mathrm{ind}_K^G 1_K}, \qquad \tau = \frac{\mathrm{ind}_K^G \overline{\mathrm{St}}}{(T_1 - 1)\mathrm{ind}_K^G \overline{\mathrm{St}}}$$

Then we have induced maps

$$\tau \xrightarrow{\varphi^-} \sigma \xrightarrow{\varphi^+} \tau \xrightarrow{\varphi^-} \sigma$$

(by abuse of notation) between the quotients that are both G-linear and $\mathcal{H}_G/(T_1 - 1) \cong E[T_2^{\pm 1}]$-linear, and $\varphi^- \circ \varphi^+ = \varphi^+ \circ \varphi^- = T_2 - 1$. By the facts above, σ and τ are free $E[T_2^{\pm 1}]$-modules (as $\mathcal{H}_G[T_1^{-1}]/(T_1-1)\mathcal{H}_G[T_1^{-1}] = \mathcal{H}_G/(T_1-1)$) and hence T_2-1 acts injectively on σ and τ, so $\tau \xrightarrow{\varphi^-} \sigma \xrightarrow{\varphi^+} \tau \xrightarrow{\varphi^-} \sigma$ may be viewed as a chain of submodules

$$(T_2-1)\tau \subseteq (T_2-1)\sigma \subseteq \tau \subseteq \sigma$$

Now

$$\frac{\tau}{(T_2-1)\tau} \cong \frac{\mathrm{ind}_K^G \overline{\mathrm{St}}}{(T_1-1, T_2-1)\mathrm{ind}_K^G \overline{\mathrm{St}}} \cong \mathrm{ind}_K^G \overline{\mathrm{St}} \otimes_{\mathcal{H}_G} \chi' \cong \mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}})$$

where the last isomorphism comes from Theorem 15. Hence both $\tau/(T_2-1)\tau$ and its subrepresentation $(T_2-1)\sigma/(T_2-1)\tau \cong \sigma/\tau$ are of finite length and

$$\left(\mathrm{ind}_K^G 1_K \otimes_{\mathcal{H}_G(V)} \chi'\right)^{ss} \cong \left(\frac{\mathrm{ind}_K^G 1_K}{(T_1-1, T_2-1)\mathrm{ind}_K^G 1_K}\right)^{ss} \cong$$

$$\cong \left(\frac{\sigma}{(T_2-1)\sigma}\right)^{ss} \cong \left(\frac{\tau}{(T_2-1)\tau}\right)^{ss} \cong \left(\mathrm{Ind}_{\overline{B}}^G(1_{\overline{B}})\right)^{ss}$$

which is what we wanted. □

Definition. An irreducible admissible representation is supercuspidal if it is not a subquotient of a principal series representation.

Corollary. *1) π is supersingular if and only if it is supercuspidal.*

2) An irreducible admissible representation π "has constant Hecke eigenvalues".

3) All principal series representations have finite length (which is 1 or 2, this is not really a corollary but was proved earlier).

11. Weights for GL_n

Let us state some results from the modular representation theory of $\mathrm{GL}_n(\mathbb{F}_p)$:

Theorem. *The irreducible (smooth) representations of $\mathrm{GL}_n(\mathbb{F}_p)$ (over E) are parametrised by equivalence classes of n-tuples $\nu \in (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$ such that $0 \le \nu_i - \nu_{i+1} \le p-1$ for all $1 \le i \le n-1$, where ν and ν' are equivalent if $\nu_1 - \nu_1' = \cdots = \nu_n - \nu_n' \in (p-1)\mathbb{Z}$. The total number of equivalence classes is $p^{n-1}(p-1)$. We write $F(\nu)$ for the representation corresponding to ν.*

We will now give an explicit description of $F(\nu)$. We may regard ν as a character $T(E) \to E^\times$ defined by

$$\mathrm{diag}(t_1, \dots, t_n) \in T(E) \longmapsto t_1^{\nu_1} \cdots t_n^{\nu_n}$$

Let

$$W(\nu) = \left\{ f \in \mathcal{O}_{\mathrm{GL}_n(E)} \mid f(t\bar{u}g) = \nu(t) f(g) \quad \forall t \in T(E),\, \bar{u} \in \overline{U}(E),\, g \in \mathrm{GL}_n(E) \right\}$$

where we let $\mathcal{O}_{\mathrm{GL}_n(E)}$ denoted the ring of regular functions of $\mathrm{GL}_n(E)$, i.e. the functions $f : \mathrm{GL}_n(E) \to E$ such $f(g)$ is a polynomial in the matrix entries g_{ij} of g and $\det(g)^{-1}$. We let $\mathrm{GL}_n(E)$ act on $W(\nu)$ via the right regular representation $(g.f)(x) = f(xg)$. This is a finite-dimensional representation of $\mathrm{GL}_n(E)$ (one can see that $W(\nu)$ is in fact the space of global sections of a line bundle on the projective variety $B(E)\backslash \mathrm{GL}_n(E)$).

It turns out that $W(\nu)^{U(E)}$ is one-dimensional (roughly, $f \in W(\nu)^{U(E)}$ then $f(1)$ determines f on the big cell, which is Zariski dense in $B(E)\backslash\mathrm{GL}_n(E)$). $F(\nu)$ is then the $\mathrm{GL}_n(E)$-, or as it turns out, the $\mathrm{GL}_n(\mathbb{F}_p)$-subrepresentation generated by $W(\nu)^{U(E)}$.

Example. 1) $n=1$: $f(x)=x^\nu$ up to scalar.

2) $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a^{\nu_1-\nu_2}\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\nu_2}$ spans $W(\nu)^{U(E)}$. The elements

$$f_i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a^i b^{\nu_1-\nu_2-i}\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\nu_2}$$

for $0\le i\le \nu_1-\nu_2$ span $W(\nu)=F(\nu)$.

Remark. 1) $\dim F(\nu)$ is not known in all cases. For $p\gg 0$, there is an algorithm to compute the dimension, based on a (proven) conjecture of Lusztig.

2) $\dim F(\nu)=1$ if and only if $\nu_1=\cdots=\nu_n$. Then $F(\nu)=\det^\nu$.

If V is a weight and $P=MN$ is standard parabolic then $V^{N(\mathbb{F}_p)}$ and $V_{\overline{N}(\mathbb{F}_p)}$ are irreducible $M(\mathbb{F}_p)$-representations and the natural map $V^{N(\mathbb{F}_p)}\to V_{\overline{N}(\mathbb{F}_p)}$ is an isomorphism. In particular, $V^{U(\mathbb{F}_p)}$ is an irreducible 1-dimensional representation (isomorphic to $V_{\overline{U}(\mathbb{F}_p)}$). Explicitly, if $P=P_{n_1,\dots,n_r}$ then $M=\prod_i \mathrm{GL}_{n_i}$ so

$$F(\nu)^{N(\mathbb{F}_p)} = F(\nu_1,\dots,\nu_{n_1})\otimes\cdots\otimes F(\nu_{n-n_r+1},\dots,\nu_n)$$

There is another parametrisation of the weights. They are in bijection with pairs (θ,M) where $\theta: T(\mathbb{F}_p)\to E^\times$ is a character and M is a standard Levi such that θ extends to $M(\mathbb{F}_p)$. Explicitly, the parametrisation is given by $V\mapsto(\theta_V,M_V)$ where $\theta_V\cong V^{U(\mathbb{F}_p)}$ and M_V is the largest standard Levi M such that $V^{U(\mathbb{F}_p)}$ are preserved by $M(\mathbb{F}_p)$ (hence by $P(\mathbb{F}_p)$). Equivalently, $V^{N(\mathbb{F}_p)}$ is one-dimensional.

Remark. $P_V(\mathbb{F}_p)=\mathrm{Stab}(V^{U(\mathbb{F}_p)}\subseteq V)$ (since the stabiliser contains $B(\mathbb{F}_p)$ it must be a standard parabolic).

Example. ($n=2$):

$$F(a,b)\longmapsto(\chi_{a,b},T)\quad\text{if }\ a\neq b$$
$$F(b,b)\longmapsto(\det{}^b,G)\quad\text{if }\ a=b$$

($n=3$):

$$F(a,b,c)\longmapsto(\chi_{a,b,c},T)\quad\text{if }\ a>b>c$$
$$F(b,b,c)\longmapsto(\chi_{b,b,c},P_{2,1})\quad\text{if }\ a=b>c$$
$$F(a,b,b)\longmapsto(\chi_{a,b,b},P_{1,2})\quad\text{if }\ a>b=c$$
$$F(b,b,b)\longmapsto(\det{}^b,G)\quad\text{if }\ a=b=c$$

12. Mod p Satake Isomorphism

Let us start by recalling the Yoneda Lemma. Let $\mathcal{C}$ be a category and let A and B be objects of $\mathcal{C}$. Suppose that

$$\mathrm{Hom}_{\mathcal{C}}(A,-)\xrightarrow{\varphi}\mathrm{Hom}_{\mathcal{C}}(B,-)$$

is a natural transformation, i.e. a collection $(\varphi_C)_{C\in Ob(\mathcal{C})}$ of maps such that for all morphisms $f : C \to D$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "\varphi_C"] \arrow[d, "f\circ-"] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f\circ-"] \\ \mathrm{Hom}_{\mathcal{C}}(A,D) \arrow[r, "\varphi_D"] & \mathrm{Hom}_{\mathcal{C}}(B,D) \end{tikzcd}$$

commutes. Then the Yoneda Lemma says that there exists a unique $g : B \to A$ such that $\varphi_C(f) = g \circ f$ for all morphisms f and objects C. Explicitly, $g = \varphi_A(id_A)$.

Let $P = MN$ be standard parabolic and V a weight.

Lemma 26. *There is a natural isomorphism of functors from smooth M-representations to E-vector spaces*

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_{\overline{P}}^G(-)) \cong \mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, -)$$

(cf. the proof of Proposition 11). We will denote the map by $f \mapsto f_M$.

Proof. We have

$$\begin{aligned} \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_{\overline{P}}^G(-)) &\cong \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}}^G(-)|_K) \cong \\ \cong \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{P}(\mathbb{Z}_p)}^K(-|_{M(\mathbb{Z}_p)})) &\cong \mathrm{Hom}_{\overline{P}(\mathbb{Z}_p)}(V, -|_{M(\mathbb{Z}_p)}) \cong \\ \cong \mathrm{Hom}_{M(\mathbb{Z}_p)}(V_{\overline{N}(\mathbb{F}_p)}, -|_{M(\mathbb{Z}_p)}) &\cong \mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, -) \end{aligned}$$

where the first, third and fifth natural isomorphisms are Frobenius reciprocity, the second comes from the Iwasawa decomposition and the fourth is the universal property of the coinvariants. ☐

Any $\varphi \in \mathcal{H}_G(V)$ induces a natural endomorphism of the functor $\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_{\overline{P}}^G(-))$, hence of $\mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, -)$. The natural endomorphisms of $\mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, -)$ are given by $\mathrm{End}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)})$ by the Yoneda Lemma, i.e. we get a unique endomorphism

$$S_G^M(\varphi) \in \mathrm{End}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}) = \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$$

where the equality is the definition of the Hecke algebra $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$. $S_G^M(\varphi)$ is characterised by

$$(f \circ \varphi)_M = f_M \circ S_G^M(\varphi)$$

S_G^M is an algebra homomorphism. The proof that it preserves multiplication we note that, for all f,

$$\begin{aligned} f_M \circ S_G^M(\varphi_1 \circ \varphi_2) &= (f \circ \varphi_1 \circ \varphi_2)_M = \\ = (f \circ \varphi_1)_M \circ S_G^M(\varphi_2) &= f_M \circ S_G^M(\varphi_1) \circ S_G^M(\varphi_2). \end{aligned}$$

The uniqueness in the Yoneda Lemma then implies that $S_G^M(\varphi_1 \circ \varphi_2) = S_G^M(\varphi_1) \circ S_G^M(\varphi_2)$. A similar argument show that S_G^M is linear as well.

Proposition 27. *We have the following explicit formula for S_G^M:*

$$S_G^M(\varphi)(m) = \sum_{\bar{n}\in\overline{N}(\mathbb{Z}_p)\backslash\overline{N}} p_{\overline{N}} \circ \varphi(\bar{n}m)$$

for all $m \in M$, where $p_{\overline{N}}$ is the projection $V \twoheadrightarrow V_{\overline{N}(\mathbb{F}_p)}$ and we are viewing φ as function $G \to \mathrm{End}_E(V)$ and $S_G^M(\varphi)$ as function $M \to \mathrm{End}_E(V_{\overline{N}(\mathbb{F}_p)})$. The sum on the right-hand side is a sum of *linear maps* $V \to V_{\overline{N}(\mathbb{F}_p)}$ *and implicit in the statement is the assertion that the sum factors through* $V \twoheadrightarrow V_{\overline{N}(\mathbb{F}_p)}$ *to a map* $V_{\overline{N}(\mathbb{F}_p)} \to V_{\overline{N}(\mathbb{F}_p)}$.

Proof. If σ is a smooth M-representation, $f : V \to \mathrm{Ind}_{\overline{P}}^{G}(\sigma)|_K$ and $f_M : V_{\overline{N}(\mathbb{F}_p)} \to \sigma|_{M(\mathbb{Z}_p)}$ are related by the equation

$$f(v)(1) = f_M(p_{\overline{N}}(v))$$

(cf. the proof of Proposition 14). By definition, $(f * \varphi)_M = f_M * S_G^M(\varphi)$. To determine $S_G^M(\varphi)$ take "$f_M = id$", so $\sigma = \mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}$ and via Frobenius reciprocity,

$$\begin{aligned} f_M : V_{\overline{N}(\mathbb{F}_p)} &\longrightarrow \mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)} \\ x &\longmapsto [1, x] \end{aligned}$$

For $v \in V$,

$$\begin{aligned} (f * \varphi)_M(p_{\overline{N}}(v)) &= (f * \varphi)(v)(1) = \\ = \sum_{g \in K \backslash G} (g^{-1}.f)(\varphi(g)v)(1) &= \sum_{g \in K \backslash G} f(\varphi(g)v)(g^{-1}) \end{aligned}$$

By the Iwasawa decomposition we have

$$K\backslash G = K\backslash(K\overline{P}) = (\overline{P} \cap K)\backslash \overline{P} = \overline{P}(\mathbb{Z}_p)\backslash \overline{P}$$

Using Lemma 28 3) below with $\overline{P} = M \ltimes \overline{N}$ and the subgroup $\overline{P}(\mathbb{Z}_p) = M(\mathbb{Z}_p)\overline{N}(\mathbb{Z}_p)$ we get

$$\sum_{g \in K\backslash G} f(\varphi(g)v)(g^{-1}) = \sum_{m \in M(\mathbb{Z}_p)\backslash M} \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} f(\varphi(\bar{n}m)v)(m^{-1}\bar{n}^{-1}) =$$

$$= \sum_{m \in M(\mathbb{Z}_p)\backslash M} \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} m^{-1}.f(\varphi(\bar{n}m)v)(1)$$

Now $f(\varphi(\bar{n}m)v)(1) = f_M(p_{\overline{N}}(\varphi(\bar{n}m)v)) = [1, p_{\overline{N}}(\varphi(\bar{n}m)v)]$ and hence

$$\sum_{m \in M(\mathbb{Z}_p)\backslash M} \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} m^{-1}.f(\varphi(\bar{n}m)v)(1) = \sum_{m \in M(\mathbb{Z}_p)\backslash M} \left[m^{-1}, \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} (p_{\overline{N}} \circ \varphi(\bar{n}m))v \right]$$

So under Frobenius reciprocity, $S_G^M(\varphi)$ corresponds to the map $V_{\overline{N}(\mathbb{F}_p)} \to \mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}|_{M(\mathbb{Z}_p)}$ given by

$$p_{\overline{N}}(v) \longmapsto \left(m \longmapsto \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} (p_{\overline{N}} \circ \varphi(\bar{n}m))v \right)$$

which is the element

$$m \longmapsto \left(p_{\overline{N}}(v) \longmapsto \sum_{\bar{n} \in \overline{N}(\mathbb{Z}_p)\backslash \overline{N}} (p_{\overline{N}} \circ \varphi(\bar{n}m))v \right)$$

of $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$, which is what we wanted to prove. $\square$

Next we state the lemma used in the proposition:

Lemma 28. *1) Let Γ be a group, $Y \lhd \Gamma$ a normal subgroup and $\Gamma_0 \subseteq \Gamma$ a subgroup. Moreover, let A be an abelian group and $\psi : \Gamma_0\backslash\Gamma \to A$ a function. Then*

$$\sum_{\gamma \in \Gamma_0\backslash\Gamma} \psi(\gamma) = \sum_{\gamma' \in \Gamma_0 Y\backslash\Gamma} \left(\sum_{y \in (Y\cap\Gamma_0)\backslash Y} \psi(y\gamma') \right)$$

2) If in addition $\Gamma = X \ltimes Y$ and $\pi : \Gamma \to X$ is the associated projection, then

$$\sum_{\gamma \in \Gamma_0\backslash\Gamma} \psi(\gamma) = \sum_{x \in \pi(\Gamma_0)\backslash X} \left(\sum_{y \in (Y\cap\Gamma_0)\backslash Y} \psi(yx) \right)$$

3) If moreover $\Gamma_0 = (\Gamma_0 \cap X)(\Gamma_0 \cap Y)$, or equivalently $\pi(\Gamma_0) = \Gamma_0 \cap X$, we get

$$\sum_{\gamma \in \Gamma_0\backslash\Gamma} \psi(\gamma) = \sum_{x \in (X\cap\Gamma_0)\backslash X} \left(\sum_{y \in (Y\cap\Gamma_0)\backslash Y} \psi(yx) \right)$$

Proof. 1) We have $\Gamma = \coprod \Gamma_0 Y \gamma'$ and $Y \cap \Gamma_0 \backslash Y \xrightarrow{\sim} \Gamma_0\backslash\Gamma_0 Y$ (induced from the inclusion $Y \hookrightarrow \Gamma_0 Y$) and hence

$$\Gamma = \coprod_{\gamma' \in \Gamma_0 Y\backslash\Gamma} \left(\coprod_{y \in (Y\cap\Gamma_0)\backslash Y} \Gamma_0 y \gamma' \right)$$

which implies the formula.

2) π induces an isomorphism $\Gamma_0 Y\backslash\Gamma \xrightarrow{\sim} \pi(\Gamma_0)\backslash X$ and hence we may substitute the outer sum to a sum over $\pi(\Gamma_0)\backslash X$.

3) Immediate from 2). □

Remark. 1) We could have defined S_G^M by the formula in Proposition 27 and then proved all the properties. However, the approach taken more efficient.

2) When $M = T$, we will write S_G for S_G^T.

3) As a special case, take $V = 1_K$, $P = B$. Then $S_G : \mathcal{H}_G(1_K) \to \mathcal{H}_T(1_{T(\mathbb{Z}_p)})$ is given by

$$\varphi \longmapsto \left(t \longmapsto \sum_{\bar{u} \in \overline{U}(\mathbb{Z}_p)\backslash\overline{U}} \varphi(\bar{u}t) \right)$$

As an aside, let us consider the classical Satake transform over $\mathbb{C}$. There we have the Hecke algebra $\mathcal{H}_G^{\mathbb{C}}$ of compactly supported functions $\varphi : G \to \mathbb{C}$ such that $\varphi(k_1 g k_2) = \varphi(g)$ for all $k_1, k_2 \in K$ and $g \in G$, and $\mathcal{H}_T^{\mathbb{C}}$ the Hecke algebra of compactly supported functions $\psi : T \to \mathbb{C}$ such that $\psi(st) = \psi(t)$ for $s \in T(\mathbb{Z}_p)$ and $t \in T$. Then classical Satake isomorphism S is the map $\mathcal{H}_G^{\mathbb{C}} \to \mathcal{H}_T^{\mathbb{C}}$

$$\varphi \longmapsto \left(t \longmapsto \delta(t)^{-1/2} \int_{\overline{U}} \varphi(\bar{u}t) d\bar{u} \right)$$

where $d\bar{u}$ is Haar measure on $\overline{U}$, normalised so that $\int_{\overline{U}} d\bar{u} = 1$, and δ is the modulus character of B, defined by

$$\mathrm{diag}(t_1, \ldots, t_n) \longmapsto |t_1|^{-(n-1)} |t_2|^{-(n-3)} \ldots |t_n|^{n-1}$$

The Weyl group $W \cong S_n$ acts on $T = \mathbb{Q}_p^\times \times \cdots \times \mathbb{Q}_p^\times$ by permutation of factors. Then the theorem of Satake is

Theorem. *(Satake, 1963) S is an injective algebra homomorphism with image $\mathcal{H}_T^W$ (the W-invariants of $\mathcal{H}_T$).*

Remark. 1) For a reference, see Cartier's article "Representations of p-adic groups: a survey" in the Corvallis volumes, or Gross's article "On the Satake isomorphism".

2) $\mathcal{H}_T^{\mathbb{C}} = \mathbb{C}[T/T(\mathbb{Z}_p)] \cong \mathbb{C}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ and $W \cong S_n$ acts by permutation of the variables.

Now let us return to our case. Let $P = B$, and define

$$T_- = \{\mathrm{diag}(t_1, \ldots, t_n) \mid \mathrm{ord}_p(t_1) \leq \cdots \leq \mathrm{ord}_p(t_n)\}$$

T^- is a submonoid of T and we may therefore defined a subalgebra $\mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)}) \subseteq \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$ by

$$\mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)}) = \left\{\psi \in \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) \mid \mathrm{supp}(\psi) \subseteq T^-\right\}$$

Then we have

Theorem 29. *$S_G = S_G^T : \mathcal{H}_G(V) \to \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$ is an injective algebra homomorphism with image $\mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)})$.*

Proof. Let us, to ease notation, put $\mathcal{H}_G = \mathcal{H}_G(V)$, $\mathcal{H}_T = \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$ and $\mathcal{H}_T^- = \mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)})$ for the duration of this proof.

Step 1: Find natural bases for $\mathcal{H}_G$ and $\mathcal{H}_T$ (cf. Theorem 13)

Put $\Lambda = \mathbb{Z}^n$ and $\Lambda_- = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \leq \cdots \leq \lambda_n\}$. For $\lambda \in \Lambda$, let $t_\lambda = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}) \in T$. The Cartan decomposition then says that $G = \coprod_{\lambda \in \Lambda_-} K t_\lambda K$. Suppose that $\varphi \in \mathcal{H}_G$ has support $K t_\lambda K$ for some $\lambda \in \Lambda_-$. As in Theorem 13 specifying φ is equivalent to specifying an operator $\varphi(t_\lambda)$ such that

$$k_1 \circ \varphi(t_\lambda) = \varphi(t_\lambda) \circ k_2$$

whenever $k_1 t_\lambda = t_\lambda k_2$. Then $k_1 \in t_\lambda K t_\lambda^{-1} \cap K$, so k_1 is of the form (a_{ij}) with $a_{ij} \in \mathbb{Z}_p$ for $i \leq j$ and $p^{\lambda_j - \lambda_i} a_{ij} \in \mathbb{Z}_p$ for $i > j$, and conversely k_2 is of the form (b_{ij}) with $b_{ij} \in \mathbb{Z}_p$ for $i \geq j$ and $p^{\lambda_i - \lambda_j} b_{ij} \in \mathbb{Z}_p$ for $i < j$. λ determines a standard parabolic $P_{-\lambda} = P_{n_1, \ldots, n_r}$ where the n_i are defined by

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots < \lambda_{n-n_r+1} = \cdots = \lambda_n$$

Put $P_\lambda = \overline{P_{-\lambda}}$. Then we have

$$mn \circ \varphi(t_\lambda) = \varphi(t_\lambda) \circ m\bar{n}$$

for $m \in M_\lambda(\mathbb{F}_p) = M_{-\lambda}(\mathbb{F}_p)$, $n \in N_{-\lambda}(\mathbb{F}_p)$ and $\bar{n} \in N_\lambda(\mathbb{F}_p)$, hence $\varphi(t_\lambda)$ factors through to a map $V_{N_\lambda(\mathbb{F}_p)} \to V^{N_{-\lambda}(\mathbb{F}_p)}$. By Section 11 we know that there is a unique such map up to scalar, and hence a unique such projection. We call the corresponding Hecke operator T_λ, and the $(T_\lambda)_{\lambda \in \Lambda_-}$ is a basis for $\mathcal{H}_G$.

For $\mathcal{H}_T$ the situation is somewhat simpler. $\mathcal{H}_T$ is the algebra of functions $\psi : T \to \mathrm{End}_E(V_{\overline{U}(\mathbb{F}_p)}) = E$ with compact support and such that $\psi(t_0 t) = t_0 \psi(t)$ for all $t_0 \in T(\mathbb{Z}_p)$ and $t \in T$. We have an isomorphism $T/T(\mathbb{Z}_p) \cong \Lambda$ given by

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto (\mathrm{ord}_p(t_1), \ldots, \mathrm{ord}_p(t_n))$$

and under this isomorphism $T_-/T(\mathbb{Z}_p)$ corresponds to Λ_-. Therefore there exists a unique $\tau_\lambda \in \mathcal{H}_T$ whose support is $t_\lambda T(\mathbb{Z}_p)$ and such that $\tau_\lambda(t_\lambda) = 1$. Then $(\tau_\lambda)_{\lambda\in\Lambda}$ is our desired basis for $\mathcal{H}_T$, and $(\tau_\lambda)_{\lambda\in\Lambda_-}$ is a basis for $\mathcal{H}_T^-$.

Step 2: S_G is injective.

If, for some $\lambda \in \Lambda_-$, $\mu \in \Lambda$,

$$(S_G T_\lambda)(t_\mu) = \sum_{\bar{u}\in\overline{U}(\mathbb{Z}_p)\backslash\overline{U}} p_{\overline{U}} \circ T_\lambda(\bar{u}t_\mu) \neq 0$$

then there is a $\bar{u} \in \overline{U}$ such that $\bar{u}t_\mu \in Kt_\lambda K$, or equivalently $\overline{U}t_\mu \cap Kt_\lambda K \neq 0$.

Fact 1: This implies that $\mu \geq \lambda$, where

$$\mu \geq \lambda \Leftrightarrow \begin{cases} \sum_{i=1}^r \mu_i \geq \sum_{i=1}^r \lambda_i & \forall r < n \quad \text{and} \\ \sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i \end{cases}$$

(note that $\geq$ is a partial order on Λ).

Fact 2: $\overline{U}t_\lambda \cap Kt_\lambda K = \overline{U}(\mathbb{Z}_p)t_\lambda$, $\lambda \in \Lambda_-$.

We deduce that

$$S_G T_\lambda = \sum_{\mu\in\Lambda,\, \mu\geq\lambda} a_\mu t_\mu$$

(from Fact 1) with $a_\lambda = 1$ (from Fact 2). Now, suppose that $S_G(\varphi) = 0$ for $\varphi \neq 0$. Choose $\lambda_0 \in \Lambda_-$, minimal with respect to $\leq$, such that $Kt_{\lambda_0}K \subseteq \operatorname{supp}(\varphi)$. Then by the above equation, $\operatorname{supp}(S_G(\varphi)) \supseteq T(\mathbb{Z}_p)t_{\lambda_0}$, a contradiction. Hence S_G is injective.

Step 3: Suppose that $\varphi \in \mathcal{H}_G$, $\mu \in \Lambda$ is such that $S_G(\varphi)(t_\mu) \neq 0$. Then $\mu_i - \mu_{i+1} \leq 1$ for all i.

Suppose that there is a k such that $\mu_k - \mu_{k+1} > 1$. Fix such a k. Let

$$\overline{U}_k = \{(a_{ij}) \in \mathrm{GL}_n(\mathbb{Q}_p) \mid a_{ii} = 1,\ a_{k+1,k} \in \mathbb{Q}_p\ a_{ij} = 0\,\text{otherwise}\}$$

and let $\overline{U}'$ be the kernel of the homomorphism $\pi : \overline{U} \to \mathbb{Q}_p$ given by $(a_{ij}) \mapsto a_{k+1,k}$. We have $\overline{U} = \overline{U}' \rtimes \overline{U}_k$. Clearly $\pi(\overline{U} \cap K) = \mathbb{Z}_p$. By 28 2),

$$S_G(\varphi)(t_\mu) = \sum_{\bar{u}\in\overline{U}(\mathbb{Z}_p)\backslash\overline{U}} p_{\overline{U}} \circ \varphi(\bar{u}t_\mu) = \sum_{\bar{u}_k\in\overline{U}_k(\mathbb{Z}_p)\backslash\overline{U}_k} \left(\sum_{\bar{u}'\in\overline{U}'(\mathbb{Z}_p)\backslash\overline{U}'} p_{\overline{U}} \circ \varphi(\bar{u}'\bar{u}_k t_\mu) \right).$$

Put $\bar{u}_k' = t_\mu^{-1}\bar{u}_k t_\mu$, by a change of variables we get that the above sum is

$$\sum_{\bar{u}_k'\in\frac{\overline{U}_k}{t_\mu^{-1}\overline{U}_k(\mathbb{Z}_p)t_\mu}} \left(\sum_{\bar{u}'\in\overline{U}'(\mathbb{Z}_p)\backslash\overline{U}'} p_{\overline{U}} \circ \varphi(\bar{u}'t_\mu\bar{u}_k') \right).$$

Next, note that $\varphi(gk) = \varphi(g) \circ k = \varphi(g)$ if $k \in K(1)$. Hence $\bar{u}_k'$ only matters modulo $\overline{U}_k \cap K(1)$. π maps $\overline{U}_k$ isomorphically onto $\mathbb{Q}_p$ and identifies $\overline{U}_k(\mathbb{Z}_p)$ with $\mathbb{Z}_p$, $t_\mu^{-1}\overline{U}_k t_\mu$ with $p^{\mu_k-\mu_{k+1}}\mathbb{Z}_p$ and $\overline{U}_k \cap K(1)$ with $p\mathbb{Z}_p$. To simplify notation somewhat, put

$$\psi(\bar{u}_k') = \sum_{\bar{u}'\in\overline{U}'(\mathbb{Z}_p)\backslash\overline{U}'} p_{\overline{U}} \circ \varphi(\bar{u}'t_\mu\bar{u}_k'),$$

i.e. it is the inner sum in the formula above it. Identifying subgroups $\overline{U}_k$ of with their image under π and using Lemma 28 2), we get

$$\sum_{x \in \mathbb{Q}_p/p^{\mu_k - \mu_{k+1}}\mathbb{Z}_p} \psi(x) = \sum_{x_2 \in \mathbb{Q}_p/p\mathbb{Z}_p} \left(\sum_{x_1 \in p\mathbb{Z}_p/p^{\mu_k - \mu_{k+1}}\mathbb{Z}_p} \psi(x_1 + x_2) \right)$$

We noted above that $\psi(x_1 + x_2) = \psi(x_2)$ for $x_1 \in p\mathbb{Z}_p$ ($\cong \overline{U}_k \cap K(1)$) and hence

$$\sum_{x_1 \in p\mathbb{Z}_p/p^{\mu_k - \mu_{k+1}}\mathbb{Z}_p} \psi(x_1 + x_2) = p^{\mu_k - \mu_{k+1} - 1}\psi(x_2) = 0$$

as $\mu_k - \mu_{k+1} > 1$, so $S_G(\varphi)(t_\mu) = 0$ which completes Step 3.

Step 4: The image of S_G is contained in $\mathcal{H}_T^-$.

Since $(T_\lambda)_{\lambda \in \Lambda_-}$ is a basis for $\mathcal{H}_G$ it is enough to show that $\mathrm{supp}(S_G(T_\lambda)) \subseteq T^-$ for all $\lambda \in \Lambda_-$. Suppose that $S_G(T_\lambda)(t_\mu) \neq 0$ for some $\mu \in \Lambda - \Lambda_-$. Fix a k such that there is a $\mu \in \Lambda - \Lambda_-$ with $S_G(T_\lambda)(t_\mu) \neq 0$ and $\mu_k - \mu_{k+1} > 0$. Put

$$\mathrm{Supp}(\lambda) = \{\mu \in \Lambda \mid S_G(T_\lambda)(t_\mu) \neq 0\}$$

This is a finite set since $S_G(T_\lambda)$ has compact support (and we assume that it is not contained in Λ_-). Consider the group homomorphism $w : \Lambda \to \mathbb{Z}^n$ defined by

$$w(\mu) = (\mu_k - \mu_{k+1}, \mu_{k+1} - \mu_{k+2}, \dots, \mu_{k-1} - \mu_k)$$

We claim that w is injective on $\mathrm{Supp}(\lambda)$. If $\mu \in \mathrm{Supp}(\lambda)$ then $\mu \geq \lambda$, in particular $\sum \mu_i = \sum \lambda_i$, and this together with $w(\mu)$ determines μ, showing the injectivity.

Next we introduce the lexicographic (total) order $\leq_\ell$ on $\mathbb{Z}^n$. Choose $\mu \in \mathrm{Supp}(\lambda)$ such that $w(\mu)$ is maximal with respect to $\leq_\ell$. In particular $\mu_k - \mu_{k+1} = 1$ (by Step 3) since we have assumed that $\mathrm{Supp}(\lambda) \subsetneq \Lambda_-$. We claim that $S_G(T_\lambda^2)(t_{2\mu}) \neq 0$. This would contradict Step 3 and hence finish the proof of Step 4. Since S_G is an algebra homomorphism we have $S_G(T_\lambda^2) = S_G(T_\lambda) * S_G(T_\lambda)$. Using the formula for $*$ and evaluating at $t_{2\mu}$ we get

$$\sum_{\mu' \in \Lambda} S_G(T_\lambda)(t_{\mu'}) \circ S_G(T_\lambda)(t_{2\mu} t_{\mu'}^{-1}),$$

identifying $\Lambda = T/T(\mathbb{Z}_p)$. If the μ'-term is nonzero, then μ', $2\mu - \mu' \in \mathrm{Supp}(\lambda)$ and hence $w(\mu')$, $w(2\mu - \mu') \leq_\ell w(\mu)$ by maximality of μ. But then

$$w(2\mu) = w(2\mu - \mu') + w(\mu') \leq_\ell w(\mu) + w(\mu) = w(2\mu)$$

so we must in fact have equality. By injectivity of w on $\mathrm{Supp}(\lambda)$ we may conclude that $\mu' = \mu$, so the sum above collapses into $(S_G(T_\lambda)(t_\mu))^2$ which is nonzero, since $S_G(T_\lambda)(t_\mu)$ is a nonzero scalar. This completes Step 4.

Step 5: S_G is surjective onto $\mathcal{H}_T^-$.

It is enough to show that $t_\mu \in \mathrm{Im}(S_G)$ for all $\mu \in \Lambda_-$ as these form a basis for $\mathcal{H}_T^-$. Fix $\mu \in \Lambda_-$. We claim that the set $\sum(\mu) = \{\lambda \in \Lambda_- \mid \lambda \geq \mu\}$ is finite. This follows from noting that for such λ, $\mu_1 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \mu_n$ and hence the claim follows. Next we claim that the matrix expressing $S_G(T_\lambda)$ in terms of the $\tau_{\lambda'}$ for λ, $\lambda' \in \sum(\mu)$ is invertible. Note that this completes Step 5.

To show this, for all $\lambda \in \sum(\mu)$

$$S_G(T_\lambda) = \sum_{\lambda \le \lambda' \in \Lambda_-} a_{\lambda'\lambda} \tau_{\lambda'}$$

with $a_{\lambda\lambda} = 1$ for all $\lambda \in \sum(\mu)$ (see just above Step 3). So if we extend $\geq$ arbitrarily to a total order on $\sum(\mu)$, the matrix $(a_{\lambda'\lambda})_{\lambda,\lambda' \in \sum(\mu)}$ is triangular with diagonal entries $= 1$. $\square$

For $1 \le i \le n$, let $\lambda_i = (0, \ldots, 0, 1, \ldots, 1) \in \Lambda_-$ with $n - i$ zeroes and i ones. Put $t_i = t_{\lambda_i} \in T^-$, $T_i = T_{\lambda_i}$ and $\tau_i = \tau_{\lambda_i} \in \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$.

Corollary. *$\mathcal{H}_G(V)$ is commutative. It is an integral domain of finite type over E. Explicitly,*

$$\mathcal{H}_G(V) = E[T_1, \ldots, T_{n-1}, T_n^{\pm 1}]$$

(canonically; these are really the T_i above and there are no relations) and $S_G(T_i) = \tau_i$.

Remark. The last part does not generalise to other groups.

Proof. $\mathcal{H}_G \cong \mathcal{H}_T^-$ with a basis τ_λ, $\lambda \in \Lambda_-$. It is easy to check that $\tau_\lambda * \tau_\mu = \tau_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. Thus the inclusion $\mathcal{H}_T^- \subseteq \mathcal{H}_T$ is the inclusion $E[\Lambda_-] \subseteq E[\Lambda]$ and we see that $\mathcal{H}_G$ is commutative. Since $\Lambda_- = \mathbb{Z}_{\ge 0}\lambda_1 \oplus \cdots \oplus \mathbb{Z}_{\ge 0}\lambda_{n-1} \oplus \mathbb{Z}\lambda_n$ as a monoid we get that $\mathcal{H}_T^- = E[T_1, \ldots, T_{n-1}, T_n^{\pm 1}]$, so it is a finite type integral domain over E. It remains to show that $S_G(T_i) = \tau_i$ for all i. We know that

$$S_G(T_\lambda) = \sum_{\lambda \le \mu \in \Lambda_-} a_\mu t_\mu$$

with $a_\lambda = 1$. It is easy to check that if $\mu \geq \lambda_i$ and $\mu \in \Lambda_-$ then in fact $\mu = \lambda_i$ and hence $S_G(T_i) = \tau_i$ as desired. $\square$

Exercise. Work out the Satake isomorphism more generally for products $\mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{Q}_p)$.

Let $M = M_{n_1,\ldots,n_r}$ be a standard Levi of GL_n and let $\overline{V}$ be a weight of $M(\mathbb{Z}_p) = M \cap K$. The Satake transform from the exercise above is an injective algebra homomorphism

$$S_M : \mathcal{H}_M(\overline{V}) \hookrightarrow \mathcal{H}_T(\overline{V}_{(\overline{U} \cap M)(\mathbb{F}_p)})$$

with image those $\psi \in \mathcal{H}_T(\overline{V}_{(\overline{U} \cap M)(\mathbb{F}_p)})$ whose support lies in the subset $T^{-,M} \subseteq T$ of those $\mathrm{diag}(t_1, \ldots, t_n) \in T$ such that $(\mathrm{ord}(t_1), \ldots, \mathrm{ord}(t_n)) \in \Lambda_{-,M}$, where $\Lambda_{-,M}$ is defined by

$$\begin{gathered} \lambda_1 \le \cdots \le \lambda_{n_1}, \\ \lambda_{n_1+1} \le \cdots \le \lambda_{n_1+n_2}, \\ \cdots \\ \lambda_{n-n_r+1} \le \cdots \le \lambda_n. \end{gathered}$$

Proposition 30. *We have a commutative diagram*

$$\begin{tikzcd} \mathcal{H}_G(V) \arrow[r, "S_G^M"] \arrow[dr, "S_G"'] & \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) \arrow[d, hook, "S_M"] \\ & \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) \end{tikzcd}$$

Hence S_G^M is injective. Moreover, S_G^M is a localisation at one element, i.e. there exists a $T \in \mathcal{H}_G(V)$ such that the induced map $\mathcal{H}_G(V)[T^{-1}] \to \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$ is an isomorphism.

Remark. $\overline{U} = (\overline{U} \cap M) \ltimes \overline{N}$ and hence $\left(V_{\overline{N}(\mathbb{F}_p)}\right)_{(\overline{U}\cap M)(\mathbb{F}_p)} = V_{\overline{U}(\mathbb{F}_p)}$.

Proof. $\mathrm{Ind}_B^G - = \mathrm{Ind}_P^G(\mathrm{Ind}_{B\cap M}^M -)$ so by Lemma 26 we have natural isomorphisms

$$\mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, \mathrm{Ind}_{B\cap M}^M -) \cong \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_B^G -) \cong \mathrm{Hom}_T(\mathrm{ind}_{T(\mathbb{Z}_p)}^T V_{\overline{U}(\mathbb{F}_p)}, -)$$

Let us denote the isomorphism from the first to the third object by $f \mapsto f_{M,T}$, then, using the notation in Lemma 26, by definition we have $(f_M)_{M,T} = f_T$. Also by definition we have, for all $\varphi \in \mathcal{H}_G(V)$ and $f \in \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_B^G -)$,

$$(f \circ \varphi)_T = f_T \circ S_G(\varphi).$$

Moreover (also by definition),

$$(f \circ \varphi)_T = ((f \circ \varphi)_M)_{M,T} = (f_M \circ S_G^M(\varphi))_{M,T} =$$
$$= (f_M)_{M,T} \circ S_M(S_G^M(\varphi)) = f_T \circ S_M(S_G^M(\varphi)).$$

Hence the uniqueness in the Yoneda Lemma implies that $S_G = S_M \circ S_G^M$, i.e. we get the commutativity of the diagram. To prove the second part, note that we have

$$\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) \cong E[\Lambda]$$
$$\mathrm{Im}(S_G) \cong E[\Lambda_-]$$
$$\mathrm{Im}(S_M) \cong E[\Lambda_{-,M}]$$

Thus S_G^M may be identified with the inclusion $E[\Lambda_-] \hookrightarrow E[\Lambda_{-,M}]$. Pick $\lambda \in \Lambda$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} < \dots$$

Then $\Lambda_{-,M} = \Lambda_- + \mathbb{Z}\lambda$ as monoids (since $\mathbb{Z}\lambda \subseteq \Lambda_{-,M}$ and for any $\lambda' \in \Lambda_{-,M}$ and $\lambda' + m\lambda \in \Lambda_-$ for $m \gg 0$) and hence

$$E[\Lambda_-]_{\tau_\lambda} = E[\Lambda_{-,M}]$$

which is what we wanted to prove. □

Remark. We could also have proven the first part of the proposition by using the explicit formulae.

Two weights. Let V and V'be weights and let $P = MN$ be a standard parabolic. Lemma 26 implies that

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G -) \cong \mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, -)$$

and similarly for V'. Any $\varphi \in \mathcal{H}_G(V, V') = \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{ind}_K^G V')$ induces a natural transformation

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V', \mathrm{Ind}_P^G -) \to \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G -)$$

By the Yoneda Lemma there is a unique $S_G^M(\varphi) \in \mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, \mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)}) = \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}, V'_{\overline{N}(\mathbb{F}_p)})$. We get the same explicit formula for $S_G^M(\varphi)$ as in Proposition 27. By uniqueness $S_G^M : \mathcal{H}_G(V, V') \to \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}, V'_{\overline{N}(\mathbb{F}_p)})$ is E-linear and whenever $\varphi \in \mathcal{H}_G(V, V')$ and $\varphi' \in \mathcal{H}_G(V', V'')$ then

$$S_G^M(\varphi' \circ \varphi) = S_G^M(\varphi') \circ S_G^M(\varphi)$$

Proposition 31. *1) S_G^M is injective.*

2) $\mathcal{H}_G(V, V') \neq 0$ if and only if $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$ as $T(\mathbb{F}_p)$-representations.

3) If $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$, then we can identify $\mathcal{H}_G(V) \cong \mathcal{H}_G(V')$ via S_G, and for all $\varphi \in \mathcal{H}_G(V, V')$ and $\varphi' \in \mathcal{H}_G(V', V)$, $\varphi \circ \varphi' = \varphi' \circ \varphi$.

Remark. We saw an instance of the last part of 3) in Proposition 19.

Proof. 1) The same argument as in Step 1 of Theorem 29 shows that for $\lambda \in \Lambda_-$,

$$\dim \{\varphi \in \mathcal{H}_G(V, V') \mid \operatorname{supp}(\varphi) \subseteq K t_\lambda K\} = \begin{cases} 1 & \text{if } V_{N_\lambda(\mathbb{F}_p)} \cong V'_{N_\lambda(\mathbb{F}_p)}, \\ 0 & \text{otherwise.} \end{cases}$$

Also, if $\operatorname{supp}(\varphi) = K t_\lambda K$, $\operatorname{supp}(S_G(\varphi) \subseteq \bigcup_{\lambda \leq \mu \in \Lambda} T(\mathbb{Z}_p) t_\mu$ ("Fact 1) and

$$S_G(\varphi)(t_\lambda) = \sum_{\bar{u} \in \overline{U}(\mathbb{Z}_p) \backslash \overline{U}} p_{\overline{U}} \circ \varphi(\bar{u} t_\lambda) = p_{\overline{U}} \circ \varphi(t_\lambda)$$

where $\varphi(t_\lambda)$ is the composition

$$V \twoheadrightarrow V_{N_\lambda(\mathbb{F}_p)} \xrightarrow{\sim} (V')^{N_{-\lambda}(\mathbb{F}_p)} \hookrightarrow V'$$

where the middle isomorphism is $M_\lambda(\mathbb{F}_p)$-linear and hence unique up to scalar (determined by φ; φ too is unique to up scalar). As $N_{-\lambda} \subseteq U$, $\operatorname{Im}(\varphi(t_\lambda)) = (V')^{N_{-\lambda}(\mathbb{F}_p)} \supseteq (V')^{U(\mathbb{F}_p)}$ so $S_G(\varphi)(t_\lambda) \neq 0$. The same argument as in Theorem 29 shows that S_G is injective. Since $S_M \circ S_G^M = S_G$ we deduce that S_G^M is injective.

2) By 1), $\mathcal{H}_G(V, V') \neq 0$ implies that $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}, V'_{\overline{U}(\mathbb{F}_p)}) \neq 0$. If $\chi, \chi' : T(\mathbb{Z}_p) \to E^\times$ are weights of $T(\mathbb{Z}_p)$, then for any $\psi \in \mathcal{H}_T(\chi, \chi')$ we have, for $t_0 \in T(\mathbb{Z}_p)$ and $t \in T$,

$$\psi(t_0 t) = \chi'(t_0)\psi(t) = \chi(t_0)\psi(t)$$

and hence $\chi = \chi'$ if $\psi \neq 0$. Thus $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$. For the converse, if $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$, pick $\lambda \in \Lambda_-$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then $N_\lambda = \overline{U}$ and from the conditions for existence of Hecke operators we see that there is a $\varphi \in \mathcal{H}_G(V, V')$ with support $K t_\lambda K$.

3) Choose an isomorphism $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$. Then $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}, V'_{\overline{U}(\mathbb{F}_p)}) \cong \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) \cong \mathcal{H}_T(V'_{\overline{U}(\mathbb{F}_p)})$ and hence

$$S_G(\varphi \circ \varphi') = S_G(\varphi) \circ S_G(\varphi') = S_G(\varphi') \circ S_G(\varphi) = S_G(\varphi' \circ \varphi)$$

by the commutativity of e.g. $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$. By injectivity of S_G, $\varphi \circ \varphi' = \varphi' \circ \varphi$. □

Remark. For $V \not\cong V'$, $\operatorname{Im}(S_G) \nsubseteq \mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)}, V'_{\overline{U}(\mathbb{F}_p)})$ in general.

13. Comparison of Compact and Parabolic Induction for GL_n

Let $P = MN$ be a standard parabolic subgroup of $\mathrm{GL}_n$and V a weight. Recall Lemma 26: for a smooth M-representation σ, we have an isomorphism

$$\operatorname{Hom}_G(\operatorname{ind}_K^G V, \operatorname{Ind}_{\overline{P}}^G(\sigma)) \cong \operatorname{Hom}_M(\operatorname{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, \sigma)$$

which is natural in σ and which we denote by $f \mapsto f_M$. By definition of S_G^M, $(f \circ \varphi)_M = f_M \circ S_G^M(\varphi)$ for $\varphi \in \mathcal{H}_G(V)$.

Let $\sigma = \mathrm{ind}_{M(\mathbb{Z}_p)}^M(V_{\overline{N}(\mathbb{F}_p)})$, and let $F_V : \mathrm{ind}_K^G V \to \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M(V_{\overline{N}(\mathbb{F}_p)}))$ be such that $(F_V)_M$ is the identity. Then for all $\varphi \in \mathcal{H}_G(V)$,

$$(F_V \circ \varphi)_M = id \circ S_G^M(\varphi) = S_G^M(\varphi) \circ id = S_G^M(\varphi) \circ (F_V)_M$$

and hence $F_V \circ \varphi = \mathrm{Ind}_{\overline{P}}^G(S_G^M(\varphi)) \circ \varphi$, i.e. F_V is $\mathcal{H}_G(V)$-linear, where $\mathcal{H}_G(V)$ acts on the codomain of F_V via $\mathrm{Ind}_{\overline{P}}^G S_G^M$. This commutes with commutes with the G-action and hence we get an induced G- and $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$-linear map

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) \to \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)})$$

Theorem 32. *This map is*

1) injective and

2) surjective if V is M-regular.

Definition. V is M-regular if $M_V \subseteq M$, where M_V is the unique largest standard Levi subgroup L such that $V^{U(\mathbb{F}_p)} \subseteq V$ is preserved by $L(\mathbb{F}_p)$ (see section 11).

Remark. 1) This generalises Theorem 16. In that case $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) = \mathcal{H}_G(V)[T_1^{-1}]$.

2) When $P = M = G$ any weight is G-regular and the theorem is trivial (the map is the identity).

3) Let $M = M_{n_1,\dots,n_r}$. Then $V = F(a)$ is M-regular if and only if $a_{n_1} > a_{n_1+1}$, $a_{n_1+n_2} > a_{n_1+n_2+1}, \dots, a_{n-n_r} > a_{n-n_r+1}$. For example when $G = \mathrm{GL}_2$, $V = F(a,b)$ is T-regular if and only if $\dim V > 1$.

4) The converse to part 2) of the theorem is true; if the map is surjective then V is M-regular (Henniart, Vignéras)

Proof. 1) S_G^M is a localisation by Proposition 30 and $\mathcal{H}_G(V)$ is an integral domain, so it is enough to show that

$$F_V : \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)})$$

is injective. Suppose not. Pick a weight $V' \hookrightarrow (\mathrm{Ker}(F_V))|_K$. By Frobenius reciprocity we get a nonzero G-linear map

$$\theta : \mathrm{ind}_K^G V' \longrightarrow \mathrm{Ker}(F_V)$$

By the same argument that we used to show that F_V is $\mathcal{H}_G(V)$-linear we get a commutative diagram

$$\begin{tikzcd}
\mathrm{ind}_K^G V \arrow[r, "F_V"] & \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}) \\
\mathrm{ind}_K^G V \arrow[u, "\theta"] \arrow[r, "F_{V'}"] & \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)}) \arrow[u, "\mathrm{Ind}_{\overline{P}} S_G^M \theta"']
\end{tikzcd}$$

By construction $F_V \circ \theta = 0$, so $(\mathrm{Ind}_{\overline{P}}^G S_G^M \theta) \circ F_{V'} = 0$. Under Lemma 26 $(F_{V'})_M$ is the identity on $\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}$, hence

$$((\mathrm{Ind}_{\overline{P}}^G S_G^M \theta) \circ F_{V'})_M = S_G^M \theta = 0$$

by naturality. Since S_G^M is injective we must have $\theta = 0$, which is a contradiction. Hence F_V is injective.

2) (sketch; similar to the case of GL_2 covered in 16)

Pick $x \in V^{U(\mathbb{F}_p)}$ nonzero. Put $f_0 = F_V([1,x]) \in \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)})$. One computes that

$$f_0(m\bar{n}k) = [m, p_{\overline{N}}(kx)]$$

for $m \in M$, $\bar{n} \in \overline{N}$ and $k \in K$.

Fact: $p_{\overline{N}}(\gamma x) \neq 0$ exactly when $\gamma \in \overline{P}(\mathbb{F}_p).N(\mathbb{F}_p)$ if and only if V is M-regular.

This implies that the support of f_0 is $\overline{P}.N(\mathbb{Z}_p)$. We have an isomorphism

$$\left\{ f \in \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)}) \mid \mathrm{supp}(f) \subseteq \overline{P}.N \right\} \xrightarrow{\sim} \mathcal{C}_c^\infty(N, \mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)})$$

by restriction to N. It maps f_0 to the function

$$n \longmapsto \begin{cases} [1,x] & \text{if } n \in N(\mathbb{Z}_p), \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 16 one then shows (using this isomorphism) that f_0 generates the left-hand side of the above under the actions of P and $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$ and that the left-hand side generates $\mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)})$ as a G-representation, which finishes the proof of surjectivity. □

Corollary. *If V is M-regular and $\chi : \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) \to E$ is an algebra homomorphism then there is an isomorphism*

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi \circ S_G^M} E \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}), \chi} E)$$

of G-representations (this generalises Theorem 15).

Proof. By the theorem we have a G- and $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$-linear isomorphism

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}_G(V)} \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)})$$

and the actions commute. Now apply $- \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}), \chi} E$ to get a G-linear isomorphism

$$\mathrm{ind}_K^G(V) \otimes_{\mathcal{H}_G(V), \chi \circ S_G^M} E \to \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}) \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}), \chi} E$$

It remains to show that

$$\mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V'_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}), \chi} E) \cong \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}) \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}), \chi} E$$

As $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$ is noetherian the kernel of χ is finitely generated and hence there is an exact sequence

$$\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})^{\oplus m} \longrightarrow \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) \xrightarrow{\chi} E \to 0$$

for some positive integer m. Applying the right exact functor $\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})} -$ and then the exact functor $\mathrm{Ind}_{\overline{P}}^G -$ gives the desired isomorphism. □

14. Supersingular Representations for GL_n

If π is an admissible G-representation and V is a weight then $\mathrm{Hom}_K(V,\pi)$ is finite-dimensional and furthermore admits an action by the commutative algebra $\mathcal{H}_G(V)$. If $\mathrm{Hom}_K(V,\pi) \neq 0$ then it contains a common eigenvector for the elements of $\mathcal{H}_G(V)$, i.e. it admits a character $\mathcal{H}_G(V) \to E$ as a submodule.

Definition. We define $\mathrm{Eval}_G(V,\pi)$ to be the set of characters $\mathcal{H}_G(V) \to E$ that occur as a submodule of $\mathrm{Hom}_K(V,\pi)$ (note that this is a finite set).

In what follows we will often identify $\mathcal{H}_G(V)$ with $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}) = E[\tau_1,\dots,\tau_{n-1},\tau_n^{\pm 1}]$ via S_G (where $\mathrm{supp}(\tau_i) = T(\mathbb{Z}_p)\mathrm{diag}(1,\dots,1,p,\dots,p)$, with $n-i$ 1's).

Lemma 33. *Let π be an irreducible admissible G-representation and V a weight. The following are equivalent:*

1) For all $\chi \in \mathrm{Eval}_G(V,\pi)$, $\chi(\tau_1) = \dots = \chi(\tau_{n-1}) = 0$.

2) For all $\chi \in \mathrm{Eval}_G(V,\pi)$ and for all standard Levi subgroups $M \neq G$, χ does not factor through $S_G^M : \mathcal{H}_G(V) \to \mathcal{H}_T(V_{\overline{N}(\mathbb{F}_p)})$.

3) $\mathrm{Hom}_K(V,\pi) \otimes_{\mathcal{H}_G(V),S_G^M} \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}) = 0$ for all standard Levi subgroups $M \neq G$.

Definition. An irreducible admissible representation is called supersingular if it satisfies the equivalent conditions 1) – 3) for all weights V.

Proof. Let us show that 1) is equivalent to 2). The proof that 2) is equivalent to 3) will be skipped and we will not need it.

Let $M = M_{n_1,\dots,n_r}$. Recall that $S_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$ is a localisation at one element. Let us identify these two algebras with their images inside $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$, we get

$$\mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)}) \subseteq \mathcal{H}_T^{-,M}(V_{\overline{U}(\mathbb{F}_p)}) \subseteq \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$$

In Proposition 30 we saw that $\mathcal{H}_T^-(V_{\overline{U}(\mathbb{F}_p)})[\tau_\lambda^{-1}] = \mathcal{H}_T^{-,M}(V_{\overline{U}(\mathbb{F}_p)})$, for any λ that satisfies

$$\lambda_1 = \dots = \lambda_{n_1} < \lambda_{n_1+1} = \dots = \lambda_{n_1+n_2} < \lambda_{n_1+n_2+1} = \dots$$

Let us take $\lambda = (0,\dots,0,1,\dots,1,\dots,r-1,\dots,r-1)$ with each $i \in \{0,\dots,r-1\}$ occurring n_{i+1} times. Thus $\tau_\lambda = \tau_{n_2+\dots+n_r}\tau_{n_3+\dots+n_r}\dots\tau_{n_r}$. Hence a character $\chi : \mathcal{H}_G(V) \cong \mathcal{H}_T^-(V_{\overline{N}(\mathbb{F}_p)}) \to E$ factors through S_G^M if and only if $\chi(\tau_\lambda) \neq 0$, i.e. if and only if $\chi(\tau_{n_i+\dots+n_r}) \neq 0$ for $1 \le i \le r$ (note that the case $i = 1$ is automatic). This gives the equivalence of 1) and 2). $\square$

15. Generalised Steinberg Representations

We wish to understand the irreducible subquotients of $\mathrm{Ind}_{\overline{B}}^G 1_{\overline{B}}$. Let P be a standard parabolic subgroup. Define

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}_{\overline{P}}^G 1_{\overline{P}}}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}}^G 1_{\overline{Q}}}$$

(note that $\mathrm{Ind}_{\overline{Q}}^G 1_{\overline{Q}} \subseteq \mathrm{Ind}_{\overline{P}}^G 1_{\overline{P}}$ when $P \subseteq Q$). As special cases, we have $\mathrm{Sp}_G = 1_G$ and Sp_B, the Steinberg representation.

Theorem 34. *(Grosse-Klönne + ϵ)*

1) Sp_P *has a unique weight and is hence admissible.*

2) Sp_P *is irreducible.*

Proof. 1) is not easy to prove (one needs to determine $(\mathrm{Sp}_P)^{I(1)}$.

2) Let $\chi : \mathcal{H}_M(1_{M(\mathbb{Z}_p)}) \to E$ denote the Hecke eigenvalues of 1_M. Then we get a nonzero M-linear map

$$\mathrm{ind}_{M(\mathbb{Z}_p)}^M 1_{M(\mathbb{Z}_p)} \otimes_{\mathcal{H}_M(1_{M(\mathbb{Z}_p)}),\chi} E \to 1_M$$

Apply $\mathrm{Ind}_{\overline{P}}^G -$ to get a G-linear map:

$$\mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M 1_{M(\mathbb{Z}_p)} \otimes_{\mathcal{H}_M(1_{M(\mathbb{Z}_p)}),\chi} E) \to \mathrm{Ind}_{\overline{P}}^G(1_M)$$

The left-hand side is isomorphic to $\mathrm{ind}_K^G V_P \otimes_{\mathcal{H}_G(V),\chi\circ S_G^M} E$ for any M-regular weight V_P such that $(V_P)_{\overline{N}(\mathbb{F}_p)} \cong 1_M$ by Theorem 32. This gives us a surjection $\mathrm{ind}_K^G V_P \twoheadrightarrow \mathrm{Sp}_P$ and hence V_P occurs in Sp_P and generates it. By 1) this implies irreducibility of Sp_P. It therefore remains to construct such a V_P. Explicitly, take

$$V_P = F\left((r-1)(p-1), \dots, (r-1)(p-1), (r-2)(p-1), \dots\right)$$

where each $(r-i)(p-1)$ occurs n_i times ($M = M_{n_1,\dots,n_r}$). This is M-regular and $(V_P)_{\overline{N}(\mathbb{F}_p)}$ is

$$F\left((r-1)(p-1), \dots, (r-1)(p-1)\right) \otimes \cdots \otimes F(0, \dots, 0) = 1_M$$

This finishes the proof (it follows from 1) that the V_P constructed above is unique; this is also easy to see from the construction). ☐

Corollary. *1) The* Sp_P *are pairwise non-isomorphic.*

2) The irreducible subquotients of $\mathrm{Ind}_{\overline{P}}^G 1_{\overline{P}}$ *are the* Sp_Q *for* $Q \supseteq P$.

3) For the unique $\chi \in \mathrm{Eval}_G(V_P, \mathrm{Sp}_P)$, $\chi(\tau_i) = 1$ *for* $1 \le i \le n$.

Proof. 1) Their weights are non-isomorphic.

2) This follows by an induction on the maximal number of elements r in a chain of parabolics $G = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_r = P$, the case of $r = 0$ being trivial. See the exercises.

3) V_P lifts to $\mathrm{Ind}_{\overline{P}}^G 1_{\overline{P}}$ and hence embeds into $\mathrm{Ind}_{\overline{B}}^G 1_{\overline{B}}$. By Lemma 26, using $(V_P)_{\overline{N}(\mathbb{F}_p)} = 1_{M(\mathbb{Z}_p)}$ and hence equal to $1_{T(\mathbb{Z}_p)}$ as a $T(\mathbb{Z}_p)$-representation, we have

$$\mathrm{Hom}_K(V_P, \mathrm{Ind}_{\overline{B}}^G 1_{\overline{B}}) = \mathrm{Hom}_{T(\mathbb{Z}_p)}(1_{T(\mathbb{Z}_p)}, 1_T)$$

By definition the action of $\mathcal{H}_G(V)$ on the left hand side agrees via S_G with the action of $\mathcal{H}_T(1_{T(\mathbb{Z}_p)})$ on the right-hand side, and since each τ_i acts as 1 on the left-hand side the result follows. ☐

Remark. Let $\pi = \mathrm{Sp}_P \otimes (\eta \circ \det)$, for $\eta : \mathbb{Q}_p^\times \to E^\times$ a smooth character. Its unique weight is $V_P \otimes (\eta \circ \det)|_K$ and its Hecke eigenvalues are given by $\chi(\tau_i) = \eta(p)^{-i}$.

16. Change of Weight for GL_n

Recall from section 11 that weights are parametrised by n-tuples $a = (a_1, \dots, a_n)$ with $0 \leq a_i - a_{i+1} \leq p-1$, with a and a' giving the same representation if and only if $p-1 \mid a_i - a_{i+1}$ for all i.

Proposition 35. *Let $V = F(a)$. Suppose that there is a k such that $a_k - a_{k+1} = 0$ (equivalent to V not being T-regular). Then*

1) There exists a unique weight $V' = F(a')$ such that $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$ and $a'_i - a'_{i+1} = 0$ if and only if $a_i - a_{i+1} = 0$ and $i \neq k$.

2) There are G-linear maps $\mathrm{ind}_K^G V \underset{\varphi^-}{\overset{\varphi^+}{\rightleftarrows}} \mathrm{ind}_K^G V'$ such that (setting $\tau_0 = 1$)

$$S_G(\varphi^+ \circ \varphi^-) = S_G(\varphi^- \circ \varphi^+) = \tau_{n-k}^2 - \tau_{n-k-1}\tau_{n-k+1}$$

3) Write $\mathcal{H}_G$ for $\mathcal{H}_G(V) \cong \mathcal{H}_G(V')$ (identified via the Satake transform). Then for any character $\chi : \mathcal{H}_G \to E$ such that $\chi(\tau_{n-k})^2 \neq \chi(\tau_{n-k-1})\chi(\tau_{n-k+1})$,

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G,\chi} E \cong \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G,\chi} E$$

Remark. This generalises Proposition 19.

Proof. 1) Recall from section 11 that $V_{\overline{U}(\mathbb{F}_p)} \cong F(a_1) \otimes \cdots \otimes F(a_n)$. Hence $V_{\overline{U}(\mathbb{F}_p)} \cong V'_{\overline{U}(\mathbb{F}_p)}$ for $V' = F(a')$ if and only if $a_i \equiv a'_i \bmod p-1$ for all i. Hence we are forced to have

$$a'_i - a'_{i+1} = \begin{cases} a_i - a_{i+1} & \text{if } i \neq k, \\ p-1 & \text{if } i = k. \end{cases}$$

Conversely, this determines a' uniquely modulo $(p-1)\mathbb{Z}$ (embedded diagonally into $\mathbb{Z}^n$) and hence we get a unique weight $V' = F(a')$.

2) By the proof of Proposition 31 1) there are φ^+ and φ^- such that $\mathrm{supp}(\varphi^\pm) = Kt_\lambda K$, where $\lambda = (0, \dots, 0, 1, \dots, 1) \in \Lambda_-$ (with k 0's) (the condition is that $V_{N_\lambda(\mathbb{F}_p)} \cong V'_{N_\lambda(\mathbb{F}_p)}$; $P_\lambda = \overline{P}_{k,n-k}$ and we have that both are isomorphic to $F(a_1, \dots a_k) \otimes F(a_{k+1}, \dots, a_n)$). The proof that $S_G(\varphi^+ \circ \varphi^-) = \tau_{n-k}^2 - \tau_{n-k-1}\tau_{n-k+1}$ (up to non-zero scalar) requires work!

3) By the proof of Proposition 31 3) $\varphi^+ : \mathrm{ind}_K^G V \to \mathrm{ind}_K^G V'$ is $\mathcal{H}_G$-linear and similarly for φ^-. Hence we get induced G-linear maps

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G,\chi} E \underset{\varphi^-}{\overset{\varphi^+}{\rightleftarrows}} \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G,\chi} E$$

The composite is $\chi(\tau_{n-k})^2 - \chi(\tau_{n-k-1})\chi(\tau_{n-k+1})$ which is a non-zero scalar and hence $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G,\chi} E \cong \mathrm{ind}_K^G V' \otimes_{\mathcal{H}_G,\chi} E$. $\square$

17. Irreducibility of Parabolic Inductions

Lemma 36. *Let $M = M_{n_1,\dots,n_r}$ be a standard Levi subgroup. The irreducible admissible M-representations are given by $\sigma = \bigotimes_{i=1}^r \sigma_i$ where the σ_i are irreducible admissible representations of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$. The weights of σ are given by $\bigotimes_{i=1}^r V_i$ where V_i runs through the weights of σ_i. We have a bijection*

$$\mathrm{Eval}_M(V,\sigma) \cong \prod_{i=1}^r \mathrm{Eval}_{\mathrm{GL}_{n_i}(\mathbb{Q}_p)}(V_i,\sigma_i)$$
$$\chi \longmapsto (\chi_i)_i$$
$$\chi(\tau_\lambda) = \chi_1(\tau_{\lambda_1,\dots,\lambda_{n_1}})\chi_2(\tau_{\lambda_{n_1+1},\dots,\lambda_{n_1+n_2}})\cdots$$

for $\lambda \in \Lambda_{-,M}$.

Proof. Skipped. □

Theorem 37. *Let $P = MN = P_{n_1,\dots,n_r}$ be a standard parabolic. Suppose that σ_i is an irreducible admissible representation of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ such that either*

a) σ_i is supersingular and $n_i > 1$ or

b) $\sigma_i = \mathrm{Sp}_{P_i} \otimes (\eta_i \circ \det)$ for some standard parabolic $P_i \subseteq \mathrm{GL}_{n_i}(\mathbb{Q}_p)$ and $\eta_i : \mathbb{Q}_p^\times \to E^\times$

Suppose moreover that $\eta_i \neq \eta_{i+1}$ if σ_i and σ_{i+1} are of type b). Then $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible.

Remark. 1) Note that any irreducible $\mathrm{GL}_1(\mathbb{Q}_p)$-representation is supersingular. Hence the condition $n_i > 1$ guarantees that a) and b) are disjoint (look at $\chi(\tau_i)$).

2) As a special case, we get that when $n_i = 1$ for all i, $\mathrm{Ind}_B^G(\eta_1 \otimes \cdots \otimes \eta_n)$ is irreducible if and only if $\eta_i \neq \eta_{i+1}$ for all i.

Proof. For admissibility, it is enough to show that each weight occurs with finite multiplicity. We have

$$\mathrm{Hom}_K(V, \mathrm{Ind}_P^G \sigma) = \mathrm{Hom}_{M(\mathbb{F}_p)}(V_{\overline{N}(\mathbb{F}_p)}, \sigma)$$

which is finite dimensional since σ is admissible.

For irreducibility, suppose that π is a non-zero subrepresentation of $\mathrm{Ind}_P^G \sigma$ and let V be a weight of π. The strategy will be two divide into two cases:

1) If V is M-regular, we show that V generates $\mathrm{Ind}_P^G \sigma$ as a G-representation and we are done.

2) If not, we change the weight to show that π contains a weight V' that is closer to being M-regular; iterate to reduce to 1)

Pick an $\mathcal{H}_G(V)$-eigenvector

$$f : \mathrm{ind}_K^G V \to \pi \subseteq \mathrm{Ind}_P^G \sigma$$

with eigenvalues χ. By Lemma 26

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G \sigma) = \mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}, \sigma)$$

as Hecke modules via the Satake transform $S_G^M : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$ which is a localisation at one element, which we may choose to be

$$\tau_\lambda = \tau_{n_2+\dots n_r}\tau_{n_3+\dots+n_r}\cdots\tau_{n_r}$$

Since τ_λ acts invertibly on the right-hand side it acts invertibly on the left-hand side, so $\chi(\tau_\lambda) \neq 0$, and hence χ extends (uniquely) to a character χ_M on $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$.

Case 1: V is M-regular

Consider f_M (notation as in Lemma 26). f_M is an $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$-eigenvector with eigenvalues χ_M, and hence induces

$$\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}),\chi_M} E \twoheadrightarrow \sigma$$

Apply $\mathrm{Ind}_P^G -$:

$$\begin{tikzcd} \mathrm{Ind}_P^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)}),\chi_M} E) \arrow[r, "\mathrm{Ind}_P^G(f_M \otimes_{\chi_M} id)"] & \mathrm{Ind}_P^G \sigma \\ \mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} E \arrow[u, "F_V \otimes_\chi id"] & \end{tikzcd}$$

where the horizontal arrow is a surjection by exactness of $\mathrm{Ind}_P^G -$ and the vertical arrow is the isomorphism of the Corollary to Theorem 32; here F_V is defined by $(F_V)_M = id$ under Lemma 26. Thus a copy of V generates $\mathrm{Ind}_P^G \sigma$ as a G-representation. We need to show that this is the same copy of V as we started with, i.e. we need to show the above composition is equal to $f \otimes_\chi id$. To do this, it is enough to show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Ind}_P^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}) \arrow[r, "\mathrm{Ind}_P^G f_M"] & \mathrm{Ind}_P^G \sigma \\ \mathrm{ind}_K^G V \arrow[u, "F_V"] \arrow[ur, "f"'] & \end{tikzcd}$$

But we have $(\mathrm{Ind}_P^G f_M \circ F_V)_M = f_M \circ (F_V)_M = f_M$, which implies $\mathrm{Ind}_P^G f_M \circ F_V = f$ as desired. This gives us that the copy of V we started with generates $\mathrm{Ind}_P^G \sigma$, and hence that $\pi = \mathrm{Ind}_P^G \sigma$.

Case 2: V is not M-regular.

We have $V = F(a_1, \ldots, a_n)$ for some integers a_i with $0 \leq a_i - a_{i+1} \leq p-1$ for all i. Since V is not M-regular there is an i such that, writing $k = n_1 + \cdots + n_i$, $a_k = a_{k+1}$. Fix such an i and hence the corresponding k. By Proposition 35 (3), provided that

$$\chi(\tau_{n-k})^2 - \chi(\tau_{n-k+1})\chi(\tau_{n-k+1})$$

we may change weight to the V' described in the proposition, which is close to being M-regular. We want to show that this holds; as i will be arbitrary with the property $a_k = a_{k+1}$ this will imply that we can apply Proposition 35 (3) as many times as we need and hence reduce to case 1).

Recall that $\chi = \chi_M|_{\mathcal{H}_G(V)}$. We know that $\chi_M(\tau_{n-k}) \neq 0$ because τ_{n-k} is a unit in $\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$. By Lemma 36,

$$V_{\overline{N}(\mathbb{F}_p)} \cong \bigotimes_{j=1}^r V_j$$

$$V_j = F(a_{n_1+\cdots+n_{j-1}+1}, \ldots, a_{n_1+\cdots+n_j})$$

and χ_M corresponds to the collection $(\chi_j)_{j=1}^r$ with $\chi_j \in \mathrm{Hom}_{\mathrm{GL}_{n_j}(\mathbb{Z}_p)}(V_j, \sigma_j)$. We have

$$\chi_M(\tau_{n-k+1}) = \chi_i(\tau_{0,\dots,0,1}) \prod_{j=i+1}^{r} \chi_j(\tau_{1,\dots,1}) = \chi_i(\tau_{0,\dots,0,1})\chi_M(\tau_{n-k})$$

and (remember that $\tau_{1,\dots,1}$ is invertible)

$$\chi_M(\tau_{n-k-1}) = \chi_{i+1}(\tau_{0,1,\dots,1}) \prod_{j=i+2}^{r} \chi_j(\tau_{1,\dots,1}) = \frac{\chi_{i+1}(\tau_{0,1,\dots,1})}{\chi_{i+1}(\tau_{1,\dots,1})}\chi_M(\tau_{n-k})$$

Hence

$$\chi(\tau_{n-k})^2 - \chi(\tau_{n-k-1})\chi(\tau_{n-k+1}) = \chi_M(\tau_{n-k})^2 - \chi_M(\tau_{n-k-1})\chi_M(\tau_{n-k+1}) =$$
$$= \chi_M(\tau_{n-k})^2 \left(1 - \chi_i(\tau_{0,\dots,0,1})\frac{\chi_{i+1}(\tau_{0,1,\dots,1})}{\chi_{i+1}(\tau_{1,\dots,1})}\right)$$

so since $\chi_M(\tau_{n-k}) \neq 0$ we see that this is zero if and only if

$$\chi_i(\tau_{0,\dots,0,1})\frac{\chi_{i+1}(\tau_{0,1,\dots,1})}{\chi_{i+1}(\tau_{1,\dots,1})} = 1$$

If σ_i is of type a) (as in the statement of the theorem) then $\chi_i(\tau_{0,\dots,0,1}) = 0$ (by definition) and hence the above does not hold, which is what we want. Similarly, if σ_{i+1} is of type a) we have $\chi_{i+1}(\tau_{0,1,\dots,1}) = 0$ and we get what we want. Thus we need to deal with the case when σ_i and σ_{i+1} are of type b), so put $\sigma_i = \mathrm{Sp}_{P_i} \otimes(\eta_i \circ \det)$ and $\sigma_{i+1} = \mathrm{Sp}_{P_{i+1}} \otimes(\eta_{i+1} \circ \det)$ for standard parabolics P_i resp. P_{i+1} of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$ resp. $\mathrm{GL}_{n_{i+1}}(\mathbb{Q}_p)$ and smooth characters η_i, η_{i+1} of $\mathbb{Q}_p^\times$. Then

$$\chi_i(\tau_{0,\dots,0,1}) = \eta_i(p)^{-1}$$

and

$$\frac{\chi_{i+1}(\tau_{0,1,\dots,1})}{\chi_{i+1}(\tau_{1,\dots,1})} = \frac{\eta_{i+1}(p)^{-(n_{i+1}-1)}}{\eta_{i+1}(p)^{-n_{i+1}}} = \eta_{i+1}(p)$$

Therefore

$$\chi_i(\tau_{0,\dots,0,1})\frac{\chi_{i+1}(\tau_{0,1,\dots,1})}{\chi_{i+1}(\tau_{1,\dots,1})} = \frac{\eta_{i+1}(p)}{\eta_i(p)}$$

which is not equal to 1 if and only $\eta_{i+1}(p) \neq \eta_i(p)$. By assumption we have that $\eta_i \neq \eta_{i+1}$.

Claim: $\eta_i|_{\mathbb{Z}_p^\times} = \eta_{i+1}|_{\mathbb{Z}_p^\times}$

Note that this implies that $\eta_i \neq \eta_{i+1}$ if and only if $\eta_i(p) \neq \eta_{i+1}(p)$, which is what we want, and hence would finish the proof of the theorem. To prove this, first recall that Sp_{P_i} has a unique weight V_{P_i} and that $(V_{P_i})_{\overline{U}_i(\mathbb{F}_p)} = 1_{T_i(\mathbb{F}_p)}$ (here we are using U_i resp. T_i to denote the upper-triangular unipotent matrices resp. the diagonal matrices of $\mathrm{GL}_{n_i}(\mathbb{Q}_p)$), hence $\mathrm{Sp}_{P_i} \otimes(\eta_i \circ \det)$ has a unique weight $V_{P_i} \otimes (\eta_i \circ \det)$ and $(V_{P_i} \otimes (\eta_i \circ \det))_{\overline{U}_i(\mathbb{F}_p)} = \eta_i \circ \det|_{T(\mathbb{Z}_p)}$. Hence we must have $V_i = V_{P_i} \otimes (\eta_i \circ \det)$ and

$$(V_i)_{\overline{U}_i(\mathbb{F}_p)} = \eta_i \circ \det|_{T(\mathbb{Z}_p)}$$

Since $V_i = F(a_{n_1+\dots+n_{i-1}+1}, \dots, a_{n_1+\dots+n_i})$ we also have

$$(V_i)_{\overline{U}(\mathbb{F}_p)} = F(a_{n_1+\dots+n_{i-1}+1}) \otimes \dots \otimes F(a_{n_1+\dots n_i})$$

and hence $(n_1 + \dots + n_i = k)$

$$\eta_i|_{\mathbb{Z}_p^\times} = F(a_{n_1+\dots+n_{i-1}+1}) = \dots = F(a_{n_1+\dots+n_i}) = F(a_k)$$

By the same argument replacing i with $i+1$, we have that $\eta_{i+1}|_{\mathbb{Z}_p^\times} = F(a_{k+1})$. Since $a_k = a_{k+1}$, this proves our claim, and finishes the proof of the theorem. $\square$

18. Classifying irreducible admissible G-representations

We want to show that any irreducible admissible G-representations is as in Theorem 37. We have seen this for $n = 1$ or 2. Let π be an irreducible admissible representation. Pick a weight $V = F(a_1, \dots, a_n)$ of π and Hecke eigenvalues $\chi \in \mathrm{Eval}_G(V, \pi)$ and write $\mathcal{H}_G = \mathcal{H}_G(V)$. Then

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G,\chi} E \twoheadrightarrow \pi$$

By the Corollary to Theorem 32,

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G,\chi} E \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)} \otimes_{\mathcal{H}_M,\chi_M} E)$$

where $P = MN = P_{n_1,\dots,n_r}$ is a standard parabolic subgroup and $\mathcal{H}_M = \mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$, provided that V is M-regular and χ factors through $S_G^M : \mathcal{H}_G \to \mathcal{H}_M$. This is of course only interesting if $P \neq G$. If $P \neq G$ and the above happens, there is a smooth M-representation σ such that $\mathrm{Ind}_{\overline{P}}^G \sigma \twoheadrightarrow \pi$. If σ is irreducible and admissible, we know that $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ and by induction each σ_i is as in Theorem 37.

Problem. 1) If $\mathrm{Ind}_{\overline{P}}^G \sigma \twoheadrightarrow \pi$, show that there is an irreducible admissible σ' such that $\mathrm{Ind}_{\overline{P}}^G \sigma' \twoheadrightarrow \pi$.

2) Show that the irreducible constituents of $\mathrm{Ind}_{\overline{P}}^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$, with each σ_i as in Theorem 37, are again as in Theorem 37.

V is M-regular if $a_{n_1+\cdots+n_i} \neq a_{n_1+\cdots+n_i+1}$ for all i. The easiest way to satisfy this is to take $r = 2$, then, for $P = P_{i,n-i}$, we want $a_i \neq a_{i+1}$. The condition that χ factors through S_G^M then becomes $\chi(\tau_{n-i}) \neq 0$. There may not be such an i. If $a_i = a_{i+1}$, we may try to change weight. By Proposition 35 we need $a_i = a_{i+1}$ and

$$\chi(\tau_{n-i}^2 - \tau_{n-i-1}\tau_{n-i+1}) \neq 0$$

Recall that when we change weight, χ does not change. When do both methods fail?

Case 1. $\chi(\tau_{n-i}) = 0$ for all i and all (V, χ) of π. If this happens, π is supersingular by definition and we are done. If not, pick (V, χ) such that $\chi(\tau_{n-i}) \neq 0$ for some i.

Case 2. Whenever we have $\chi(\tau_{n-i}) \neq 0$, we also have $a_i = a_{i+1}$ and $\chi(\tau_{n-i}^2 - \tau_{n-i-1}\tau_{n-i+1}) = 0$ (and there is such an i). Arguing inductively, we see that $\chi(\tau_{n-i}) \neq 0$ for all i and hence $a_1 = \cdots = a_n$. Denote the common value of the a_i by α and put $\chi(\tau_{n-i}) = \zeta_{n-i}$. Then we have $V = \det^\alpha$ and from the relation $\zeta_{n-i}^2 = \zeta_{n-i-1}\zeta_{n-i+1}$ for all i (with $\zeta_0 = 1$) we deduce by induction that $\zeta_j = \zeta_1^j$ for all j. If we replace π by $\pi \otimes (\eta \circ \det)$, where η is the character defined by $\eta(x) = \bar{x}^{-\alpha}$ for $x \in \mathbb{Z}_p^\times$ and $\eta(p) = \zeta_1$, we may without loss of generality assume that $\alpha = 0$ and $\zeta_1 = 1$, i.e. $V = 1_K$ and $\chi(\tau_i) = 1$ for all i.

We are then faced with

Problem. 3) Suppose that $1_K \hookrightarrow \pi|_K$ with eigenvalues $\chi(\tau_i) = 1$ for all i. Show that π is as in Theorem 37.

Remark. It is not too hard to see that in fact π then has to be 1_G.

Let us first sketch the solution of Problem 2. For simplicity, let us take $r = 2$, the proof in the general case is similar but notationally more complicated. By transitivity of parabolic induction we have

$$\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \sigma_2) = \mathrm{Ind}_{\overline{P_{12}}}^{G}(\tau_1 \otimes \cdots \otimes \tau_k \otimes \tau_1' \otimes \cdots \otimes \tau_\ell')$$

where $P_1 = M_1N_1$ resp. $P_2 = M_2N_2$ are standard parabolic subgroups of $\mathrm{GL}_{n_1}(\mathbb{Q}_p)$ resp. $\mathrm{GL}_{n_2}(\mathbb{Q}_p)$ and P_{12} denotes the standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ with Levi subgroup $M_1 \times M_2$, and we have written $\sigma_1 = \mathrm{Ind}_{\overline{P_1}}^{\mathrm{GL}_{n_1}(\mathbb{Q}_p)}(\tau_1 \otimes \cdots \otimes \tau_k)$ and $\sigma_2 = \mathrm{Ind}_{\overline{P_2}}^{\mathrm{GL}_{n_2}(\mathbb{Q}_p)}(\tau_1' \otimes \cdots \otimes \tau_\ell')$ as in Theorem 37. Thus, $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \sigma_2)$ is itself as in Theorem 37 (and hence irreducible and equal to π) unless $\tau_k = \mathrm{Sp}_{Q_1} \otimes (\eta \circ \det)$ and $\tau_1' = \mathrm{Sp}_{Q_2} \otimes (\eta \circ \det)$ for some character η. Note that this means that neither τ_{k-1} nor τ_2' are of the form $\mathrm{Sp} \otimes (\eta \circ \det)$, since σ_1 and σ_2 are as in Theorem 37. Let us, in order to further simplify notation, assume that $\eta = 1$. Then (for appropriate m_1,m_2)

$$\mathrm{Ind}_{\overline{Q}_1}^{\mathrm{GL}_{m_1}(\mathbb{Q}_p)} 1_{\overline{Q}_1} \twoheadrightarrow \mathrm{Sp}_{Q_1} = \tau_k$$

$$\mathrm{Ind}_{\overline{Q}_2}^{\mathrm{GL}_{m_2}(\mathbb{Q}_p)} 1_{\overline{Q}_2} \twoheadrightarrow \mathrm{Sp}_{Q_2} = \tau_1'$$

so $\mathrm{Ind}_{\overline{P}}^{G}(\sigma_1 \otimes \sigma_2)$ is a quotient of

$$\mathrm{Ind}_{\overline{P}_{12}}^{G}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \mathrm{Ind}_{\overline{Q}_1}^{\mathrm{GL}_{m_1}(\mathbb{Q}_p)} 1_{\overline{Q}_1} \otimes \mathrm{Ind}_{\overline{Q}_2}^{\mathrm{GL}_{m_2}(\mathbb{Q}_p)} 1_{\overline{Q}_2} \otimes \tau_2' \otimes \cdots \otimes \tau_\ell')$$

which is the same as

$$\mathrm{Ind}_{\overline{P}_{12}}^{G}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \mathrm{Ind}_{\overline{Q}_{12}}^{\mathrm{GL}_{m_1+m_2}(\mathbb{Q}_p)} 1_{\overline{Q}_{12}} \otimes \tau_2' \otimes \cdots \otimes \tau_\ell')$$

where Q_{12} is the parabolic associated to Q_1 and Q_2 in the same way that we got P_{12} from P_1 and P_2. We know by the Corollary to Theorem 34 that the irreducible subquotients of $\mathrm{Ind}_{\overline{Q}_{12}}^{\mathrm{GL}_{m_1+m_2}(\mathbb{Q}_p)} 1_{\overline{Q}_{12}}$ are of the form Sp_R for parabolics $R \supseteq Q_{12}$ so π is a subquotient of some

$$\mathrm{Ind}_{\overline{P}_{12}}^{G}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \mathrm{Sp}_R \otimes \tau_2' \otimes \cdots \otimes \tau_\ell')$$

(by exactness of parabolic induction. But these are now of the form in Theorem 37, hence irreducible, since neither τ_{k-1} nor τ_2' are generalised Steinberg representations. Hence $\pi \cong \mathrm{Ind}_{\overline{P}_{12}}^{G}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \mathrm{Sp}_R \otimes \tau_2' \otimes \cdots \otimes \tau_\ell')$ which is what we wanted to show.

Ordinary parts. Let $P = MN$ be a standard parabolic. If π is a smooth G-representation then the $\overline{N}$-coinvariants $\pi_{\overline{N}}$ is a smooth M-representation (the "Jacquet module"). The association $\pi \mapsto \pi_{\overline{N}}$ is a (right exact) functor. We will write it as $-_{\overline{N}}$.

Lemma 38. *Let π be a smooth G-representation and σ a smooth M-representation. Then there is a natural isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_{\overline{P}}^{G}\sigma) \cong \mathrm{Hom}_M(\pi_{\overline{N}}, \sigma)$$

Proof. By Frobenius reciprocity we have a natural isomorphism $\mathrm{Hom}_G(\pi, \mathrm{Ind}_{\overline{P}}^{G}\sigma) \cong \mathrm{Hom}_{\overline{P}}(\pi, \sigma)$. The universal property of coinvariants gives a natural isomorphism $\mathrm{Hom}_{\overline{P}}(\pi, \sigma) \cong \mathrm{Hom}_M(\pi_{\overline{N}}, \sigma)$ and the composition is our desired natural isomorphism. $\square$

Remark. Since $-_{\overline{N}}$ is a left adjoint, we see that it is right exact.

For smooth complex representations one has

(1) If π is admissible then $\pi_{\overline{N}}$ is admissible (in fact Emerton proved that this remains true in characteristic p).
(2) There is a natural isomorphism $\mathrm{Hom}_G(\mathrm{Ind}_P^G\sigma,\pi)\cong \mathrm{Hom}_M(\sigma,\pi_{\overline{N}})$ (due to Bernstein).

Note the Ind_P^G rather than $\mathrm{Ind}_{\overline{P}}^G$ in Property 2). Property 2) implies that $-_{\overline{N}}$ is left exact for complex representations. However, in characteristic p $-_{\overline{N}}$ is not left exact (and hence Property 2) must fail). Emerton defined a functor Ord_P from smooth G-representations to smooth M-representations in characteristic p which is left exact.

Fact. *1) If π is admissible then $\mathrm{Ord}_P\pi$ is admissible.*

2) If π and σ are admissible then we have a natural isomorphism $\mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G\sigma,\pi)\cong\mathrm{Hom}_M(\sigma,\mathrm{Ord}_P\pi)$.

3) If σ has a central character then $\mathrm{Hom}_{\mathrm{G}}(\mathrm{Ind}_{\overline{\mathrm{P}}}^{\mathrm{G}}\sigma,\pi)\hookrightarrow\mathrm{Hom}_{\mathrm{M}}(\sigma,\mathrm{Ord}_{\mathrm{P}}\pi)$.

4) If σ is admissible then $\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G\sigma)=\sigma$.

Definition. Let $P=MN=P_{n_1,\dots,n_r}$ be a standard parabolic subgroup as above. We will let Z_M denote the center of M (isomorphic to $(\mathbb{Q}_p^\times)^r$). We define

$$M^+=\left\{m\in M\mid mN(\mathbb{Z}_p)m^{-1}\subseteq N(\mathbb{Z}_p)\right\}$$

M^+ acts on $\pi^{N(\mathbb{Z}_p)}$ by the following action (which we call the Hecke action):

$$m.x=\sum_{n\in N(\mathbb{Z}_p)/mN(\mathbb{Z}_p)m^{-1}} nmx$$

for $m\in M^+$ and $x\in\pi^{N(\mathbb{Z}_p)}$.

Then $\mathrm{Ord}_P\pi=\mathrm{Map}_{M^+}(M,\pi^{N(\mathbb{Z}_p)})_{Z_M-finite}=\mathrm{Map}_{M^+\cap Z_M}(Z_M,\pi^{N(\mathbb{Z}_p)})_{Z_M-finite}$, where Map_{M^+} (resp. $\mathrm{Map}_{M^+\cap Z_M}$) denotes M^+-equivariant (resp. $M^+\cap Z_M$-equivariant) functions and the subscript Z_M-finite means those functions whose Z_M-orbit spans a finite-dimensional vector space. By evaluation at 1, $\mathrm{Ord}_P\pi$ embeds into $\pi^{N(\mathbb{Z}_p)}$.

Now we are in a position to solve Problem 1). Recall that we have a surjection (see the paragraph above Problem 1))

$$\mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}\otimes_{\mathcal{H}_M,\chi_M}E)\twoheadrightarrow\pi$$

$\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}\otimes_{\mathcal{H}_M,\chi_M}E$ might not be admissible, but it has a central character. By Fact 3) above we get

$$0\neq\mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}\otimes_{\mathcal{H}_M,\chi_M}E),\pi)\hookrightarrow\mathrm{Hom}_M(\mathrm{ind}_{M(\mathbb{Z}_p)}^M V_{\overline{N}(\mathbb{F}_p)}\otimes_{\mathcal{H}_M,\chi_M}E,\mathrm{Ord}_P\pi)$$

and hence $\mathrm{Ord}_P\pi\neq 0$. $\mathrm{Ord}_P\pi$ is admissible by Fact 1) (since π is admissible) and therefore contains an irreducible subrepresentation σ by Lemma 22. By Frobenius reciprocity, the non-zero map $\sigma\hookrightarrow\mathrm{Ord}_P\pi$ gives a non-zero map $\mathrm{Ind}_{\overline{P}}^G\sigma\twoheadrightarrow\pi$ (necessarily surjective since π irreducible), which is what we wanted since σ is admissible.

It remains to deal with Problem 3). Recall that we wish to show that if $1_K\hookrightarrow\pi|_K$ with Hecke eigenvalues χ such that $\chi(\tau_i)=1$ for all i, π is of the form in Theorem 37. We will show that either $\pi=1_G$ (in which case we are done) or $\mathrm{Ord}_P\pi\neq 0$ for some standard parabolic $P\neq G$. In the latter case we get a surjection $\mathrm{Ind}_{\overline{P}}^G\sigma\twoheadrightarrow\pi$ coming from an irreducible admissible subrepresentation $\sigma\hookrightarrow\mathrm{Ord}_P\pi$ as above, and we are done by the solution to Problem 2).

Remark. 1) See notes at http://www.math.toronto.edu/~herzig/ihp.pdf, pages 24–25 for the case $n = 2$.

2) There is another proof due to Abe, mentioned in the notes above when $n = 2$.

We will not say more about the proof. All in all, modulo some details, we have established

Theorem 39. *Any irreducible admissible representation of G is of the form given in Theorem 37.*

Corollary. *Let π be an irreducible admissible G-representation.*

1) π has constant Hecke eigenvalues in the sense that all weights V of π and all Hecke eigenvalues $\mathrm{Eval}_G(V,\pi)$ the n-tuple $(\chi(\tau_1),\dots,\chi(\tau_n))$ is the same (in fact there exists a more canonical and precise version of this statement).

2) Uniqueness: if $\mathrm{Ind}_P^G(\sigma_1\otimes\cdots\otimes\sigma_r)\cong\mathrm{Ind}_{P'}^G(\sigma_1'\otimes\cdots\otimes\sigma_{r'}')$ and both are as in Theorem 37, then $P=P'$ (hence $r=r'$) and $\sigma_i\cong\sigma_i'$ for all i.

Proof. 1) By Theorem 39 we can take $\pi\cong\mathrm{Ind}_P^G\sigma$, $\sigma=\sigma_1\otimes\cdots\otimes\sigma_r$, as in Theorem 37. Let V be weight of π. We have a $\mathcal{H}_G(V)$-equivariant isomorphism $\mathrm{Hom}_K(V,\mathrm{Ind}_P^G\sigma)\cong\mathrm{Hom}_{M(\mathbb{Z}_p)}(V_{\overline{N}(\mathbb{F}_p)},\sigma)$ where $\mathcal{H}_G(V)$ acts on the right-hand side via $S_G^M:\mathcal{H}_G(V)\hookrightarrow\mathcal{H}_M(V_{\overline{N}(\mathbb{F}_p)})$. Now each σ_i has constant eigenvalues (it is supersingular or generalised Steinberg), hence σ has constant eigenvalues by Lemma 36, so from the isomorphism $\mathrm{Hom}_K(V,\mathrm{Ind}_P^G\sigma)\cong\mathrm{Hom}_{M(\mathbb{Z}_p)}(V_{\overline{N}(\mathbb{F}_p)},\sigma)$ we deduce that $\pi=\mathrm{Ind}_P^G\sigma$ has constant eigenvalues as desired.

2) Let $\pi=\mathrm{Ind}_P^G\sigma$, $\sigma=\sigma_1\otimes\cdots\otimes\sigma_r$ be as in Theorem 37. First let us compute the constant Hecke eigenvalues of π using Lemma 36: In the i-th block, we have

$$\chi(\tau_{j+n_{i+1}+\cdots+n_r})=\begin{cases}0 & \text{if } \sigma_i \text{ of type (a) and } j\neq 0,n_i,\\ \neq 0 & \text{if } \sigma_i \text{ of type (a) and } j=0,n_i,\\ \chi(\tau_{n_{i+1}+\cdots+n_r})\eta_i(p)^{-1}\neq 0 & \text{if } \sigma_i \text{ of type (b).}\end{cases}$$

Thus the zeroes determine the supersingular blocks of P. For $1\le k\le n$, suppose the k^{th} diagonal entry does not belong to a supersingular block. We wish to determine η_i. Have

$$\eta_i(p)=\frac{\chi(\tau_{n-k})}{\chi(\tau_{n-k+1})}$$

by above. Also, $\eta_i|_{\mathbb{Z}_p^\times}=F(a_k)$ for all weights $V=F(a_1,\dots,a_n)$ of π (see the end of Theorem 37). Hence the weights and the Hecke eigenvalues determine all characters η_i, so as $\eta_i\neq\eta_{i+1}$ this determines all generalised Steinberg blocks and hence we get P. Thus if $\mathrm{Ind}_P^G(\sigma_1\otimes\cdots\otimes\sigma_r)\cong\mathrm{Ind}_{P'}^G(\sigma_1'\otimes\cdots\otimes\sigma_{r'}')$ as in the statement of 2), we have proven that $P=P'$. Thus, we can apply Ord_P to both sides. By Fact 4),

$$\sigma_1\otimes\cdots\otimes\sigma_r=\mathrm{Ord}_P(\mathrm{Ind}_P^G(\sigma_1\otimes\cdots\otimes\sigma_r))\cong\mathrm{Ord}_P(\mathrm{Ind}_P^G(\sigma_1'\otimes\cdots\otimes\sigma_r'))=\sigma_1'\otimes\cdots\otimes\sigma_r'$$

and so we deduce from this that $\sigma_i\cong\sigma_i'$ by Lemma 36. □

Corollary. *Let $P=MN$ be a standard parabolic subgroup and σ an irreducible admissible M-representation. Then $\mathrm{Ind}_P^G\sigma$ has finite length, all irreducible subquotients occur with multiplicity one and they all have the same Hecke eigenvalues.*

Proof. Write $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$. By Theorem 39 each σ_i is as in Theorem 37, so in fact this was Problem 2) of the proof of Theorem 39. □

Remark. The constant Hecke eigenvalues in the Corollary factor through $\mathcal{H}_M$: It is enough to show this for a subrepresentation, in which case it is clear.

Definition. Let π be an irreducible admissible G-representation. π is said to be supercuspidal if π does not occur in $\mathrm{Ind}_P^G \sigma$ for any $P \neq G$ parabolic and σ irreducible admissible representation of M.

Corollary. *Let π be an irreducible admissible G-representation.*

1) π is supercuspidal if and only if it is supersingular.

2) ("supercuspidal support") There exists a unique standard parabolic subgroup $P = MN$ and unique σ_i irreducible, admissible and supercuspidal (up to isomorphism) such that π occurs in $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$.

Proof. 1) Assume that π is supersingular. If π occurs in $\mathrm{Ind}_P^G \sigma$, with σ irreducible and admissible, then the Hecke eigenvalues factor through $\mathcal{H}_M$ by the remark above. Hence we must have $M = G$ and so π is supercuspidal.

For the converse, assume that π is supercuspidal. By Theorem 39 $\pi = \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ as in Theorem 37. Since π is supercuspidal, $P = G$ so $\pi = \sigma_1$ and so π is either supersingular or $\mathrm{Sp}_Q \otimes (\eta \circ \det)$ for some parabolic Q and character η. But the latter occurs in $\mathrm{Ind}_B^G(\eta \circ \det)$ and is therefore not supercuspidal, so π is supersingular.

2) Consider $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ with σ_i supersingular/supercuspidal for all i. Grouping together adjacent characters among σ_i which are the same we may rewrite (by transitivity of parabolic induction) $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ as $\mathrm{Ind}_Q^G(\sigma_1' \otimes \cdots \otimes \sigma_{r'}')$ where each σ_i' is either supersingular or $\mathrm{Ind}_B^{\mathrm{GL}_{n_i'}(\mathbb{Q}_p)}(\eta_i \circ \det)$, and if σ_i' and σ_{i+1}' are both of the second type, then $\eta_i \neq \eta_{i+1}$. By exactness of parabolic induction, the Corollary to Theorem 34 and Theorem 37 the irreducible constituents of these are of the form $\mathrm{Ind}_Q^G(\sigma_1'' \otimes \cdots \otimes \sigma_{r'}'')$ with $\sigma_i' = \sigma_i''$ if σ_i' supersingular and $\sigma_i'' = \mathrm{Sp}_R \otimes (\eta_i \circ \det)$. By Theorem 39 these irreducible constituents, for all P and σ_i, exhaust the irreducible admissible representations of G, and by part 2) of the first Corollary to Theorem 39 each irreducible admissible representation of G occur only in one $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ (note that the decomposition as in Theorem 37 determines P and the σ_i uniquely). □