Linear Algebraic Groups

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Introduction.

**Algebraic group:** a group that is also an algebraic variety such that the group operations are maps of varieties.

*Example.* $G = \text{GL}_n(k)$, $k = \overline{k}$

**Goal:** to understand the structure of reductive/semisimple affine algebraic groups over algebraically closed fields $k$ (not necessarily of characteristic 0). Roughly, they are classified by their Dynkin diagrams, which are associated graphs.

Within $G$ are maximal, connected, solvable subgroups, called the Borel subgroups.

*Example.* In $G = \text{GL}_n(k)$, a Borel subgroup $B$ is given by the upper triangular matrices.

A fundamental fact is that the Borels are conjugate in $G$, and much of the structure of $G$ is grounded in those of the $B$. (Thus, it is important to study solvable algebraic groups). $B$ decomposes as

$$B = T \ltimes U$$

where $T \cong \mathbb{G}_m^n$ is a maximal torus and $U$ is unipotent.

*Example.* With $G = \text{GL}_n(k)$, we can take $T$ consisting of all diagonal matrices with $U$ the upper triangular matrices with 1’s along the diagonal.

$G$ acts on its Lie algebra $\mathfrak{g} = T_1 G$. This action restricts to a semisimple action of $T$ on $\mathfrak{g}$. From the nontrivial eigenspaces, we get characters $T \to k^\times$ called the roots. The roots give a root system, which allows us to define the Dynkin diagrams.

*Example.* $G = \text{GL}_n(k)$. $\mathfrak{g} = M_n(k)$ and the action of $G$ on $\mathfrak{g}$ is by conjugation. The roots are given by

$$\text{diag}(x_1, \ldots, x_n) \mapsto x_i x_j^{-1}$$

for $1 \leq i \neq j \leq n$.

Main References:
- Springer’s *Linear Algebraic Groups*, second edition
- Polo’s course notes at [www.math.jussieu.fr/~polo/M2](http://www.math.jussieu.fr/~polo/M2)
- Borel’s *Linear Algebraic Groups*
0. Algebraic geometry (review).

We suppose $k = \overline{k}$. Possible additional references for this section: Milne’s notes on Algebraic Geometry, Mumford’s Red Book.

0.1 Zariski topology on $k^n$.

If $I \subset k[x_1, \ldots, x_n]$ is an ideal, then $V(I) := \{x \in k^n \mid f(x) = 0 \ \forall f \in I\}$. Closed subsets are defined to be the $V(I)$. We have

$$\bigcap_{\alpha} V(I_\alpha) = V(\sum I_\alpha)$$

$$V(I) \cup V(J) = V(I \cap J)$$

Note: this topology is not $T_2$ (i.e., Hausdorff). For example, when $n = 1$ this is the finite complement topology.

0.2 Nullstellensatz.

Theorem 1 (Nullstellensatz).

(i) \{radical ideals $I$ in $k[x_1, \ldots, x_n]$\} $\xleftrightarrow{V_I} \{\text{closed subsets in } k^n\}$ are inverse bijections, where $I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall x \in X\}$

(ii) $I, V$ are inclusion-reversing

(iii) If $I \leftrightarrow X$, then $I$ prime $\iff X$ irreducible.

It follows that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form

$m_a = I(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n)$, $a \in k$. 
0.3 Some topology.

$X$ is a topological space.

$X$ is **irreducible** if $X = C_1 \cup C_2$, for closed sets $C_1, C_2$ implies that $C_i = X$ for some $i$.

\[\iff\] any two non-empty open sets intersect

\[\iff\] any non-empty open set is dense

**Facts.**

- $X$ irreducible $\implies$ $X$ connected.
- If $Y \subset X$, then $Y$ irreducible $\iff Y$ irreducible.

$X$ is **noetherian** if any chain of closed subsets $C_1 \supset C_2 \supset \cdots$ stabilises. If $X$ is noetherian, any irreducible subset is contained in a maximal irreducible subset (which is automatically closed), an **irreducible component**. $X$ is the union of its finitely many irreducible components:

\[X = X_1 \cup \cdots \cup X_n\]

**Fact.** The Zariski topology on $k^n$ is noetherian and compact (a consequence of Nullstellensatz).

0.4 Functions on closed subsets of $k^n$

$X \subset k^n$ is a closed subset.

\[X = \{a \in k^n \mid \{a\} \subset X \iff m_a \supset I(X)\} \leftrightarrow \{\text{maximal ideals in } k[x_1, \ldots, x_n]/I(X)\}\]

Define the **coordinate ring** of $X$ to be $k[X] := k[x_1, \ldots, x_n]/I(X)$. The coordinate ring is a reduced, finitely-generated $k$-algebra and can be regarded as the restriction of polynomial functions on $k^n$ to $X$.

- $X$ irreducible $\iff k[X]$ integral domain
- The closed subsets of $X$ are in bijection with the radical ideals of $k[X]$.

**Definition 2.** For a non-empty open $U \subset X$, define

\[O_X(U) := \{f : U \to k \mid \forall x \in U, \ \exists x \in V \subset U, V \text{ open, and } \exists p, q \in k[x_1, \ldots, x_n] \text{ such that } f(y) = \frac{p(y)}{q(y)} \ \forall y \in V\}\]

$O_X$ is a sheaf of $k$-valued functions on $X$:

- for all $U$, $O_X(U)$ is a $k$-subalgebra of \{set-theoretic functions $U \to k$\}
- $U \subset V$, then $f \in O_X(V) \implies f|_U \in O_X(U)$;
- if $U = \bigcup U_\alpha$, $f : U \to k$ function, then $f|_{U_\alpha} \in O_X(U_\alpha)$ $\forall \alpha \implies f \in O_X(U)$.

**Facts.**

- $O_X(X) \cong k[X]$
- If $f \in O_X(X)$, $D(f) := \{x \in X \mid f(x) \neq 0\}$ is open and these sets form a basis for the topology. $O_X(D(f)) \cong k[X]_f$. 


Definitions 3. A **ringed space** is a pair \((X, \mathcal{F}_X)\) of a topological space \(X\) and a sheaf of \(k\)-valued functions on \(X\). A morphism \((X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)\) of ringed spaces is a continuous map \(\phi : X \to Y\) such that
\[
\forall V \subset Y \text{ open}, \forall f \in \mathcal{F}_Y(V), \ f \circ \phi \in \mathcal{F}_X(\phi^{-1}(V))
\]

An **affine variety** (over \(k\)) is a pair \((X, \mathcal{O}_X)\) for a closed subset \(X \subset k^n\) for some \(n\) (with \(\mathcal{O}_X\) as above). **Affine \(n\)-space** is defined as \(A^n := (k^n, \mathcal{O}_{k^n})\).

**Theorem 4.** \(X \mapsto k[X], \phi \mapsto \phi^*\) gives an equivalence of categories
\[
\{\text{affine varieties over } k\}^{\text{op}} \to \{\text{reduced finitely-generated } k\text{-algebras}\}
\]

If \(\phi : X \to Y\) is a morphism of varieties, then \(\phi^* : k[Y] \to k[X]\) here is \(f + I(Y) \mapsto f \circ \phi + I(X)\). The inverse functor is given by mapping \(A\) to \(\text{m-Spec}(A)\), the spectrum of maximal ideals of \(A\), along with the appropriate topology and sheaf.

### 0.5 Products.

**Proposition 5.** \(A, B\) finitely-generated \(k\)-algebras. If \(A, B\) are reduced (resp. integral domains), then so is \(A \otimes_k B\).

From the above theorem and proposition, we get that if \(X, Y\) are affine varieties, then \(\text{m-Spec}(k[X] \otimes_k k[Y])\) is a product of \(X\) and \(Y\) in the category of affine varieties.

**Remark 6.** \(X \times Y\) is the usual product as a set, but not as topological spaces (the topology is finer).

**Definition 7.** A **prevariety** is a ringed space \((X, \mathcal{F}_X)\) such that \(X = U_1 \cup \cdots \cup U_n\) with the \(U_i\) open and the \((U_i, \mathcal{F}_{|U_i})\) isomorphic to affine varieties. \(X\) is compact and noetherian. (This is too general of a construction. Gluing two copies of \(A^1\) along \(A^1 - \{0\}\) (a pathological space) gives an example of a prevariety.)

Products in the category of prevarieties exist: if \(X = \bigcup_{i=1}^n U_i, Y = \bigcup_{j=1}^m V_j\) (\(U_i, V_j\) affine open), then \(X \times Y = \bigcup_{i,j} U_i \times V_j\), where each \(U_i \times V_j\) is the product above. As before, this gives the usual products of sets but not topological spaces.

**Definition 8.** A **prevariety** is a **variety** if the diagonal \(\Delta_X \subset X \times X\) is a closed subset. (This is like being \(T_2\)!)

- Affine varieties are varieties; \(X, Y\) varieties \(\implies\) \(X \times Y\) variety.
- If is \(Y\) a variety, then the graph of a morphism \(X \to Y\) is closed in \(X \times Y\).
- If \(Y\) is a variety, \(f, g : X \to Y\), then \(f = g\) if \(f, g\) agree on a dense subset.
- If \(X, Y\) are irreducible, then so is \(X \times Y\).
0.6 Subvarieties.

Let $X$ be a variety and $Y \subset X$ a locally closed subset (i.e., $Y$ is the intersection of a closed and an open set, or, equivalently, $Y$ is open in $\overline{Y}$). There is a unique sheaf $\mathcal{O}_Y$ on $Y$ such that $(Y, \mathcal{O}_Y)$ is a prevariety and the inclusion $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism such that

for all morphisms $f : Z \to X$ such that $f(Z) \subset Y$, $f$ factors through the inclusion $Y \to X$.

Concretely,

$$\mathcal{O}_Y(V) = \{ f : V \to k \mid \forall x \in V, \exists U \subset X, x \in U \text{ open, and } \exists \tilde{f} \in \mathcal{O}_X(U) \text{ such that } f|_{U \cap V} = \tilde{f}|_{U \cap V} \}.$$

Remarks 9. $Y, X$ as above.

- If $Y \subset X$ is open, then $\mathcal{O}_Y = \mathcal{O}_X|_Y$.
- $Y$ is a variety ($\Delta_Y = \Delta_X \cap (Y \times Y)$)
- If $X$ is affine and $Y$ is closed, then $Y$ is affine with $k[Y] \cong k[X]/I(Y)$
- If $X$ is affine and $Y = D(f)$ is basic open, then $Y$ is affine with $k[Y] \cong k[X]/(f)$. (Note that general open subsets of affine varieties need not be affine (e.g., $\mathbb{A}^2 - \{0\} \subset \mathbb{A}^2$.)

It’s easy to see from the above definitions that if $X, Y$ are varieties and $Z \subset X, W \subset Y$ are locally closed, then $Z \times W \subset X \times Y$ is locally closed and the subvariety structure on $Z \times W$ inside the product $X \times Y$ agrees with the product structure on the product of subvarieties $Z, W$.

Theorem 10. Let $\phi : X \to Y$ be a morphism of affine varieties.

(i) $\phi^* : k[Y] \to k[X]$ surjective $\iff$ $\phi$ is a closed immersion (i.e., an isomorphism onto a closed subvariety)

(ii) $\phi^* : k[Y] \to k[X]$ is injective $\iff$ $\overline{\phi(X)} = Y$ (i.e., $\phi$ is dominant)

0.7 Projective varieties.

$\mathbb{P}^n = \frac{k^{n+1} - \{0\}}{k^*}$ as a set. The Zariski topology on $\mathbb{P}^n$ is given by defining, for all homogeneous ideals $I, V(I)$ to be a closed set. For $U \subset \mathbb{P}^n$ open,

$$\mathcal{O}_{\mathbb{P}^n}(U) := \{ f : U \to k \mid \forall x \in U, \exists F, G \in k[x_0, \ldots, x_n], \text{ homogeneous of the same degree such that } f(y) = \frac{F(y)}{G(y)}, \text{ for all } y \text{ in a neighbourhood of } x \}.$$

Let $U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\} = \mathbb{P}^n - V((x_i))$, which is open. $\mathbb{A}^n \to U_i$ given by

$$x \mapsto (x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n)$$

gives an isomorphism of ringed spaces, which implies that $\mathbb{P}^n$ is a prevariety; in fact, it is an irreducible variety.

Definitions 11. A projective variety is a closed subvariety of $\mathbb{P}^n$. A quasi-projective variety is a locally closed subvariety of $\mathbb{P}^n$.

Facts.

- The natural map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ is a morphism
- $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$
0.8 Dimension.

Here is an irreducible variety. The function field of $X$ is $k(X) := \varinjlim \mathcal{O}_X(U)$, the germs of regular functions.

**Facts.**
- For $U \subset X$ open, $k(U) = k(X)$.
- For $U \subset X$ irreducible affine, $k(U)$ is the fraction field of $k[U]$.
- $k(X)$ is a finitely-generated field extension of $k$.

**Definition 12.** The dimension of $X$ is $\dim X := \text{tr.deg}_k k(X)$.

**Theorem 13.** If $X$ is affine, then $\dim X = \text{Krull dimension of } k[X]$ (which is the maximum length of chains of $C_0 \subsetneq \cdots \subsetneq C_n$ of irreducible closed subsets).

**Facts.**
- $\dim \mathbb{A}^n = n = \dim \mathbb{P}^n$
- If $Y \subsetneq X$ is closed and irreducible, then $\dim Y < \dim X$
- $\dim(X \times Y) = \dim X + \dim Y$

For general varieties $X$, define $\dim X := \max\{\dim Y \mid Y$ is an irreducible component$\}$.

0.9 Constructible sets.

A subset $A \subset X$ of a topological space is **constructible** if it is the union of finitely many locally closed subsets. Constructible sets are stable under finite unions and intersection, taking complements, and taking inverse images under continuous maps.

**Theorem 14** (Chevalley). Let $\phi : X \to Y$ be a morphism of varieties.

(i) $\phi(X)$ contains a nonempty open subset of its closure.

(ii) $\phi(X)$ is constructible.

0.10 Other examples.

- A finite dimensional $k$-vector space is an affine variety: fix a basis to get a bijection $V \sim k^n$, giving $V$ the corresponding structure (which is actually independent of the basis chosen). Intrinsically, we can define the topology and functions using polynomials in linear forms of $V$, that is, from $\text{Sym}(V^*) = \bigoplus_{n=0}^{\infty} \text{Sym}^n(V^*)$: $k[V] := \text{Sym}(V^*)$.

- Similarly, $\mathbb{P}V = \mathbb{P}^{V-\{0\}}$. As above, use a linear isomorphism $V \sim k^{n+1}$ to get the structure of a projective space; or, intrinsically, use homogeneous elements of $\text{Sym}(V^*)$.
1. Algebraic groups: beginnings.

1.1 Preliminaries.

We will only consider the category of affine algebraic groups, a.k.a. linear algebraic groups. In future, by “algebraic group” we will mean “affine algebraic group”. There are three descriptions of the category:

(1) **Objects:** affine varieties $G$ over $k$ with morphisms $\mu : G \times G \to G$ (multiplication), $i : G \to G$ (inversion), and $\epsilon : A^0 \to G$ (i.e., a distinguished point $e \in G$) such that the group axioms hold, i.e., that the following diagrams commute.

\[
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mu \times \text{id}} & G \\
\downarrow{\text{id} \times \mu} & & \downarrow{\mu} \\
G \times G & \xrightarrow{\mu} & G
\end{array}
\] \quad
\begin{array}{ccc}
G \times A^0 & \xrightarrow{\text{id} \times \epsilon} & G \\
\downarrow{\mu} & & \downarrow{\mu} \\
G & \xrightarrow{i} & A^0
\end{array}
\quad
\begin{array}{ccc}
G \xrightarrow{(\text{id},i)} & G \times G & \xrightarrow{(i,\text{id})} G \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
A^0 & \xrightarrow{\epsilon} & A^0
\end{array}

**Maps:** morphisms of varieties compatible with the above structure maps.

(2) **Objects:** commutative Hopf $k$-algebras, which are reduced, commutative, finitely-generated $k$-algebras $A$ with morphisms $\Delta : A \to A \otimes A$ (co-multiplication), $i : A \to A$ (co-inverse, also called antipode), and $\epsilon : A \to k$ (co-unit) such that the co-group axioms hold, i.e., that the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\Delta \otimes \text{id}} & A \otimes A \\
\downarrow{\text{id} \otimes \Delta} & & \downarrow{\Delta} \\
A \otimes A & \xrightarrow{\epsilon \otimes \text{id}} & k \otimes A \\
\downarrow{\Delta} & & \downarrow{\Delta} \\
A & \xrightarrow{\epsilon} & k
\end{array}
\] \quad
\begin{array}{ccc}
A & \xrightarrow{(i,\text{id})} & A \otimes A \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
A & \xrightarrow{(\text{id},i)} & A
\end{array}

**Maps:** $k$-algebra morphisms compatible with the above structure maps.

(3) **Objects:** functors

\[
\left( \text{reduced finitely-generated (commutative) } k\text{-algebras} \right) \to \left( \text{groups} \right)
\]
that are representable as set-valued functors;

Maps: natural transformations.

Here are the relationships:

\(1) \leftrightarrow (2) : \ G \mapsto A = k[G] \) gives an equivalence of categories. Note that \(k[G \times G] = k[G] \otimes k[G]\).

\(2) \leftrightarrow (3) : \ A \mapsto \text{Hom}_{\text{alg}}(A, -) \) gives an equivalence of categories by Yoneda’s lemma.

Examples.

- \(G = \mathbb{A}^1 =: \mathbb{G}_a\)
  
  In (1): \(\mu : (x, y) \mapsto x + y\) (sum of projections), \(i : x \mapsto -x\), \(\epsilon : * \mapsto 0\)
  
  In (2): \(A = k[T], \quad \Delta(T) = T \otimes 1 + 1 \otimes T, \quad i(T) = -T, \quad \epsilon(T) = 0\)
  
  In (3): the functor \(\text{Hom}_{\text{alg}}(k[T], -)\) sends an algebra \(R\) to its additive group \((R, +)\).

- \(G = \mathbb{A}^1 - \{0\} =: \mathbb{G}_m = \text{GL}_1\)
  
  In (1): \(\mu : (x, y) \mapsto xy\) (product of projections), \(i : x \mapsto x^{-1}\), \(\epsilon : * \mapsto 1\)
  
  In (2): \(A = k[T, T^{-1}], \quad \Delta(T) = T \otimes T, \quad i(T) = T^{-1}, \quad \epsilon(T) = 1\)
  
  In (3): the functor \(\text{Hom}_{\text{alg}}(k[T, T^{-1}], -)\) sends an algebra \(R\) to its group of units \((R, \times)\).

- \(G = \text{GL}_n\)
  
  In (1): \(\text{GL}_n(k) \subset M_n(k) \cong k^{n^2}\) with the usual operations is the basic open set given by \(\det \neq 0\)
  
  In (2): \(A = k[T_{ij}, \det(T_{ij})^{-1}]_{1 \leq i, j \leq n}, \quad \Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}\)
  
  In (3): the functor \(R \mapsto \text{GL}_n(R)\)

- \(G = V\) finite-dimensional \(k\)-vector space
  
  Given by the functor \(R \mapsto (V \otimes_k R, +)\)

- \(G = \text{GL}(V), \) for a finite-dimensional \(k\)-vector space \(V\)
  
  Given by the functor \(R \mapsto \text{GL}(V \otimes_k R)\)

Examples of morphisms.

- For \(\lambda \in k^\times, \ x \mapsto \lambda x\) is an automorphism of \(\mathbb{G}_a\)

Exercise. Show that \(\text{Aut}(\mathbb{G}_a) \cong k^\times\). Note that \(\text{End}(\mathbb{G}_a)\) can be larger, as we have the Frobenius \(x \mapsto x^p\) when \(\text{char } k = p > 0\).

- For \(n \in \mathbb{Z}, \ x \mapsto x^n\) gives an automorphism of \(\mathbb{G}_m\).
- \(g \mapsto \det g\) gives a morphism \(\text{GL}_n \to \mathbb{G}_m\).

Note that if \(G, H\) are algebraic groups, then so is \(G \times H\) (in the obvious way).

1.2 Subgroups.

A locally closed subgroup \(H \leq G\) is a locally closed subvariety that is also a subgroup. \(H\) has a unique structure as an algebraic group such that the inclusion \(H \to G\) is a morphism (it is given
Examples. Closed subgroups of $GL_n$:

- $G = SL_n$, $(\det = 1)$
- $G = D_n$, diagonal matrices $(T_{ij} = 0 \ \forall \ i \neq j)$
- $G = B_n$, upper-triangular matrices $(T_{ij} = 0 \ \forall \ i > j)$
- $G = U_n$, unipotent matrices (upper-triangular with 1’s along the diagonal)
- $G = O_n$ or $Sp_n$, for a particular $J \in GL_n$ with $J^tJ = \pm J$, these are the matrices $g$ with $g^tJg = J$
- $G = SO_n = O_n \cap SL_n$

Exercise. $D_n \cong G^n_m$. Multiplication $(d, n) \mapsto dn$ gives an isomorphism $D_n \times U_n \to B_n$ as varieties.

(Actually, $B_n$ is a semidirect product of the two, with $U_n \triangleleft B_n$.)

Remark 15. $G_a$, $G_m$, and $GL_n$ are irreducible (latter is dense in $A^{n^2}$). $SL_n$ is irreducible, as it is defined by the irreducible polynomial $\det - 1$. In fact, $SO_n, Sp_n$ are also irreducible.

Lemma 16.

(a) If $H \leq G$ is an (abstract) subgroup, then $\overline{H}$ is a (closed) subgroup.

(b) If $H \leq G$ is a locally closed subgroup, then $H$ is closed.

(c) If $\phi : G \to H$ is a morphism of algebraic groups, then $\ker \phi$, $\operatorname{im} \phi$ are closed subgroups.

Proof.

(a). Multiplication by $g$ is an isomorphism of varieties $G \to G$: $g\overline{H} = \overline{gH}$ and $\overline{H}g = \overline{Hg}$

(b). $H \subset \overline{H}$ is open and $\overline{H} \subset G$ is closed, so without loss of generality suppose that $H \subset G$ is open. Since the complement of $H$ is a union of cosets of $H$, which are open since $H$ is, it follows that $H$ is closed.

(c). $\ker \phi$ is clearly a closed subgroup. $\operatorname{im} \phi = \phi(G)$ contains a nonempty open subset $U \subset \overline{\phi(G)}$ by Chevalley; hence, $\phi(G) = \bigcup_{h \in \phi(G)} hU$ is open in $\overline{\phi(G)}$ and so $\phi(G)$ is closed by (b).

Lemma 17. The connected component $G^0$ of the identity $e \in G$ is irreducible. The irreducible and connected components of $G^0$ coincide and they are the cosets of $G^0$. $G^0$ is an open normal subgroup (and thus has finite index).

Proof. Let $X$ be an irreducible component containing $e$ (which must be closed). Then $X \cdot X^{-1} = \mu(X \times X^{-1})$ is irreducible and contains $X$; hence, $X = X \cdot X^{-1}$ is a subgroup as it is closed under inverse and multiplication. So $G = \bigsqcup_{gX \in G/X} gX$ gives a decomposition of $G$ into its irreducible components. Since $G$ has a finite number of irreducible components, it follows that $(G : X) < \infty$ and $X$ is open. Hence, the cosets $gX$ are the connected components: $X = G^0$. Moreover, $G^0$ is normal since $gG^0g^{-1}$ is another connected component containing $e$.

Corollary 18. $G$ connected $\iff$ $G$ irreducible

Exercise. $\phi : G \to H \implies \phi(G^0) = \phi(G)^0$
1.3 Commutators.

**Proposition 19.** If \(H, K\) are closed, connected subgroups of \(G\), then

\[
[H, K] = \langle [h, k] = hkh^{-1}k^{-1} \mid h \in H, k \in K \rangle
\]

is closed and connected. (Actually, we just need one of \(H, K\) to be connected. Moreover, without any of the connected hypotheses, Borel shows that \([H, K]\) is closed.)

**Lemma 20.** Let \(\{X_{\alpha}\}_{\alpha \in I}\) be a collection of irreducible varieties and \(\{\phi_{\alpha} : X_{\alpha} \to G\}\) a collection of morphisms into \(G\) such that \(e \in Y_{\alpha} := \phi_{\alpha}(X_{\alpha})\) for all \(\alpha\). Then the subgroup \(H\) of \(G\) generated by the \(Y_{\alpha}\) is connected and closed. Furthermore, \(\exists \alpha_1, \ldots, \alpha_n \in I, \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}\) such that \(H = Y_{\epsilon_1}^{\alpha_1} \cdots Y_{\epsilon_n}^{\alpha_n}\).

**Proof of Lemma.** Without loss of generality suppose that \(\phi_{\alpha}^{-1} = i \circ \phi_{\alpha} : X_{\alpha} \to G\) is also among the maps for all \(\alpha\). For \(n \geq 1\) and \(a \in I^n\), write \(Y_a := Y_{\alpha_1}^{\epsilon_1} \cdots Y_{\alpha_n}^{\epsilon_n} \subset G\). \(Y_a\) is irreducible, and so \(Y_b \subset Y_a\). Choose \(n, a\) such that \(\dim Y_a\) is maximal. Then for all \(m, b \in I^m\),

\[
Y_a \subset Y_{a,b} \subset Y_a \cdot Y_b = Y_{(a,b)}
\]

(second inclusion as in Lemma 16(a)) which by maximality implies that \(Y_a = Y_{(a,b)}\) and \(Y_b \subset Y_a\). In particular, this gives that \(Y_a \cdot Y_a \subset Y_{(a,a)} = Y_a\) and \(Y_a^{-1} \subset Y_a\).

\(Y_a\) is a subgroup. By Chevalley, there is a nonempty \(U \subset Y_a\) open in \(Y_a\).

**Claim:** \(Y_a = U \cdot U\) (\(\Rightarrow Y_a = Y_a \cdot Y_a = Y_{(a,a)} \Rightarrow\) done.)

\[
g \in Y_a \Rightarrow gU^{-1} \cap U \neq \emptyset \Rightarrow g \in U \cdot U
\]

**Proof of Proposition.** For \(k \in K\), consider the morphisms \(\phi_k : H \to G, h \mapsto [h, k]\). Note that \(\phi_k(e) = e\).

**Corollary 21.** If \(\{H_{\alpha}\}\) are connected closed subgroups, then so is the subgroup generated by them.

**Corollary 22.** If \(G\) is connected, then its derived subgroup \(\mathfrak{D}G := [G, G]\) is closed and connected.

**Definitions 23.** Inductively define \(\mathfrak{D}^n G := \mathfrak{D}(\mathfrak{D}^{n-1} G) = [\mathfrak{D}^{n-1} G, \mathfrak{D}^{n-1} G]\) with \(\mathfrak{D}^0 G = G\).

\[
G \supseteq \mathfrak{D}G \supseteq \mathfrak{D}^2G \supseteq \cdots
\]

is the **derived series** of \(G\), with each group a normal subgroup in the previous (even in \(G\)). \(G\) is **solvable** if \(\mathfrak{D}^n G = 1\) for some \(n \geq 0\). Now, inductively define \(\mathcal{C}^n G := [G, \mathcal{C}^{n-1} G]\) with \(\mathcal{C}^0 G = G\).

\[
G \supseteq \mathcal{C}G \supseteq \mathcal{C}^2G \supseteq \cdots
\]

is the **descending central series** of \(G\), with each group normal in the previous (even in \(G\)). \(G\) is **nilpotent** if \(\mathcal{C}^n G = 1\) for some \(n \geq 0\).
Recall the following facts of group theory:

- nilpotent $\implies$ solvable
- $G$ solvable (resp. nilpotent) $\implies$ subgroups, quotients of $G$ are solvable (resp. nilpotent)
- If $N \trianglelefteq G$, then $N$ and $G/N$ solvable $\implies G$ solvable.

**Examples.**

- $B_n$ is solvable. ($\mathfrak{D}B_n = U_n$)
- $U_n$ is nilpotent.

### 1.4 $G$-spaces.

A $G$-space is a variety $X$ with an action of $G$ on $X$ (as a set) such that $G \times X \to X$ is a morphism of varieties. For each $x \in X$ we have a morphism $f_x : G \to X$ be given by $g \mapsto gx$, and for each $g \in G$ we have an isomorphism $t_g : X \to X$ given by $x \mapsto gx$. $\text{Stab}_G(x) = f_x^{-1}(\{x\})$ is a closed subgroup.

**Examples.**

- $G$ acts on itself by $g \ast x = gx$ or $xg^{-1}$ or $gxg^{-1}$. (Note that in the case of the last action, $\text{Stab}(x) = Z_G(x)$ is closed and so the center $Z_G = \bigcap_{x \in G} Z_G(x)$ is closed.)
- $\text{GL}(V) \times V \to V$, $(g, x) \mapsto g(x)$
- $\text{GL}(V) \times PV \to PV$ (exercise)

**Proposition 24.**

(a) Orbits are locally closed (so each orbit is a subvariety and is itself a $G$-space).

(b) There exists a closed orbit.

**Proof.**

(a). Let $Gx$ be an orbit, which is the image of $f_x$. By Chevalley, there is a nonempty $U \subset Gx$ open in $\overline{Gx}$. Then $Gx = \bigcup_{g \in G} gU$ is open in $\overline{Gx}$.

(b). Since $X$ is noetherian, we can choose an orbit $Gx$ such that $\overline{Gx}$ is minimal (with respect to inclusion). We will show that $Gx$ is closed. Suppose otherwise. Then $\overline{Gx} - Gx$ is nonempty, closed in $\overline{Gx}$ by (a), and $G$-stable (by the usual argument); let $y$ be an element in the difference. But then $\overline{Gy} \subset \overline{Gx}$. Contradiction. Hence, $Gx$ is closed.

**Lemma 25.** If $G$ is irreducible, then $G$ preserves all irreducible components of $X$.

**Exercise.**

Suppose $\theta : G \times X \to X$ gives an affine $G$-space. Then $G$ acts linearly on $k[X]$ by

$$(g \cdot f)(x) := f(g^{-1}x), \quad \text{i.e.,} \quad g \cdot f = t_{g^{-1}}^*(f)$$
**Definitions 26.** Suppose a group $G$ acts linearly on a vector space $W$. Say the action is **locally finite** if $W$ is the union of finite-dimensional $G$-stable subspaces. If $G$ is an algebraic group, say the action is **locally algebraic** if it is locally finite and, for any finite-dimensional $G$-stable subspace $V$, the action $\theta : G \times V \to V$ is a morphism.

**Proposition 27.** The action of $G$ on $k[X]$ is locally algebraic. Moreover, for all finite-dimensional $G$-stable $V \subset k[X]$, then $\theta^*(V) \subset k[G] \otimes V$.

**Proof.** $t_{g^{-1}}$ factors as

$$t_{g^{-1}} : X \to G \times X \xrightarrow{\theta} X \quad x \mapsto (g^{-1}, x)$$

$$t_{g^{-1}}^* : k[X] \xrightarrow{\theta^*} k[G] \otimes k[X] \xrightarrow{(ev_{g^{-1}, \text{id}})} k[X]$$

Fix $f \in k[X]$ and write $\theta^*(f) = \sum_{i=1}^n h_i \otimes f_i$, so

$$g \cdot f = t_{g^{-1}}^*(f) = \sum_{i=1}^n h_i(g^{-1})f_i$$

Hence, the $G$-orbit of $f$ is contained in $\sum_{i=1}^n kf_i$, implying local finiteness.

Let $V \subset k[X]$ be finite-dimensional and $G$-stable, and pick basis $(e_i)_{i=1}^n$. Extend the $e_i$ to a basis $\{e_i\}_i \cup \{e'_\alpha\}_\alpha$ of $k[X]$. Write

$$\theta^* e_i = \sum_j h_{ij} \otimes e_j + \sum_{\alpha} h'_{i\alpha} \otimes e'_\alpha$$

$$\Rightarrow g \cdot e_i = \sum_j h_{ij}(g^{-1})e_j + \sum_{\alpha} h'_{i\alpha}(g^{-1})e'_\alpha \in V$$

$$\Rightarrow h'_{i\alpha}(g^{-1}) = 0 \quad \forall \ g, i, \alpha$$

$$\Rightarrow h'_{i\alpha} = 0 \quad \forall \ i, \alpha$$

Hence, $\theta^*(V) \subset k[G] \otimes V$. Moreover, we see that $G \times V \to V$ is a morphism, as it is given by

$$(g, \sum_i \lambda_i e_i) \mapsto \sum_{i,j} \lambda_j h_{ij}(g^{-1})e_j$$

It follows that the action of $G$ on $k[X]$ is locally algebraic.

**Theorem 28** (Analogue of Cayley’s Theorem). Any algebraic group is isomorphic to a closed subgroup of some $GL_n$.

**Proof.** $G$ acts on itself by right translation, so $(g \cdot f)(\gamma) = f(\gamma g)$. By Proposition 27 we know that this gives a locally algebraic action on $k[G]$. Let $f_1, \ldots, f_n$ be generators of $k[G]$. Without loss of generality, the $f_i$ are linearly independent and $V = \sum_{i=1}^n kf_i$ is $G$-stable. Write

$$g \cdot f_i = \sum_j h_{ji}(g^{-1})f_j = \sum_j h'_{ji}(g)f_j$$
where $h_{ji} \in k[G]$ and $h'_{ji} : g \mapsto h_{ji}(g^{-1})$. It follows that $\phi : G \to \text{GL}(V)$ given by $g \mapsto (h'_{ij}(g))$ is a morphism of algebraic groups. It remains to show that $\phi$ is a closed immersion.

We have $h'_{ij} \in \text{im} \phi^*$ for all $i, j$, as they are the image of projections. Moreover,$$
f_i(g) = (g \cdot f_i)(e) = \sum_j h'_{ji}(g) f_j(e) \implies f_i \in \sum_j k h'_{ji} \subset \text{im} \phi^*
$$

Since the $f_i$ generate $k[G]$, it follows that $\phi^*$ is surjective; that is, $\phi$ is a closed immersion. \qed

1.5 Jordan Decomposition.

Let $V$ be a finite-dimensional $k$-vector space. $\alpha \in \text{GL}(V)$ is semisimple if it is diagonalisable, and is unipotent if $1$ is its only eigenvalue. If $\alpha, \beta$ commute then

$$\alpha \text{ and } \beta \text{ semisimple (resp. unipotent) } \implies \alpha \beta \text{ semisimple (resp. unipotent) }$$

Proposition 29. $\alpha \in \text{GL}(V)$

(i) $\exists! \alpha_s$ (semisimple), $\alpha_u$ (unipotent) $\in \text{GL}(V)$ such that $\alpha = \alpha_s \alpha_u = \alpha_u \alpha_s$.

(ii) $\exists p_s(x), p_u(x) \in k[X]$ such that $\alpha_s = p_s(\alpha), \alpha_u = p_u(\alpha)$.

(iii) If $W \subset V$ is an $\alpha$-stable subspace, then

$$\begin{align*}
(\alpha|_W)_s &= \alpha_s|_W, \quad (\alpha|_{V/W})_s = \alpha_s|_{V/W} \\
(\alpha|_W)_u &= \alpha_u|_W, \quad (\alpha|_{V/W})_u = \alpha_u|_{V/W}
\end{align*}$$

(iv) If $f : V_1 \to V_2$ linear with $\alpha_i \in \text{GL}(V_i)$ for $i = 1, 2$, then

$$f \circ \alpha_1 = \alpha_2 \circ f \implies \begin{cases} f \circ (\alpha_1)_s = (\alpha_2)_s \circ f \\ f \circ (\alpha_1)_u = (\alpha_2)_u \circ f \end{cases}$$

(v) If $\alpha_i \in \text{GL}(V_i)$ for $i = 1, 2$, then

$$\begin{align*}
(\alpha_1 \otimes \alpha_2)_s &= (\alpha_1)_s \otimes (\alpha_2)_s \\
(\alpha_1 \otimes \alpha_2)_u &= (\alpha_1)_u \otimes (\alpha_2)_u
\end{align*}$$

Proof sketch.

(i) – existence:

A Jordan block for an eigenvalue $\lambda$ decomposes as

$$\begin{pmatrix}
\lambda & 1 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
& & & \lambda
\end{pmatrix} = \begin{pmatrix}
\lambda \\
& \ddots \\
& & \ddots \\
& & & \lambda
\end{pmatrix} \begin{pmatrix}
1 & \lambda^{-1} \\
& \ddots & \ddots \\
& & \ddots & \lambda^{-1} \\
& & & 1
\end{pmatrix}$$
The left factor is semisimple and the right is unipotent, and so they both commute.

(i) – uniqueness:
If \( \alpha = \alpha_a \alpha_u = \alpha'_a \alpha'_u \), then \( \alpha_a^{-1} \alpha'_a = \alpha_u^{-1} \alpha'_u \) is both unipotent and semisimple, and thus is the identity.

(ii): This follows from the Chinese Remainder Theorem.

(iii): Use (ii) + uniqueness.

(iv): Since \( \exists \) \( \alpha \), (ii) – uniqueness:
If \( \alpha = \alpha_a \alpha_u \), then \( \alpha_a^{-1} \alpha'_a = \alpha_u^{-1} \alpha'_u \) is both unipotent and semisimple, and thus is the identity.

Proposition 32. Suppose that for all algebraic \( V \), there is a \( \alpha_V \in GL(V) \) such that

(i) \( \alpha_{k_0} = id_{k_0} \), where \( k_0 \) is the one-dimensional trivial representation.

(ii) \( \alpha_V \otimes_W = \alpha_V \otimes \alpha_W \)

(iii) If \( f : V \to W \) is a map of \( G \)-representations, then \( \alpha_W \circ f = f \circ \alpha_V \).

Then \( \exists ! g \in G \) such that \( \alpha_V = g_V \) for all \( V \).

Proof. From (iii), if \( W \hookrightarrow V \) is a \( G \)-stable subspace, then \( \alpha_V|_W = \alpha_W \). If \( V \) is a local algebraic \( G \)-representation, then \( \exists ! \alpha_V \) such that \( \alpha_V|_W = \alpha_W \) for all finite-dimensional \( G \)-stable \( W \subset V \). Note that (ii), (iii) still hold for locally algebraic representations. Also note that from (iii) it follows that \( \alpha_V \otimes_W = \alpha_V \otimes \alpha_W \). Define \( \alpha = \alpha_{k[G]} \in GL(k[G]) \), where \( G \) acts on \( k[G] \) by \( (gf)(\lambda) = f(\lambda g) \).

Claim. \( \alpha \) is a ring automorphism.

\[ m : k[G] \otimes k[G] \to k[G] \] is a map of locally algebraic \( G \)-representations: \( f_1(\cdot g)f_2(\cdot g) = (f_1f_2)(\cdot g) \).
Thus, by (ii) and (iii), \( \alpha \circ m = m \circ (\alpha \otimes \alpha) \), and so \( \alpha(f_1f_2) = \alpha(f_1)\alpha(f_2) \).

Definition 30. An (algebraic) \( G \)-representation is a linear \( G \)-action on a finite-dimension \( k \)-vector space such that \( G \times V \to V \) is a morphism of varieties, which is equivalent to \( G \to GL(V) \) being a morphism of algebraic groups. Note that if \( G \to GL(V) \) is given by \( g \mapsto (h_{ij}(g)) \), then \( G \times V \to V \) is given by \( (g, \sum_i \lambda_i e_i) \mapsto \sum_i \lambda_i h_{ij}(g)e_j \).

Lemma 31. Suppose \( \rho : G \to GL(V) \) is an algebraic representation. Then there is a unique \( G \)-linear map \( \eta : V \to V \otimes k[G] \) such that \( (1 \otimes \text{ev}_g) \circ \eta = \rho(g) \) for all \( g \in G \). Moreover, \( \eta \) is injective and \( \eta \circ h = (1 \otimes h) \circ \eta \) for all \( h \in G \), i.e. as map of \( G \)-representations \( \eta : V \to V_0 \otimes k[G] \), where \( V_0 \) is \( V \) with the trivial \( G \)-action and \( G \) acts on \( k[G] \) by right translation.

Proof. Suppose \( \eta(e_i) = \sum_j f_{ij} \otimes e_j \) for some \( f_{ij} \in k[G] \). Then \( (1 \otimes \text{ev}_g) \circ \eta = \rho(g) \) for all \( g \) implies that \( f_{ij} = h_{ij} \) in the notation above, so \( \eta \) is unique, and conversely it shows that \( \eta \) exists. Moreover, \( \eta \) is injective since \( \rho(g) \) is injective.
To see that \( \eta \circ h = (1 \otimes h) \circ \eta \) holds, it suffices to check it after evaluating it at any \( v \in V \) and then applying \( 1 \otimes \text{ev}_g \) on both sides. We get equality, since \( \rho(g) \rho(h)(v) = \rho(gh)(v) \).
Therefore, the composition $k[G] \xrightarrow{\alpha} k[G] \xrightarrow{ev_g} k$ is a ring homomorphism and is equal to $ev_g$ for some unique $g$.

Claim. $\alpha(f) = gf$. \textit{i.e., } $\alpha = g k[G]$.

By above $\alpha(f)(e) = f(g)$. Also, if $\ell(\lambda)(f) := f(\lambda^{-1} \cdot)$, then $\ell(\lambda) : k[G] \rightarrow k[G]$ is $G$-linear by (iii):

$$\alpha \circ \ell(\lambda) = \ell(\lambda) \circ \alpha \implies \alpha(f)(\lambda^{-1}) = f(\lambda^{-1}g) \implies \alpha(f) = gf$$

Now if $V$ is a $G$-rep, $\eta : V \hookrightarrow V_0 \otimes k[G]$ is $G$-linear, by Lemma $31$ and so

$$\alpha_{V_0 \otimes k[G]} \circ \eta = \eta \circ \alpha_V$$

Since

$$\alpha_{V_0 \otimes k[G]} = \alpha_{V_0} \otimes \alpha_{k[G]} = id_{V_0} \otimes g_{k[G]} = g_{V_0 \otimes k[G]}$$

and

$$g_{V_0 \otimes k[G]} \circ \eta = \eta \circ g_V$$

and the fact that $\eta$ is injective, it follows that $\alpha_V = g_V$. ($g$ is unique, as $G \rightarrow GL(k[G])$ is injective. \textit{Exercise!})

\textbf{Theorem 33.} Let $G$ be an algebraic group.

(i). Fix $g \in G$. For all $G$-representations $V$, let $\alpha_V := (g_V)_s$. If $f : V \rightarrow W$ is $G$-linear, then $f \circ g_V = g_W \circ f$ implies that $f \circ \alpha_V = \alpha_W \circ f$ by Proposition $29$. Also, $\alpha_{id} = id$ and

$$\alpha_{V \otimes W} = (g_V \otimes g_W)_s = (g_V \otimes g_W)_s = \alpha_V \otimes \alpha_W$$

(the last equality following from Proposition $29$). By Proposition $32$, there is a unique $g_s \in G$ such that $\alpha_V = (g_s)_V$ for all $V$, \textit{i.e., } $\rho(g_s) = \rho(g)_s$. Similarly for $g_u$. From a closed immersion $G \hookrightarrow GL(V)$, from Theorem $28$, we see that $g = g_s g_u = g_u g_s$.

(ii). Given $\phi : G \rightarrow H$, let $\rho : H \rightarrow GL(V)$ be a closed immersion. Then

$$\rho(\phi(g_s)) = \rho(\phi(g)_s) = \rho(\phi(g)_s)$$

where the first equality is by (i) for $G$ (as $\phi \circ \rho$ makes $V$ into a $G$-representation) and the second equality is by (i) for $H$. \hfill \square

\textbf{Exercise.} What is the Jordan decomposition in $G_a$? How about in a finite group?
Remark 34. $F : (G\text{-representations}) \rightarrow (k\text{-vector spaces})$ denotes the forgetful functor, then Proposition 32 says that

$$G \cong \text{Aut}^\otimes(F)$$

where the left side is the group of natural isomorphisms $F \rightarrow F$ respecting $\otimes$. 
2. Diagonalisable and elementary unipotent groups.

2.1 Unipotent and semisimple subsets.

Definitions 35.

\[ G_s := \{ g \in G \mid g = g_s \} \]
\[ G_u := \{ g \in G \mid g = g_u \} \]

Note that \( G_s \cap G_u = \{ e \} \) and \( G_u \) is a closed subset of \( G \) (embedding \( G \) into a \( \text{GL}_n \), \( G_u \) is the closed subset consisting of \( g \) such that \( (g - I)^n = 0 \). \( G_s \), however, need not be closed (as in the case \( G = B_2 \))).

Corollary 36. If \( gh = hg \) and \( g, h \in G^* \), then \( gh, g^{-1} \in G^* \), where \( ^* = s, u \).

Proposition 37. If \( G \) is commutative, then \( G_s, G_u \) are closed subgroups and \( \mu : G_s \times G_u \to G \) is an isomorphism of algebraic groups.

Remark 38. This will be generalised to connected nilpotent groups in Proposition 131.

Proof. \( G_s, G_u \) are subgroups by Corollary 36 and \( G_u \) is closed by a remark above. Without loss of generality, \( G \subset \text{GL}(V) \) is a closed subgroup for some \( V \). As \( G \) is commutative, \( V = \bigoplus_{\lambda : G_s \to k^*} V_{\lambda} \) (a direct sum of eigenspaces for \( G_s \)) and \( G \) preserves each \( V_{\lambda} \). Hence, we can choose a basis for each \( V_{\lambda} \) such that the \( G \)-action is upper-triangular (commuting matrices are simultaneously upper-triangular-isable), and so \( G \subset B_n \) and \( G_s = G \cap D_n \). Then \( G \to B_n \) followed by projecting to the diagonal \( D_n \) gives a morphism \( G \to G_s \), \( g \mapsto g_s \); hence, \( g \mapsto (g_s, g_s^{-1}g) \) gives a morphism \( G \to G_s \times G_u \), one inverse to \( \mu \).

Definition 39. \( G \) is unipotent if \( G = G_u \).

Example. \( U_n \) is unipotent, and so is \( G_a \) (as \( G_a \cong U_2 \)).

Proposition 40. If \( G \) is unipotent and \( \phi : G \to \text{GL}_n \), then there is a \( \gamma \in \text{GL}_n \) such that \( \text{im}(\gamma \phi \gamma^{-1}) \subset U_n \).

Proof. We prove this by induction on \( n \). Suppose that this true for \( m < n \), let \( V \) be an \( n \)-dimensional vector space, and \( \phi : G \to \text{GL}(V) \). Suppose that there is a \( G \)-invariant subspace \( 0 \nsubseteq W_1 \nsubseteq V \). Let \( W_2 \) be complementary to \( W_1 \), so that \( V = W_1 \oplus W_2 \), and let \( \phi_i : G \to \text{GL}(V_i) \) be the induced morphisms for \( i = 1, 2 \), so that \( \phi = \phi_1 \oplus \phi_2 \). Since \( n > \dim W_1, \dim W_2 \), there are \( \gamma_1, \gamma_2 \in \text{GL}(V) \)
such that $\text{im} \ (\gamma_i \phi_i \gamma_i^{-1})$ consists of unipotent elements for $i = 1, 2$. If $\gamma = \gamma_1 \oplus \gamma_2$, then it follows that $\text{im} \ (\gamma \phi \gamma^{-1})$ consists of unipotent elements as well.

Now, suppose that there does not exist such a $W_1$, so that $V$ is irreducible. For $g \in G$

$$\text{tr} \ (\phi(g)) = n \implies \forall h \in G \quad \text{tr} \ ((\phi(g) - 1)\phi(h)) = \text{tr} \ (\phi(gh)) - \text{tr} \ (\phi(h)) = n - n = 0$$

$$\implies \forall x \in \text{End}(V) \quad \text{tr} \ ((\phi(g) - 1)x) = 0, \text{ by Burnside's theorem}$$

$$\implies \phi(g) - 1 = 0$$

$$\implies \phi(g) = 1$$

$$\implies \text{im} \phi = 1$$

(Recall that Burnside’s Theorem says that $G$ spans $\text{End}(V)$ as a vector space.)

**Remark 41.** Here’s a sketch proof of Burnside’s theorem, which works for any abstract subgroup $G$ of $\text{GL}(V)$ even: let $A$ be the $k$-span of $G$ inside $\text{End}(V)$. This is a $k$-subalgebra of $\text{End}(V)$ acting irreducibly on $V$.

We’ll prove more generally that any (possibly non-commutative) $k$-algebra $A$ with $\dim_k A < n^2$ cannot have an irreducible module of $k$-dimension $n$. By replacing $A$ by $A/\text{rad}(A)$, where $\text{rad}(A)$ is the Jacobson radical of $A$, we may assume WLOG that $A$ is semisimple. Then $A \cong \prod_{i=1}^r M_{n_i}(k)$ by the Artin-Wedderburn theorem (since $k$ is algebraically closed!). Now the irreducible modules of this ring are precisely the modules $k^{n_i}$ with $A$ acting naturally via the $i$-th projection. Hence any irreducible module has dimension $n_i \leq \sqrt{\dim_k A} < n$.

**Corollary 42.** Any irreducible representation of a unipotent group is trivial.

**Corollary 43.** Any unipotent $G$ is nilpotent.

**Proof.** $U_n$ is nilpotent.

**Remark 44.** The converse is not true; any torus is nilpotent (the definition of a torus to come immediately). More generally we will see that any connected nilpotent group is a product of a torus and a connected unipotent group.

### 2.2 Diagonalisable groups and tori.

**Definitions 45.** $G$ is **diagonalisable** if $G$ is isomorphic to a closed subgroup of $D_n \cong G_m^n (n \geq 0)$. $G$ is a **torus** if $G \cong D_n$ ($n \geq 0$). The **character group** of $G$ is

$$X^*(G) := \text{Hom}(G, G_m) \quad (\text{morphisms of algebraic groups})$$

It is an abelian group under multiplication $((\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g))$ and is a subgroup of $k[G]^\times$.

Recall the following result:

**Proposition 46** (Dedekind). Suppose $X^*(G)$ is a linearly independent subset of $k[G]$.

The proof shows in fact that characters are linearly independent for any (abstract) group.
Proof. Suppose that \( \sum_{i=1}^{n} \lambda_i \chi_i = 0 \) in \( k[G] \), \( \lambda_i \in k \). Without loss of generality, \( n \geq 2 \) is minimal among all possible nontrivial linear combinations (so that \( \lambda_i \neq 0 \ \forall i \)). Then

\[
\forall g, h, \quad \begin{cases} 0 = \sum \lambda_i \chi_i(g) \chi_i(h) \\ 0 = \sum \lambda_i \chi_i(g) \chi_n(h) \end{cases} \implies \forall h, \quad 0 = \sum_{i=1}^{n-1} \lambda_i [\chi_i(h) - \chi_n(h)] \chi_i
\]

By the minimality of \( n \), we must have that the coefficients are all 0; that is, \( \forall i, h \ \chi_i(h) = \chi_n(h) \implies \chi_i = \chi_n \). We still arrive at a contradiction. \( \square \)

**Proposition 47.** The following are equivalent:

(i) \( G \) is diagonalisable.

(ii) \( X^*(G) \) is a basis of \( k[G] \) and \( X^*(G) \) is finitely-generated.

(iii) \( G \) is commutative and \( G = G_s \).

(iv) Any \( G \)-representation is a direct sum of 1-dimensional representations.

**Proof.**

(i) \( \Rightarrow \) (ii): Fix an embedding \( G \hookrightarrow D_n \). \( k[D_n] = k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] \) – as seen from restricting \( T_{ij}, \det(T_{ij})^{-1} \in k[GL_n] \) – has a basis of monomials \( T_1^{a_1} \cdots T_n^{a_n} \), \( a_i \in \mathbb{Z} \), each of which is in \( X^*(G) \):

\[
\text{diag}(x_1, \ldots, x_n) \mapsto x_1^{a_1} \cdots x_n^{a_n}
\]

Hence, \( X^*(D_n) \cong \mathbb{Z}^n \) (by Proposition 46). The closed immersion \( G \hookrightarrow D_n \) gives a surjection \( k[D^n] \twoheadrightarrow k[G] \), inducing a map \( X^*(D_n) \twoheadrightarrow X^*(G) \). \( \chi \mapsto \chi|_G \). \( \text{im} (X^*(D_n) \twoheadrightarrow X^*(G)) \) spans \( k[G] \) and is contained in \( X^*(G) \), which is linearly independent. Hence, \( X^*(G) \) is a basis of \( k[G] \) and we have the surjection

\[
\mathbb{Z}^n \cong X^*(D_n) \twoheadrightarrow X^*(G)
\]

implying the finite-generation.

(ii) \( \Rightarrow \) (iii): Say \( \chi_1, \ldots, \chi_n \) are generators of \( X^*(G) \). Define a morphism \( \phi : G \to GL_n \) by \( g \mapsto \text{diag}(\chi_1(g), \ldots, \chi_n(g)) \).

\[
g \in \ker \phi \implies \chi_i(g) = 1 \ \forall i \\
\implies \chi(g) = 1 \ \forall \chi \in X^*(G) \\
\implies f(g) = 0 \ \forall f \in M_e = \{ g = \sum_{\chi} \lambda_{\chi} \chi \in k[X] \mid 0 = g(e) = \sum_{\chi} \lambda_{\chi} \}
\]

\[
\implies M_e \subseteq M_g \\
\implies M_e = M_g \\
\implies g = e
\]

So \( \phi \) is injective, which implies that \( G \) is commutative and \( G = G_s \).
(iii) ⇒ (iv): Let \( \phi : G \to \text{GL}_n \) be a representation. \( \text{im} \phi \) is a commuting set of diagonalisable elements, which means we can simultaneously diagonalise them.

(iv) ⇒ (i): Pick \( \phi : G \hookrightarrow \text{GL}_n \) (Theorem 28). By (iii), without loss of generality, suppose that \( \text{im} \phi \subset D_n \). Hence, \( \phi : G \hookrightarrow D_n \).

**Corollary 48.** Subgroups and images under morphisms of diagonalisable groups are diagonalisable.

**Proof.** (iii).

**Observations:**

- \( \text{char } k = p \implies X^*(G) \) has no \( p \)-torsion.
- \( k[G] \cong k[X^*(G)] \) as algebras (\( k[X^*(G)] \) being a group algebra).
- For \( \chi \in X^*(G) \),
  \[
  \Delta(\chi) = \chi \otimes \chi, \quad i(\chi) = \chi^{-1}, \quad \epsilon(\chi) = 1
  \]
  Indeed,
  
  \[
  \Delta(\chi)(g_1, g_2) = \chi(g_1g_2) = \chi(g_1)\chi(g_2) = (\chi \otimes \chi)(g_1, g_2)
  \]
  
  \[
  i(\chi)(g) = \chi(g^{-1}) = \chi(g)^{-1} = \chi^{-1}(g)
  \]
  
  \[
  \epsilon(\chi) = \chi(e) = 1
  \]

**Theorem 49.** Let \( p = \text{char } k \).

\[
\begin{array}{ccc}
\text{(diagonalisable algebraic groups)} & \xrightarrow{X^*} & \text{(finitely-generated abelian groups (with no } p \text{-torsion if } p > 0) } \\
\downarrow & & \uparrow \\
G & \longrightarrow & X^*(G) \\
\downarrow & & \uparrow \\
H & \longrightarrow & X^*(H)
\end{array}
\]

is a (contravariant) equivalence of categories.

**Proof.** It is well-defined by the above. We will define an inverse functor \( F \). Given \( X \cong \mathbb{Z} \bigoplus \bigoplus_{i=1}^s \mathbb{Z}/n_i \mathbb{Z} \) from the category on the right, we have that its group algebra \( k[X] \) is finitely-generated and reduced:

\[
k[X] \cong k[\mathbb{Z}] \otimes \bigotimes_{i=1}^s k[\mathbb{Z}/n_i \mathbb{Z}] \cong k[T^{\pm 1}] \otimes \bigotimes_{i=1}^s k[T]/(T^{n_i} - 1)
\]

Moreover, \( k[X] \) is a Hopf algebra, which is easily checked, defining

\[
\Delta : e_x \mapsto e_x \otimes e_x, \quad i : e_x \mapsto e_{x^{-1}} = e_x^{-1}, \quad \epsilon : e_x \mapsto 1
\]

where \( X \) has been written multiplicatively and \( k[X] = \bigoplus_{x \in X} ke_x \). Define \( F \) by \( F(X) = m\text{-Spec}(k[X]) \).

Above, we saw that \( FX^*(G)) \cong G \) as algebraic groups.
\[ X^*(F(X)) = \text{Hom}(F(X), \mathbb{G}_m) \]
\[ = \text{Hom}_{\text{Hopf-alg}}(k[T, T^{-1}], k[X]) \]
\[ = \{ \lambda \in k[X] \times \text{(corresponding to the images of } T) \mid \Delta(\lambda) = \lambda \otimes \lambda \} \]

For an element above, write \( \lambda = \sum_{x \in X} \lambda_x e_x \) (almost all of the \( \lambda_x \in k \) of course being zero). Then
\[ \Delta(\lambda) = \sum_x \lambda_x (e_x \otimes e_x) \quad \text{and} \quad \lambda \otimes \lambda = \sum_{x, x'} \lambda_x \lambda_{x'} (e_x \otimes e'_x) \]

Hence,
\[ \lambda_x \lambda_{x'} = \begin{cases} 
\lambda_x, & x = x' \\
0, & x \neq x'
\end{cases} \]

So, \( \lambda_x \neq 0 \) for an unique \( x \in X \), and
\[ \lambda_x^2 = \lambda \implies \lambda_x = 1 \implies \lambda = e_x \in X \]

Thus we have \( X^*(F(X)) \cong X \) as abelian groups. The two functors are inverse on maps as well, as is easily checked.

\( \blacksquare \)

**Corollary 50.**

(i) The diagonalisable groups are the groups \( \mathbb{G}_m^r \times H \), where \( H \) is a finite group of order prime to \( p \).

(ii) For a diagonalisable group \( G \),

\( G \) is a torus \( \iff \) \( G \) is connected \( \iff \) \( X^*(G) \) is free abelian

**Proof.** Define \( \mu_n := \ker(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m) \), which is diagonalisable. If \( (n, p) = 1 \), then \( k[\mu_n] = k[T]/(T^n - 1) \) \( (T^n - 1) \) is separable) and \( X^*(\mu_n) \cong \mathbb{Z}/n\mathbb{Z} \). Since \( X^*(\mathbb{G}_m) \cong \mathbb{Z} \) and \( X^*(G \times H) \cong X^*(G) \oplus X^*(H) \), the result follows from Theorem [49].

**Corollary 51.** \( \text{Aut}(D_n) \cong \text{GL}_n(\mathbb{Z}) \)

**Fact/Exercise.** If \( G \) is diagonalisable, then
\[ G \times X^*(G) \to \mathbb{G}_m, \quad (g, \chi) \mapsto \chi(g) \]

is a “perfect bilinear pairing”, i.e., it induces isomorphisms \( X^*(G) \xrightarrow{\sim} \text{Hom}(G, \mathbb{G}_m) \) and \( G \xrightarrow{\sim} \text{Hom}_\mathbb{Z}(X^*(G), \mathbb{G}_m) \) (as abelian groups). Moreover, it induces inverse bijections
\[ \{ \text{closed subgroups of } G \} \leftrightarrow \{ \text{subgroups } Y \text{ of } X^*(G) \text{ such that } X^*(G)/Y \text{ has no } p\text{-torsion} \} \]
\[ H \mapsto H^\perp \]
\[ Y^\perp \leftrightarrow Y \]

\( \blacksquare \)
Fact. Say
\[ 1 \to G_1 \to G_2 \to G_3 \to 1 \]
is exact if the sequence is set-theoretically exact and the induced sequence of lie algebras
\[ 0 \to \text{Lie} G_1 \to \text{Lie} G_2 \to \text{Lie} G_3 \to 0 \]
is exact. (See Definition 92.) Suppose the \( G_i \) are diagonalisable, so that \( \text{Lie} G_i \cong \text{Hom}_{\mathbb{Z}}(X^*(G_i), k) \). Then the sequence of the \( G_i \) is exact if and only if
\[ 0 \to X^*(G_3) \to X^*(G_2) \to X^*(G_1) \to 0 \]

Remark 52.
\[ 1 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 1 \]
is set-theoretically exact, but
\[ 0 \to X^*(\mathbb{G}_m) \xrightarrow{p} X^*(\mathbb{G}_m) \to X^*(\mu_p) \to 0 \]
is not if \( \text{char} \, k = p \) (in which case \( X^*(\mu_p) = 0 \)).

Definition. The group of cocharacters of \( G \) are
\[ X_*(G) := \text{Hom}(\mathbb{G}_m, G) \]
If \( G \) is abelian, then \( X_*(G) \) is an abelian group.

Proposition 53. If \( T \) is a torus, then \( X_*(T), X^*(T) \) are free abelian and
\[ X^*(T) \times X_*(T) \to \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}, \quad (\chi, \lambda) \mapsto \chi \circ \lambda \]
is a perfect pairing.

Proof.
\[ X_*(T) = \text{Hom}(\mathbb{G}_m, T) \cong \text{Hom}(X^*(T), \mathbb{Z}). \]
The isomorphism follows from Theorem 49. Since \( X^*(T) \) is finitely-generated free abelian by Corollary 50 we have that \( X_*(T) \cong \text{Hom}(X^*(T), \mathbb{Z}) \) is free abelian as well. Moreover, since
\[ \text{Hom}(X, \mathbb{Z}) \times X \to \mathbb{Z}, \quad (\alpha, x) \mapsto \alpha(x) \]
is a perfect pairing for any finitely-generated free abelian \( X \), it follows from the isomorphism above that the pairing in question is also perfect.

Proposition 54 (Rigidity of diagonalisable groups). Let \( G, H \) be diagonalisable groups and \( V \) a connected affine variety. If \( \phi : G \times V \to H \) is a morphism of varieties such that \( \phi_v : G \to H \), \( g \mapsto \phi(g, v) \) is a morphism of algebraic groups for all \( v \in V \), then \( \phi_v \) is independent of \( v \).
Proof. Under $\phi^* : k[H] \to k[G] \otimes k[V]$, for $\chi \in X^*(H)$, write
\[ \phi^*(\chi) = \sum_{\chi' \in X^*(G)} \chi' \otimes f_{\chi\chi'} \]
Then
\[ \phi^*_v(\chi) = \sum_{\chi'} f_{\chi\chi'}(v) \chi \in X^*(G) \quad \Rightarrow \quad \forall \chi', v \quad f_{\chi\chi'}(v) \in \{0, 1\} \]
\[ \Rightarrow \quad \forall \chi' \quad f_{\chi\chi'}^2 = f_{\chi\chi'} \]
\[ \Rightarrow \quad \forall \chi' \quad V = V(f_{\chi\chi'}) \sqcup V(1 - f_{\chi\chi'}) \]
\[ \Rightarrow \quad \forall \chi' \quad f_{\chi\chi'} \text{ is constant, since } V \text{ is connected} \]
\[ \Rightarrow \quad \forall \phi_0 \text{ is independent of } v \]

Corollary 55. Suppose that $H \subset G$ is a closed diagonalisable subgroup. Then $N_G(H)^0 = Z_G(H)^0$ and $N_G(H)/Z_G(H)$ is finite. ($N_G(H), Z_G(H)$ are easily seen to be closed subgroups.)

Proof. Applying the above proposition to the morphism
\[ H \times N_G(H)^0 \to H, \quad (h, n) \mapsto nhn^{-1} \]
we get that $nhn^{-1} = h$ for all $h, n$. Hence
\[ N_G(H)^0 \subset Z_G(H) \subset N_G(H) \]
and the corollary immediately follows.

2.3 Elementary unipotent groups.

Define $A(G) := \text{Hom}(G, G_\alpha)$, which is an abelian group under addition of maps; actually, it is an $R$-module, where $R = \text{End}(G_\alpha)$. Note that $A(G_\alpha^n) \cong R^n$. $R = \text{End}(G_\alpha)$ can be identified with
\[ \{ f \in k[G_\alpha] = k[x] \mid f(x + y) = f(x) + f(y) \text{ in } k[x,y] \} = \begin{cases} \{ \lambda x \mid \lambda \in k \} , & \text{char } k = p = 0 \\ \{ \sum \lambda_i x^p^i \mid \lambda_i \in k \} , & \text{char } k = p > 0 \end{cases} \]
Accordingly,
\[ R \cong \begin{cases} k , & p = 0 \\ \text{noncommutative polynomial ring over } k , & p > 0 \end{cases} \]

Proposition 56. $G$ is an algebraic group. The following are equivalent:

(i) $G$ is isomorphic to a closed subgroup of $G_\alpha^n$ ($n \geq 0$).

(ii) $A(G)$ is a finitely-generated $R$-module and generates $k[G]$ as a $k$-algebra.
(iii) $G$ is commutative and $G = G_u$ (and $G^p = 1$ if $p > 0$).

**Definition 57.** If one of the above conditions holds, then $G$ is **elementary unipotent**. Note that (iii) rules out $\mathbb{Z}/p^n\mathbb{Z}$ as elementary unipotent when $n > 1$.

**Theorem 58.**

( elementary unipotent groups ) $\xrightarrow{\mathcal{A}}$ ( finitely-generated $R$-modules )

is an equivalence of categories.

**Proof.** For the inverse functor, see Springer 14.3.6. \qed

**Corollary 59.**

(i) The elementary unipotent groups are $G^n_a$ if $p = 0$, and $G^n_a \times (\mathbb{Z}/p\mathbb{Z})^s$ if $p > 0$

(ii) For an elementary unipotent group $G$,

$$G$$ is isomorphic to a $G^n_a$ if $G$ is connected $\iff A(G)$ is free

**Theorem 60.** Suppose $G$ is a connected algebraic group of dimension 1, then $G \cong G_a$ or $G_m$.

**Proof.**

**Claim:** $G$ is commutative.

Fix $\gamma \in G$ and consider $\phi : G \to G$ given by $g \mapsto g\gamma g^{-1}$. Then $\phi(G)$ is irreducible and closed, which implies that $\phi(G) = \{ \gamma \}$ or $\phi(G) = G$. Now, either $\phi(G) = \{ \gamma \}$ for all $\gamma \in G$, in which case $G$ is commutative and the claim is true, or $\phi(G) = G$ for at least one $\gamma$. Suppose the second case holds with a particular $\gamma$ and fix an embedding $G \hookrightarrow \text{GL}_n$. Consider the morphism $\psi : G \to \mathbb{A}^{n+1}$ which takes $g$ to the coefficients of the characteristic polynomial of $g$, $\det(T \cdot \text{id} - g)$. $\psi$ is constant on the conjugacy class $\phi(G)$, implying that $\psi$ is constant. Hence, every $g \in G$, $e$ included, has the same characteristic polynomial: $(T - 1)^n$. Thus

$$G = G_u \implies G$$ is nilpotent $\implies G \supseteq [G,G] \implies [G,G] = 1 \implies G$$ is commutative

Now, by Proposition 57

$$G \cong G_s \times G_u \implies G = G_s \text{ or } G = G_u$$

as dimension is additive. In the former case, $G \cong G_m$ by Corollary 50. In the latter, if we can prove that $G$ is elementary unipotent, then $G \cong G_a$ by Corollary 59; we must show that $G^p = 1$ when $p > 0$ by Proposition 56. Suppose that $G^p \neq 1$, so that $G^p = G$. Then $G = G^p = G^{p^2} = \ldots$.

But $(g - 1)^n = 0$ in $\text{GL}_n$ and so for $p^r \geq n$,

$$0 = (g - 1)^{p^r} = g^{p^r} - 1 \implies g^{p^r} = 1 \implies \{ e \} = G^{p^r} = G$$

which is a contradiction. \qed
3. Lie algebras.

If $X$ is a variety and $x \in X$, then the local ring at $x$ is

$$
O_{X,x} := \lim_{\substack{U \text{ open} \\ U \ni x}} O_X(U) = \text{germs of functions at } x = \{(f,U) \mid f \in O_X(U)\} / \sim
$$

where $(f,U) \sim (f',U')$ if there is an open neighbourhood $V \subset U \cap U'$ of $x$ for which $f|_V = f'|_V$. There is a well-defined ring morphism $ev_x : O_{X,x} \to k$ given by evaluating at $x$: $[(f,U)] \mapsto f(x)$. $O_{X,x}$ is a local ring (hence the name) with unique maximal ideal $m_x =: \ker ev_x = \{(f,U) \mid f(x) = 0\}$

for if $f \notin m_x$, then $f^{-1}$ is defined near $x$, implying that $f \in O_{X,x}^\times$.

**Fact.** If $X$ is affine and $x$ corresponds to the maximal ideal $m \subset k[X]$ (via Nullstellensatz), then $O_{X,x} \cong k[X]/m$. By choosing an affine chart in $X$ at $x$, we see in general that $O_{X,x}$ is noetherian.

3.1 Tangent Spaces.

Analogous to the case of manifolds, the tangent space to a variety $X$ at a point $x$ is

$$
T_xX := \text{Der}_k(O_{X,x},k) = \{\delta : O_{X,x} \to k \mid \delta \text{ is } k\text{-linear, } \delta(fg) = f(x)\delta(g) + g(x)\delta(f)\}
$$

(so $k$ is viewed as a $O_{X,x}$-module via $ev_x$.) $T_xX$ is a $k$-vector space.

**Lemma 61.** Let $A$ be a $k$-algebra, $\epsilon : A \to k$ a $k$-algebra morphism, and $m = \ker \epsilon$. Then

$$
\text{Der}_k(A,k) \cong (m/m^2)^*, \quad \delta \mapsto \delta|_m
$$

**Proof.** An inverse map is given by sending $\lambda$ to a derivation defined by $x \mapsto \begin{cases} 0, & x = 1 \\ \lambda(x), & x \in m \end{cases}$. Checking this is an exercise.

Hence, $T_xX \cong (m_x/m_x^2)^*$ is finite-dimensional.

**Examples.**

- If $X = \mathbb{A}^n$, then $T_xX$ has basis

$$
\left. \frac{\partial}{\partial x_1} \right|_x, \ldots, \left. \frac{\partial}{\partial x_n} \right|_x
$$
For a finite-dimensional $k$-vector space $V$, $T_x(V) \cong V$.

**Definition 62.** $X$ is smooth at $x$ if $\dim T_x X = \dim X$. Moreover, $X$ is smooth if it is smooth at every point. From the above example, we see that $\mathbb{A}^n$ is smooth.

If $\phi : X \to Y$ we get $\phi^* : \mathcal{O}_Y,\phi(x) \to \mathcal{O}_{X,x}$ and hence

$$d\phi : T_x X \to T_{\phi(x)} Y, \quad \delta \mapsto \delta \circ \phi^*$$

**Remark 63.** If $U \subset X$ is an open neighbourhood of $x$, then $d(U \hookrightarrow X) : T_x U \sim \to T_x X$. More generally, if $X \subset Y$ is a locally closed subvariety, then $T_x X$ embeds into $T_x Y$.

**Theorem 64.**

$$\dim T_x X \geq \dim X$$

with equality holding for all $x$ in some open dense subset.

Note that if $X$ is affine and $x$ corresponds to $m \subset k[X]$, then the natural map $k[X] \to k[X]_m = \mathcal{O}_{X,x}$ induces an isomorphism

$$T_x X \cong \text{Der}_k(k[X], k), \quad (k \text{ being viewed as a } k[X]-\text{modules via } ev_x)$$

which is isomorphic to $(m/m^2)^*$ by Lemma 61. So, we can work without localising.

**Remark 65.** If $G$ is an algebraic group, then $G$ is smooth by Theorem 64 since

$$d(\ell_g : x \mapsto gx) : T_{\gamma} G \sim \to T_{\gamma\gamma} G$$

The same holds for homogeneous $G$-spaces (i.e., $G$-spaces for which the $G$-action is transitive).

### 3.2 Lie algebras.

**Definition 66.** A Lie algebra is a $k$-vector space $L$ together with a bilinear map $[,] : L \times L \to L$ such that

(i) $[x, x] = 0 \quad \forall x \in L$ \quad (\implies [x, y] = -[y, x])

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$

**Examples.**

- If $A$ is an associative $k$-algebra (maybe non-unital), then $[a, b] := ab - ba$ gives $A$ the structure of a Lie algebra.
- Take $A = \text{End}(V)$ and as above define $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$.
- For $L$ an arbitrary $k$-vector space, define $[,] = 0$. When $[,] = 0$ a Lie algebra is said to be abelian.

We will construct a functor

$$(\text{algebraic groups}) \overset{\text{Lie}}{\longrightarrow} (\text{Lie algebras})$$
As a vector space, \( \text{Lie } G = T_e G \). \( \dim \text{Lie } G = \dim G \) by above remarks.

The following is another way to think about \( T_e G \). Recall that we can identify \( G \) with the functor
\[
R \mapsto \text{Hom}_{\text{alg}}(k[G], R) := G(R)
\]
where \( k[G] \) is a reduced finite-dimensional commutative Hopf \( k \)-algebra. The Hopf (i.e., co-group) structure on \( R \) induces a group structure on \( G(R) \), even when \( R \) is not reduced.

**Lemma 67.**
\[
\text{Lie } G \cong \ker \left( G(k[\epsilon]/(\epsilon^2)) \to G(k) \right)
\]
as abelian groups.

**Proof.** Write the algebra morphism \( \theta : k[G] \to k[\epsilon]/(\epsilon^2) \) as given by \( f \mapsto \text{ev}_e(f) + \delta(f) \cdot \epsilon \) for some \( \delta : k[G] \to k \). \( \delta \) is a derivation. \( \square \)

**Examples.**
- For \( G = \text{GL}_n \), \( G(R) = \text{GL}_n(R) \), and we have
\[
\text{Lie } G = \ker \left( \text{GL}_n(k[\epsilon]/(\epsilon^2)) \to \text{GL}_n(k) \right) = \{ I + A\epsilon \mid A \in M_n(k) \} \sim M_n(k)
\]
Explicitly, the isomorphism \( \text{Lie } \text{GL}_n \to M_n(k) \) is given by \( \delta \mapsto (\partial(T_{ij})) \).
- Intrinsically, for a finite-dimensional \( k \)-vector space \( V \): Since \( \text{GL}(V) \) is an open subset of \( \text{End}(V) \), we have
\[
\text{Lie } \text{GL}(V) \sim T_I(\text{End } V) \sim \text{End } V
\]

**Definition 68.** A **left-invariant vector field** on \( G \) is an element \( D \in \text{Der}_k(k[G], k[G]) \) such that the diagram
\[
\begin{array}{ccc}
  k[G] & \xrightarrow{D} & k[G] \\
  \downarrow \Delta & & \downarrow \Delta \\
  k[G] \otimes k[G] & \xrightarrow{\text{id} \otimes D} & k[G] \otimes k[G]
\end{array}
\]
commutes.

For a fixed \( D \), for \( g \in G \), define \( \delta_g := \text{ev}_g \circ D \in T_g G \).

Evaluating \( \Delta \circ D \) at \( (g_1, g_2) \) gives \( \delta_{g_1 g_2} \)

Evaluating \( (\text{id} \otimes D) \circ \Delta \) at \( (g_1, g_2) \) gives \( \delta_{g_2} \circ \ell_{g_1}^* = d\ell_{g_1}(\delta_{g_2}) \)

Hence \( D \in \text{Der}_k(k[G], k[G]) \) being left-invariant is equivalent to \( \delta_{g_1, g_2} = d\ell_{g_1}(\delta_{g_2}) \) for all \( g_1, g_2 \in G \).

Define
\[
\mathcal{D}_G := \text{ vector space of left-invariant vector fields on } G.
\]

**Theorem 69.**
\[
\mathcal{D}_G \to \text{Lie } G, \quad D \mapsto \delta_e = \text{ev}_e \circ D
\]
is a linear isomorphism.
Proof. We shall prove that \( \delta \mapsto (\text{id} \otimes \delta) \circ \Delta \) is an inverse morphism. Fix \( \delta \in \text{Lie} G \), set \( D = (\text{id}, \delta) \circ \Delta : k[G] \to k[G] \), and check that \((\text{id}, \delta)\) is a \( k \)-derivation \( k[G] \otimes k[G] \to k[G] \), where \( k[G] \) is viewed as a \( k[G] \otimes k[G] \)-module via \( \text{id} \otimes \text{ev} \). First, we shall check that \( D \in \mathcal{D}_G \):

\[
D(fh) = (\text{id} \otimes \delta)(\Delta(fh)) \\
= (\text{id} \otimes \delta)(\Delta(f) \cdot \Delta(h)) \\
= (\text{id} \otimes \text{ev}_e)(\Delta f) \cdot (\text{id} \otimes \delta)(\Delta h) + (\text{id} \otimes \text{ev}_e)(\Delta h) \cdot (\text{id} \otimes \delta)(\Delta f) \\
= f \cdot D(h) + h \cdot D(f).
\]

Next, we show that \( D \) is left-invariant:

\[
(\text{id} \otimes D) \circ \Delta = (\text{id} \otimes ((\text{id} \otimes \delta) \circ \Delta)) \circ \Delta \\
= (\text{id} \otimes (\text{id} \otimes \delta)) \circ (\text{id} \otimes \Delta) \circ \Delta \\
= (\text{id} \otimes (\text{id} \otimes \delta)) \circ (\Delta \otimes \text{id}) \circ \Delta \quad ("\text{co-associativity}") \\
= \Delta \circ (\text{id} \otimes \delta) \circ \Delta \quad (\text{easily checked}) \\
= \Delta \circ D.
\]

Lastly, we show that the maps are inverse:

\[
\delta \mapsto (\text{id} \otimes \delta) \circ \Delta \mapsto \text{ev}_e \circ (\text{id} \otimes \delta) \circ \Delta = \delta \circ (\text{ev}_e \otimes \text{id}) \circ \Delta = \delta \\
D \mapsto \text{ev}_e \circ D \mapsto (\text{id} \otimes \text{ev}_e) \circ (\text{id} \otimes D) \circ D = (\text{id} \otimes \text{ev}_e) \circ \Delta \circ D = D.
\]

Since \( \text{Hom}_k(k[G], k[G]) \) is an associative algebra, there is a natural candidate for a Lie bracket on \( \mathcal{D}_G \subset \text{Hom}_k(k[G], k[G]) \): \([D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1\). We must check that \([\mathcal{D}_G, \mathcal{D}_G] \subset \mathcal{D}_G\). Let \( D_1, D_2 \in \mathcal{D}_G \). Since

\[
[D_1, D_2](fh) = D_1(D_2(fh)) - D_2(D_1(fh)) \\
= D_1(f \cdot D_2(h) + h \cdot D_2(f)) - D_2(f \cdot D_1(h) + h \cdot D_1(f)) \\
= D_1(f \cdot D_2(h)) + D_1(h \cdot D_2(f)) - D_2(f \cdot D_1(h)) - D_2(h \cdot D_1(f)) \\
= fD_1(D_2(h)) + D_2(h)D_1(f) + hD_1(D_2(f)) + D_2(f)D_1(h) \\
- fD_2(D_1(h)) + D_1(h)D_2(f) - hD_2(D_1(f)) + D_1(f)D_2(h) \\
= f\left(D_1(D_2(h)) - fD_2(D_1(h))\right) + h\left(D_1(D_2(f)) - hD_2(D_1(f))\right) \\
= f \cdot [D_1, D_2](h) + h \cdot [D_1, D_2](f)
\]

we have that \([D_1, D_2]\) is a derivation. Moreover,

\[
(\text{id} \otimes [D_1, D_2]) \circ \Delta = (\text{id} \otimes (D_1 \circ D_2)) \circ \Delta - (\text{id} \otimes (D_2 \circ D_1)) \circ \Delta \\
= (\text{id} \otimes D_1) \circ (\text{id} \otimes D_2) \circ \Delta - (\text{id} \otimes D_2) \circ (\text{id} \otimes D_1) \circ \Delta \\
= (\text{id} \otimes D_1) \circ \Delta \circ D_2 - (\text{id} \otimes D_2) \circ \Delta \circ D_1 \\
= \Delta \circ D_1 \circ D_2 - \Delta \circ D_2 \circ D_1 \\
= \Delta \circ [D_1, D_2]
\]

\[\square\]
and so \([D_1, D_2]\) is left-invariant. Accordingly, \([\mathcal{D}_G, \mathcal{D}_G] \subset \mathcal{D}_G\), and thus by the above theorem Lie \(G\) becomes a Lie algebra.

**Remark 70.** If \(p > 0\), then \(\mathcal{D}_G\) is also stable under \(D \mapsto D^p\) (composition with itself \(p\)-times).

**Proposition 71.** If \(\delta_1, \delta_2 \in \text{Lie } G\), then \([\delta_1, \delta_2] : k[G] \rightarrow k\) is given by \([\delta_1, \delta_2] = ((\delta_1, \delta_2) - (\delta_2, \delta_1)) \circ \Delta\).

**Proof.** Let \(D_i = (\text{id} \otimes \delta_i) \circ \Delta\) for \(i = 1, 2\). Then

\[
[\delta_1, \delta_2] = \text{ev}_e \circ [D_1, D_2] = \text{ev}_e \circ D_1 \circ D_2 - \text{ev}_e \circ D_2 \circ D_1 = \delta_1 \circ (\text{id} \otimes \delta_2) \circ \Delta - \delta_2 \circ (\text{id} \otimes \delta_1) \circ \Delta = (\delta_1 \otimes \delta_2) \circ \Delta - (\delta_2 \otimes \delta_1) \circ \Delta = ((\delta_1 \otimes \delta_2) - (\delta_2 \otimes \delta_1)) \circ \Delta.
\]

\[\square\]

**Corollary 72.** If \(\phi : G \rightarrow H\) is a morphism of algebraic groups, then \(d\phi : \text{Lie } G \rightarrow \text{Lie } H\) is a morphism of Lie algebras (i.e., brackets are preserved).

**Proof.**

\[
d\phi([\delta_1, \delta_2]) = [\delta_1, \delta_2] \circ \phi^* = (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1) \circ \Delta \circ \phi^*, \quad \text{(by the above Prop.)}
\]

\[= (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1) \circ (\phi^* \otimes \phi^*) \circ \Delta = (\delta_1 \circ \phi^*, \delta_2 \circ \phi^*) \circ \Delta - (\delta_2 \circ \phi^*, \delta_1 \circ \phi^*) \circ \Delta
\]

\[= (d\phi(\delta_1), d\phi(\delta_2)) \circ \Delta - (d\phi(\delta_2), d\phi(\delta_1)) \circ \Delta = [d\phi(\delta_1), d\phi(\delta_2)].
\]

\[\square\]

**Corollary 73.** If \(G\) is commutative, then so too is \(\text{Lie } G\) (i.e., \([\cdot, \cdot] = 0\)).

**Example.** We have that \(\phi : \text{Lie } \text{GL}_n \cong M_n(k)\) is given by \(\phi : \delta \mapsto (\delta(T_{ij}))\). Since

\[
[\delta_1, \delta_2](T_{ij}) = (\delta_1, \delta_2)(\Delta T_{ij}) - (\delta_2, \delta_1)(\Delta T_{ij})
\]

\[= \sum_{l=1}^{n} \delta_1(T_{il}) \delta_2(T_{lj}) - \sum_{l=1}^{n} \delta_2(T_{il}) \delta_1(T_{lj}) = (\phi(\delta_1) \phi(\delta_2))_{ij} - (\phi(\delta_2) \phi(\delta_1))_{ij}
\]

Hence,

\[
\phi([\delta_1, \delta_2]) = \phi(\delta_1) \phi(\delta_2) - \phi(\delta_2) \phi(\delta_1)
\]

and so in identifying \(\text{Lie } \text{GL}_n\) with \(M_n(k)\), we can also identify the Lie bracket with the usual one on \(M_n(k)\): \([A, B] = AB - BA\). Similarly, the Lie bracket on \(\text{Lie } \text{GL}(V) \cong \text{End}(V)\) can be identified with the commutator. \[\square\]
Remark 74. If \( \phi : G \to H \) is a closed immersion, then \( \phi^* \) is surjective, and so \( d\phi : \text{Lie } G \to \text{Lie } H \) is injective. Hence, if \( G \hookrightarrow \text{GL}_n \), then the above example determines \([\cdot, \cdot]\) on \( \text{Lie } G \).

Examples.
- \( \text{Lie } \text{SL}_n = \) trace 0 matrices in \( M_n(k) \)
- \( \text{Lie } B_n = \) upper-triangular matrices in \( M_n(k) \)
- \( \text{Lie } U_n = \) upper-triangular matrices in \( M_n(k) \) with 1’s along diagonal
- \( \text{Lie } D_n = \) diagonal matrices in \( M_n(k) \)

Exercise. If \( G \) is diagonal, show that \( \text{Lie } G \cong \text{Hom}_\mathbb{Z}(X^*(G), k) \).

### 3.3 Adjoint representation.

\( G \) acts on itself by conjugation: for \( x \in G \),
\[
c_x : G \to G, \quad g \mapsto xgx^{-1}
\]
is a morphism. \( \text{Ad}(x) := dc_x : \text{Lie } G \to \text{Lie } G \) is a Lie algebra endomorphism such that
\[
\text{Ad}(e) = \text{id}, \quad \text{Ad}(xy) = \text{Ad}(x) \circ \text{Ad}(y)
\]
Hence, we have a morphism of groups
\[
\text{Ad} : G \to \text{GL}(\text{Lie } G)
\]

**Proposition 75.** \( \text{Ad} \) is an algebraic representation of \( G \).

**Proof.** We must show that
\[
\theta : G \times \text{Lie } G \to \text{Lie } G, \quad (x, \delta) \mapsto \text{Ad}(x)(\delta) = dc_x(\delta) = \delta \circ c^*_x
\]
is a morphism of varieties. It is enough to show that \( \lambda \circ \theta \) is a morphism for all \( \lambda \in (\text{Lie } G)^* \). Given such a \( \lambda \), since \( (\text{Lie } G)^* \cong m/m^2 \) we must have \( \lambda(\delta) = \delta(f) \) for some \( f \in m \). Accordingly, for any \( f \in m \) we must show that
\[
(x, \delta) \mapsto \delta(c^*_x f)
\]
is a morphism. Recall from the proof of Proposition 27 that \( c^*_x f = \sum h_i(x)f_i \) for some \( f_i, h_i \in k[G] \), which implies that
\[
(x, \delta) \mapsto \delta(c^*_x f) = \sum h_i(x)\delta(f_i)
\]
is a morphism as \( x \mapsto h_i(x) \) and \( \delta \mapsto \delta(f_i) \) are morphisms.

**Exercises.**
- Show that \( \text{ad} := d(\text{Ad}) : \text{Lie } G \to \text{End}(\text{Lie } G) \) is
\[
\delta_1 \mapsto (\delta_2 \mapsto [\delta_1, \delta_2])
\]
This is hard, but is easiest to manage in reducing to the case of \( \text{GL}_n \) using an embedding \( G \hookrightarrow \text{GL}_n \).
- Show that \( d(\text{det} : \text{GL}_n \to \text{GL}_1) : M_n(k) \to k \) is the trace map.
3.4 Some derivatives.

If $X_1, X_2$ are varieties with points $x_1 \in X_1$ and $x_2 \in X_2$, then the morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\pi_1} & X_1 \\
\downarrow & & \downarrow \\
X_1 \times X_2 & \xrightarrow{i_{x_1}} & (x_1, x) \\
\downarrow & & \downarrow \\
X_1 \times X_2 & \xrightarrow{i_{x_2}} & (x, x_2) \\
X_2 & \xrightarrow{\pi_2} & X_2
\end{array}
$$

induce inverse isomorphisms $T_{x_1}X_1 \oplus T_{x_2}X_2 \cong T_{(x_1, x_2)}(X_1 \times X_2)$. In particular, for algebraic groups $G_1, G_2$ we have inverse isomorphisms

$$
\text{Lie } G_1 \oplus \text{Lie } G_2 \cong \text{Lie } (G_1 \times G_2)
$$

**Proposition 76.**

(i) $d(\mu : G \times G \to G) = (\text{Lie } G \oplus \text{Lie } G \xrightarrow{(X, Y) \mapsto X + Y} \text{Lie } G)$

(ii) $d(i : G \to G) = (\text{Lie } G \xrightarrow{X \mapsto -X} \text{Lie } G)$

*Proof.*

(i). It is enough to show that $d\mu$ is the identity on each factor. Since $\text{id}_G$ can be factored as

$$
G \xrightarrow{i_e} G \times G \xrightarrow{\mu} G
$$

where $i_e : x \mapsto (e, x)$ or $x \mapsto (x, e)$, we are done.

(ii). Since $x \mapsto e$ can be factored $G \xrightarrow{(\text{id}, i)} G \times G \xrightarrow{\mu} G$. From (i) we have that $0 : \text{Lie } G \to \text{Lie } G$ can factored as

$$
\text{Lie } G \xrightarrow{(\text{id}, d)} \text{Lie } G \oplus \text{Lie } G \xrightarrow{+} \text{Lie } G
$$

**Remark 77.** The open immersion $G^0 \hookrightarrow G$ induces an isomorphism $\text{Lie } G^0 \xrightarrow{\sim} \text{Lie } G$.

**Proposition 78** (Derivative of a linear map). If $V, W$ be vector spaces and $f : V \to W$ a linear map (hence a morphism), then, for all $v \in V$, we have the commutative diagram

$$
\begin{array}{ccc}
T_vV & \xrightarrow{T_v(f)} & T_{f(v)}W \\
\downarrow & & \downarrow \\
V & \xrightarrow{f} & W
\end{array}
$$

*Proof.* Exercise.
Proposition 79. Suppose that $\sigma : G \rightarrow \text{GL}(V)$ is a representation and $v \in V$. Define $o_v : G \rightarrow V$ by $g \mapsto \sigma(g)v$. Then
\[ do_v(X) = d\sigma(X)(v) \]
in $T_vV \cong V$, for all $X \in \text{Lie} G$.

Proof. Factor $o_v$ as
\[ G \xrightarrow{\phi} \text{GL}(V) \times V \xrightarrow{\psi} V \]
$g \mapsto (\sigma(g), v)$
\[ (A, w) \mapsto Aw \]
$d\phi = (d\sigma, 0) : \text{Lie} G \rightarrow \text{End} V \oplus V$. By Proposition 78, under the identification $V \cong T_vV$, we have that the derivative at $(e, v)$ of the first component of $\psi$, which sends $A \rightarrow Av$, is the same map. The result follows.

Proposition 80. Suppose that $\rho_i : G \rightarrow \text{GL}(V_i)$ are algebraic representations for $i = 1, 2$. Then the derivative of $\rho_1 \otimes \rho_2 : G \rightarrow \text{GL}(V_1 \otimes V_2)$ is
\[ d(\rho_1 \otimes \rho_2)(X) = d\rho_1(X) \otimes \text{id} + \text{id} \otimes d\rho_2(X) \]
(i.e., $X(v_1 \otimes v_2) = (Xv_1) \otimes v_2 + v_1 \otimes (Xv_2)$.) Similarly for $V_1 \otimes \cdots \otimes V_n$, $\text{Sym}^n V$, $\Lambda^n V$.

Proof. We have the commutative diagram
\[
\begin{array}{c}
\rho_1 \otimes \rho_2 : G \longrightarrow \text{GL}(V_1) \times \text{GL}(V_2) \longrightarrow \text{GL}(V_1 \otimes V_2) \\
\downarrow \text{open} \quad \downarrow \text{open} \\
\text{End}(V_1) \times \text{End}(V_2) \xrightarrow{\phi} \text{End}(V_1 \otimes V_2)
\end{array}
\]
where $\phi : (A, B) \mapsto A \otimes B$. (Note that $\phi$ being a morphism implies that $\rho_1 \otimes \rho_2$ is.) Computing $d\phi$ component-wise at $(1, 1)$, we get that $d\phi|_{\text{End}(V_1)}$ is the derivative of the linear map $\text{End}(V_1) \rightarrow \text{End}(V_1 \otimes V_2)$ given by $A \mapsto A \otimes 1$, which is the same map; likewise for $d\phi|_{\text{End}(V_2)}$. Hence,
\[ d\phi(A, B) = A \otimes 1 + 1 \otimes B \]
and we are done.

Exercise. If $\rho : G \rightarrow \text{GL}(V)$ is an algebraic representation, then so is $\rho^\vee : G \rightarrow \text{GL}(V^*)$, given by $\rho^\vee(g) = \rho(g^{-1})^*$. (Here, $V^*$ is the dual vector space.) Moreover, $d\rho^\vee(X) = -d\rho(X)^*$.

Proposition 81 (Adjoint representation for $\text{GL}(V)$). For $g \in \text{GL}(V)$, $A \in \text{Lie} \text{GL}(V) \cong \text{End}(V)$,
\[ \text{Ad}(g)A = gAg^{-1} \]

Proof. This follows from Proposition 78 by considering the linear map $f : \text{End}(V) \rightarrow \text{End}(V)$ given by $A \mapsto gAg^{-1}$ and noting that $\text{GL}(V)$ is open in $\text{End}(V)$.

Exercise. Deduce that, for $\text{GL}(V)$, $\text{ad}(A)(B) = AB - BA$. 

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3.5 Separable morphisms.

Let \( \phi : X \to Y \) be a dominant morphism of irreducible varieties (i.e., \( \overline{\phi(X)} = Y \)). From the induced maps \( O_Y(V) \to O_X(\phi^{-1}(V)) \) – note that \( \phi^{-1}(V) \neq \emptyset \), as \( \phi \) is dominant – given by \( f \mapsto f \circ \phi \), we get a morphism of fields \( \phi^* : k(Y) \to k(X) \). That is, \( k(X) \) is a finitely-generated field extension of \( k(Y) \).

Remark 82. This field extension has transcendence degree \( \dim X - \dim Y \), and hence is algebraic if and only if \( \dim X = \dim Y \).

Definition 83. A dominant \( \phi \) is separable if \( \phi^* : k(Y) \to k(X) \) is a separable field extension.

Recall.
- An algebraic field extension \( E/F \) being separable means that every \( \alpha \in E \) has a minimal polynomial without repeated roots.
- A finitely-generated field extension \( E/F \) is separable if it is of the form

\[
\begin{array}{c}
E \\
\text{finite separable} \\
F(x_1, \ldots, x_n) \\
\text{x_1, \ldots, x_n algebraically independent} \\
F
\end{array}
\]

Facts.
- If \( E'/E \) and \( E/F \) are separable then \( E'/F \) is separable.
- If \( \text{char } k = 0 \), all extensions are separable; in characteristic 0 being dominant is equivalent to being separable. (As an example, if \( \text{char } k = p > 0 \), then \( F(t^{1/p})/F(t) \) is never separable.)
- The composition of separable morphisms is separable.

Example. If \( p > 0 \), then \( \mathbb{G}_m^p \to \mathbb{G}_m \) is not separable.

Theorem 84. Let \( \phi : X \to Y \) be a morphism between irreducible varieties. The following are equivalent:

(i) \( \phi \) is separable.

(ii) There is a dense open set \( U \subset X \) such that \( d\phi_x : T_xX \to T_{\phi(x)}Y \) is surjective for all \( x \in U \).

(iii) There is an \( x \in X \) such that \( X \) is smooth at \( x \), \( Y \) is smooth at \( \phi(x) \), and \( d\phi_x \) is surjective.

Corollary 85. If \( X, Y \) are irreducible, smooth varieties, then \( \phi : X \to Y \)

is separable \( \iff d\phi_x \) is surjective for all \( x \) \( \iff d\phi_x \) is surjective for one \( x \)

Remark 86. The corollary applies in particular if \( X, Y \) are connected algebraic groups or homogeneous spaces.
3.6 Fibres of morphisms.

Theorem 87. Let $\phi : X \to Y$ be a dominant morphism between irreducible varieties and let $r := \dim X - \dim Y \geq 0$.

(i) For all $y \in \phi(X)$, $\dim \phi^{-1}(y) \geq r$.

(ii) There is a nonempty open subset $V \subset Y$ such that for all irreducible closed $Z \subset Y$ and for all irreducible components $Z' \subset \phi^{-1}(Z)$ with $Z' \cap \phi^{-1}(V) \neq \emptyset$, $\dim Z' = \dim Z + r$ (which implies that $\dim \phi^{-1}(y) = r$ for all $y \in V$). If $r = 0$, $|\phi^{-1}(y)| = [k(X) : k(Y)]$, for all $y \in V$.

Theorem 88. If $\phi : X \to Y$ is a dominant morphism between irreducible varieties, then there is a nonempty open $V \subset Y$ such that $\phi^{-1}(V) \xrightarrow{\phi \times \text{id}_Z} V \times Z$ is an open map.

Corollary 89. If $\phi : X \to Y$ is a $G$-equivariant morphism of homogeneous $G$-spaces,

(i) For all varieties $Z$, $\phi \times \text{id}_Z : X \times Z \to Y \times Z$ is an open map.

(ii) For all closed, irreducible $Z \subset Y$ and for all irreducible components $Z' \subset \phi^{-1}(Z)$, $\dim Z' = \dim Z + r$. (In particular, all fibres are equidimensional of dimension $r$.)

(iii) $\phi$ is an isomorphism if and only if $\phi$ is bijective and $d\phi_x$ is an isomorphism for one (or, equivalently, all) $x$.

(In this statement it’s easy to reduce to the irreducible case.)

Corollary 90. For all $G$-spaces, $\dim \text{Stab}_G(x) + \dim(Gx) = \dim G$.

Proof. Apply the above to $G \to Gx$. \qed

Corollary 91. Let $\phi : G \to H$ be a surjective morphism of algebraic groups.

(i) $\phi$ is open

(ii) $\dim G = \dim H + \dim \ker \phi$

(iii) $\phi$ is an isomorphism $\iff$ $\phi$ and $d\phi$ are bijective $\iff$ $\phi$ is bijective and separable

Proof. They are homogeneous $G$-spaces by left-translation, $H$ via $\phi$. \qed
Definition 92. A sequence of algebraic groups

\[ 1 \to K \xrightarrow{\phi} G \xrightarrow{\psi} H \to 1 \]

is exact if

(i) it is exact as sequence of abstract groups and

(ii)

\[ 0 \to \text{Lie } K \xrightarrow{d\phi} \text{Lie } G \xrightarrow{d\psi} \text{Lie } H \to 0 \]

is an exact sequence of Lie algebras (i.e., of vector spaces).

Exercise.

(a) Show that \( \phi \) is a closed immersion if and only if \( \phi \) is injective and \( d\phi \) injective.

(b) Suppose that \( G \) is connected. Show that \( \psi \) is separable if and only if \( \psi \) is surjective and \( d\psi \) surjective.

(c) Suppose that \( G \) is connected. Deduce that the sequence is exact if and only if (i) as above and

(ii') \( \phi \) is a closed immersion and \( \psi \) is separable.

(d) If the characteristic of \( k \) is 0, show that (i) implies (ii). (Hint: reduce to the case when \( G \) is connected.)

Theorem 93 (Weak form of Zariski’s Main Theorem). If \( \phi : X \to Y \) is a morphism between irreducible varieties such that \( Y \) is smooth, and \( \phi \) is birational (i.e., \( k(Y) = k(X) \)) and bijective, then \( \phi \) is an isomorphism.

3.7 Semisimple automorphisms.

Our goal is to show that semisimple conjugacy classes are closed, and to deduce some related results. The following definition is introduced purely for this purpose.

Definition 94. An automorphism \( \sigma : G \to G \) is semisimple if there is a \( G \hookrightarrow \text{GL}_n \) and a semisimple element \( s \in \text{GL}_n \) such that \( \sigma(g) = sgs^{-1} \) for all \( g \in G \).

Example. If \( s \in G_s \), then the inner automorphism \( g \mapsto sgs^{-1} \) is semisimple.

Example. Here’s an example that is not inner. Consider \( G = G_m^n \cong D_n \leq \text{GL}_n \). Then any “permutation automorphism” \( G_m^n \to G_m^n \) is semisimple, at least provided the characteristic is 0 or \( p > n \).

Definitions 95. Given a semisimple automorphism of \( G \), define

\[ G_\sigma := \{ g \in G \mid \sigma(g) = g \}, \text{ which is a closed subgroup} \]

\[ g_\sigma := \{ X \in g := \text{Lie } G \mid d\sigma(X) = X \} \]

Let \( \tau : G \to G, \ g \mapsto \sigma(g)g^{-1} \). Then \( G_\sigma = \tau^{-1}(e) \) and \( d\tau = d\sigma - \text{id} \) by Proposition 76, which implies that \( \ker d\tau = g_\sigma \). Since \( G_\sigma \hookrightarrow G \xrightarrow{\tau} G \) is constant, we have

\[ d\tau(\text{Lie } G_\sigma) = 0 \implies \text{Lie } G_\sigma \subset g_\sigma \]
Lemma 96.

\[ \text{Lie } G_{\sigma} = g_{\sigma} \iff G \xrightarrow{\tau} \tau(G) \text{ is separable} \iff d\tau : \text{Lie } G \to T_e(\tau(G)) \text{ is surjective} \]

Proof. \( \tau \) is a \( G \)-map of homogeneous spaces, acting by \( x \ast g = \sigma(x)gx^{-1} \) on the codomain. \( \tau(G) \) is smooth and is, by Proposition 24, locally closed. Hence, by Theorem 84

\( \tau \) is separable \iff \( d\tau \) is surjective
\iff \( \dim g_{\sigma} = \dim \ker d\tau = \dim G - \dim \tau(G) = \dim G_{\sigma} = \dim \text{Lie } G_{\sigma} \)
\iff \( g_{\sigma} = \text{Lie } G_{\sigma} \)

\( \square \)

Proposition 97. \( \tau(G) \) is closed and \( \text{Lie } G_{\sigma} = g_{\sigma} \).

Proof. Without loss of generality \( G \subset \text{GL}_n \) is a closed subgroup and \( \sigma(g) = sgs^{-1} \) for some semisimple \( s \in \text{GL}_n \). Without loss of generality, \( s \) is diagonal with

\[ s = a_1I_{m_1} \times \cdots \times a_nI_{m_n} \]

with the \( a_i \) distinct and \( n = m_1 + \cdots + m_n \). Then, extending \( \tau, \sigma \) to \( \text{GL}_n \), we have

\[ (\text{GL}_n)_\sigma = \text{GL}_{m_1} \times \cdots \times \text{GL}_{m_n} \quad \text{and} \quad (\mathfrak{gl}_n)_\sigma = M_{m_1} \times \cdots \times M_{m_n} \]

So, \( \text{Lie } (\text{GL}_n)_\sigma = (\mathfrak{gl}_n)_\sigma \). Hence

\[ \begin{array}{c}
\mathfrak{gl}_n \xrightarrow{d\tau} T_e(\tau(\text{GL}_n)) \\
\downarrow \\
\mathfrak{g} \xrightarrow{d\tau} T_e(\tau(G))
\end{array} \]

So, if \( X \in T_e(\tau(G)) \), there is \( Y \in \mathfrak{gl}_n \) such that \( X = d\tau(Y) = (d\sigma - 1)Y \). But, since \( d\sigma : A \mapsto sAs^{-1} \) acts semisimply on \( \mathfrak{gl}_n \) and preserves \( \mathfrak{g} \), we can write \( \mathfrak{gl}_n = \mathfrak{g} \oplus V \), with \( V \) a \( d\sigma \)-stable complement. Without loss of generality, \( Y \in \mathfrak{g} \), so \( d\tau \) is surjective and \( \text{Lie } G_{\sigma} = g_{\sigma} \).

Consider \( S := \{ x \in \text{GL}_n \mid (i), (ii), (iii) \} \) where

(i) \( xGx^{-1} = G \), which implies that \( \text{Ad}(x) \) preserves \( \mathfrak{g} \)

(ii) \( m(x) = 0 \), where \( m(T) = \prod_i (T - a_i) \) is the minimal polynomial of \( s \) on \( k^n \)

(iii) \( \text{Ad}(x) \) has the same characteristic polynomial on \( \mathfrak{g} \) as \( \text{Ad}(s) \)

Note that \( s \in S, S \) is closed (check), and if \( x \in S \) then (ii) implies that \( x \) is semisimple. \( G \) acts on \( S \) by conjugation. Define \( G_x, g_x \) as \( G_{\sigma}, g_{\sigma} \) were defined. Then

\[ g_x = \{ X \in \mathfrak{g} \mid \text{Ad}(x)X = X \} \]

and

\[ \dim g_x = \text{multiplicity of eigenvalue 1 in Ad}(x) \text{ on } g \]

and

\[ \dim G_x = \dim G_{\sigma} \]
by what we proved above. The stabilisers of the $G$-action on $S$ (conjugation) all $G_x, x \in S$, and have the same dimension. This implies that the orbits of $G$ on $S$ all have the same dimension, which further gives that all orbits are closed (Proposition. 23) in $S$ and hence in $G$. We have

$$\text{orbit of } s = \{gsg^{-1} \mid g \in G\} = \{g\sigma(g^{-1})s \mid g \in G\}$$

and that the map from the orbit to $\tau(G)$ given by $z \mapsto sz^{-1}$ is an isomorphism.

**Corollary 98.** If $s \in G_s$, then $\text{cl}_G(s)$, the conjugacy class of $s$, is closed and

$$G \to \text{cl}_G(s), \quad g \mapsto gsg^{-1}$$

is separable.

**Remark 99.** The conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $B_2$ is not closed!

**Proposition 100.** If a torus $D$ is a closed subgroup of a connected $G$, then $\text{Lie } Z_G(D) = \mathfrak{z}_D(D)$, where

$$Z_G(D) = \{g \in G \mid dgd^{-1} = g \quad \forall d \in D\}$$

is the centraliser of $D$ in $G$, and

$$\mathfrak{z}_D(D) = \{X \in \mathfrak{g} \mid \text{Ad}(d)(X) = X \quad \forall d \in D\}$$

**Note:** $Z_G(D) = \bigcap_{d \in D} G_d$ and $\mathfrak{z}_D(D) = \bigcap_{d \in D} \mathfrak{g}_d (G_d, \mathfrak{g}_d$ as above) since, for $d \in G_s$ and $\text{Lie } G_d = \mathfrak{g}_d$ by above.

**Proof.** Use induction on $\text{dim } G$. When $G = 1$ this is trivial.

Case 1: If $\mathfrak{z}_D(D) = \mathfrak{g}$, then $\mathfrak{g}_d = \mathfrak{g}$ for all $d \in D$ so $G_d = G$ for all $d \in D$, implying that $Z_G(D) = G$.

Case 2: Otherwise, there exists $d \in D$ such that $\mathfrak{g}_d \not\subseteq \mathfrak{g}$. Hence, $G_d \subset G$. Also have $D \subseteq G_0^D$, as $D$ is connected. Note that $Z_G^0(D) = Z_G(D) \cap G_0^D$ has finite index in $Z_G(D) \cap G_d = Z_G(D)$ and so their Lie algebras coincide. By induction,

$$\text{Lie } Z_G(D) = \text{Lie } Z_G^0(D) = \mathfrak{z}_{\text{Lie } G_0^D}(D) = \mathfrak{z}_{\mathfrak{g}_d}(D) = \mathfrak{z}_D(D) \cap \mathfrak{g}_d = \mathfrak{z}_D(D)$$

**Proposition 101.** If $G$ is connected, nilpotent, then $G_s \subset Z_G$ (which implies that $G_s$ is a subgroup).

**Proof.** Pick $s \in G_s$ and set $\sigma : g \mapsto sgs^{-1}$ and $\tau : g \mapsto \sigma(g)g^{-1} = [s, g]$. Since $G$ is nilpotent, there is an $n > 0$ such that $\tau^n(g) = [s, [s, \ldots, [s, g] \cdots]] = e$ for all $g \in G$ and so

$$\tau^n = e \implies d\tau^n = 0
\implies d\tau = d\sigma - 1 \text{ is nilpotent, but is also semisimple by above, since } d\sigma \text{ is semisimple}
\implies d\tau = 0
\implies \tau(G) = \{e\} \text{ as } G \xrightarrow{\tau} \tau(G) \text{ is separable}
\implies sgs^{-1} = g \text{ for all } g \in G$$
4. Quotients.

4.1 Existence and uniqueness as a variety.

Given a closed subgroup \( H \subset G \), we want to give the coset space \( G/H \) the structure of a variety such that \( \pi : G \to G/H, \ g \mapsto gH \) is a morphism satisfying a natural universal property.

**Proposition 102.** There is a \( G \)-representation \( V \) and a subspace \( W \subset V \) such that

\[
H = \{ g \in G \mid gW \subset W \} \quad \text{and} \quad \mathfrak{h} = \text{Lie } H = \{ X \in \mathfrak{g} \mid XW \subset W \}
\]

(We only need the characterisation of \( \mathfrak{h} \) when \( \text{char } k > 0 \).)

**Proof.** Let \( I = I_G(H) \), so that \( 0 \to I \to k[G] \to k[H] \to 0 \). Since \( k[G] \) is noetherian, \( I \) is finitely-generated; say, \( I = (f_1, \ldots, f_n) \). Let \( V \supset \sum k f_i \) be a finite-dimensional \( G \)-stable subspace of \( k[G] \) (with \( G \) acting by right translation). This gives a \( G \)-representation \( \rho : G \to \text{GL}(V) \). Let \( W = V \cap I \).

If \( g \in H \), then \( \rho(g)I \subset I \implies \rho(g)W \subset W \). Conversely,

\[
\rho(g)W \subset W \implies \rho(g)(f_i) \in I \quad \forall i
\]

\[
\implies \rho(g)I \subset I, \quad \text{as } \rho(g) \text{ is a ring morphism } k[G] \to k[G]
\]

\[
\implies g \in H \quad \text{(easy exercise. Note that } \rho(g)I = I_G(Hg^{-1})\text{)}
\]

Moreover, if \( X \in \mathfrak{h} \), then \( d\rho(X)W \subset W \) from the above. For the converse \( d\rho(X)W \subset W \implies X \in \mathfrak{h} \), we first need a lemma.

**Lemma 103.** \( d\rho(X)f = D_X(f) \quad \forall X \in \mathfrak{g}, f \in V \)

**Proof.** We know (Proposition 79) that \( d\rho(X)f = d\sigma_f(X) \), identifying \( V \) with \( T_f V \), where

\[
\sigma_f : G \to V, \quad g \mapsto \rho(g)f
\]

That is, for all \( f^\vee \in V^* \)

\[
\langle d\rho(X)f, f^\vee \rangle = \langle d\sigma_f(X), f^\vee \rangle
\]

Extend any \( f^\vee \) to \( k[G]^* \) arbitrarily. We need to show that

\[
\langle d\sigma_f(X), f^\vee \rangle = \langle D_X(f), f^\vee \rangle
\]

or, equivalently,

\[
X(\sigma_f^*(f^\vee)) = \langle d\sigma_f(X), f^\vee \rangle = \langle D_X(f), f^\vee \rangle = (1, X)\Delta f, f^\vee = (f^\vee, X)\Delta f.
\]
We have
\[ o_f^*(f^\vee) = f^\vee \circ o_f : g \mapsto (\rho(g)f, f^\vee) = \langle f(\cdot g), f^\vee \rangle = \langle (\id, \ev_g)\Delta f, f^\vee \rangle = (f^\vee, \ev_g)\Delta f \]
and so
\[ o_f^*(f^\vee) = (f^\vee, \id)\Delta f \implies X(o_f^*(f^\vee)) = (f^\vee, X)\Delta f \]

Now,
\[ d\rho(X)W \subset W \implies D_X(f_i) \in I \quad \forall i \]
\[ \implies D_X(I) \subset I \quad \text{(as } D_X \text{ is a derivation)} \]
\[ \implies X(I) = 0 \quad \text{easy exercise} \]
which implies that \( X \) factors through \( k[H] \):

\[ \begin{array}{ccc}
  k[G] & \longrightarrow & k[H] \\
  \downarrow & & \downarrow \\
  \uparrow X & & \uparrow \Xi \\
  k & \longrightarrow & k \\
\end{array} \]

It is easy to see that \( \Xi \) is a derivation, which means that \( X \in \mathfrak{h} \).

**Corollary 104.** We can even demand \( \dim W = 1 \) in Proposition 102 above.

**Proof.** Let \( d = \dim W, V' = \Lambda^d V, \) and \( W' = \Lambda^d W \), which has dimension 1 and is contained in \( V' \).

We have actions
\[ g(v_1 \wedge \cdots \wedge v_d) = gv_1 \wedge \cdots \wedge gv_d \]
\[ X(v_1 \wedge \cdots \wedge v_d) = (Xv_1 \wedge \cdots \wedge v_d) + (v_1 \wedge Xv_2 \wedge \cdots \wedge v_d) + \cdots + (v_1 \wedge \cdots \wedge Xv_d) \]

We need to show that
\[ gW' \subset W' \iff gW \subset W \]
\[ XW' \subset W' \iff XW \subset W \]
which is just a lemma in linear algebra (see Springer).

**Corollary 105.** There is a quasiprojective homogeneous space \( X \) for \( G \) and \( x \in X \) such that

(i) \( \Stab_G(x) = H \)

(ii) If \( o_x : G \to X, g \mapsto gx \), then

\[ 0 \to \Lie H \to \Lie G \xrightarrow{d o_x} T_x X \to 0 \]

is exact.
Note that (ii) follows from (i) if \( \text{char } k = 0 \) (use Corollaries 85 and 89.)

**Proof.** Take a line \( W \subset V \) as in the corollary above. Let \( x = [W] \in PV \) and let \( X = Gx \subset PV \). \( X \) is a subvariety and is a quasiprojective homogeneous space. Then (i) is clear.

**Exercise.** The natural map \( \phi : V - \{0\} \to PV \) induces an isomorphism

\[
V/x \cong T_x V/x \cong T_x (PV)
\]

for all \( x \in PV \) and \( v \in \phi^{-1}(x) \). (Hint:

\[
\begin{align*}
k^\times \xrightarrow{\lambda \mapsto \lambda v} & V - \{0\} \\
\phi & \cong PV
\end{align*}
\]

is constant. Use an affine chart in \( PV \) to prove that \( d\phi \) is surjective.)

**Claim.** \( \ker(d\phi) = \mathfrak{h} \) (then (ii) follows by dimension considerations.)

Fix \( v \in \phi^{-1}(x) \).

\[
\begin{align*}
\phi & \circ d \circ \phi : G \xrightarrow{\rho(g),v} \text{GL}(V) \times (V - \{0\}) \xrightarrow{(\rho(g),v) \mapsto \rho(\phi(g)v)} PV \\
d\phi & \circ d \circ \phi : g \xrightarrow{\phi(\rho(g)), \rho(g)v} \text{End}(V) \oplus V \xrightarrow{(\rho(\phi(\rho(g))), \rho(g)v) \mapsto [\rho(\phi(\rho(g)))]} V/x.
\end{align*}
\]

We have

\[
[d\phi(X)v] = 0 \iff XW \subset W \iff X \in \mathfrak{h}
\]

**Definition 106.** If \( H \subset G \) is a closed subgroup (not necessarily normal). A quotient of \( G \) by \( H \) is a variety \( G/H \) together with a morphism \( \pi : G \to G/H \) such that

(i) \( \pi \) is constant on \( H \)-cosets, i.e., \( \pi(g) = \pi(gh) \) for all \( g \in G, h \in H \), and

(ii) if \( G \to X \) is a morphism that is constant on \( H \)-cosets, then there exists a unique morphism \( G/H \to X \) such that

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & G/H \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & X
\end{array}
\]

commutes. Hence, if a quotient exists, it is unique up to unique isomorphism.

**Theorem 107.** A quotient of \( G \) by \( H \) exists; it is quasiprojective. Moreover,

(i) \( \pi : G \to G/H \) is surjective whose fibers are the \( H \)-cosets.

(ii) \( G/H \) is a homogeneous \( G \)-space under

\[
G \times G/H \to G/H, \quad (g, \pi(\gamma)) \mapsto \pi(g\gamma)
\]

**Proof.** Let \( G/H = \{ \text{cosets } gH \} \) as a set with natural surjection \( \pi : G \to G/H \) and give it the quotient topology (so that \( G/H \) is the quotient in the category of topological spaces). \( \pi \) is open. For \( U \subset G/H \) let \( \mathcal{O}_{G/H}(U) := \{ f : U \to k \mid f \circ \pi \in \mathcal{O}_G(\pi^{-1}(U)) \} \). Easy check: \( \mathcal{O}_{G/H} \) is a sheaf of \( k \)-valued functions on \( G/H \) and so \( (G/H, \mathcal{O}_{G/H}) \) is a ringed space.
If $\phi : G \to X$ is a morphism constant on $H$-cosets, then we get

$$\begin{array}{c}
G \xrightarrow{\pi} G/H \\
\phi \downarrow \\
X
\end{array}$$

in the category of ringed spaces.

By the second corollary 105 to Proposition 102 there is a quasiprojective homogeneous space $X$ of $G$ and $x \in X$ such that

1. $\text{Stab}_G(x) = H$
2. If $s_x : G \to X$, $g \mapsto gx$, then

$$0 \to \text{Lie} H \to \text{Lie} G \xrightarrow{d_{s_x}} T_x X \to 0$$

is exact.

Since $s_x$ is constant on $H$-cosets, we get a map $\psi : G/H \to X$ of ringed spaces (from the above universal property). $\psi$ is necessarily given by $gH \mapsto gx$ and is bijective. If we show that $\psi$ is an isomorphism of ringed spaces and that $(G/H, \mathcal{O}_{G/H})$ is a variety, then the theorem follows.

$\psi$ is a homeomorphism:

We need only show that $\psi$ is open. If $U \subset G/H$ is open then

$$\psi(U) = \psi(\pi(\pi^{-1}(U))) = \phi(\pi^{-1}(U))$$

is open, as $\phi$ is an open map (by Corollary 89).

$\psi$ gives an isomorphism of sheaves:

We must show that for $V \subset X$ open

$$\mathcal{O}_X(V) \to \mathcal{O}_{G/H}(\psi^{-1}(V))$$

is an isomorphism of rings. Clearly it is injective. To get surjectivity we need that for all $f : V \to k$

$$f \circ \phi : \phi^{-1}(V) \to k \text{ regular} \implies f \text{ regular}$$

Since

$$\begin{array}{c}
G \xrightarrow{\pi} G/H \\
\phi \downarrow \\
X \xleftarrow{\psi}
\end{array}$$

and $\psi$ is a homeomorphism, we need only focus on $(X, \phi)$. A lemma:

**Lemma 108.** Let $X, Y$ be irreducible varieties and $f : X \to Y$ a map of sets. If $f$ is a morphism, then the graph $\Gamma_f \subset X \times Y$ is closed. The converse is true if $X$ is smooth if $\Gamma_f$ is irreducible, and $\Gamma_f \to X$ is separable.
Proof.

(⇒:) If $f$ is a morphism, then $\Gamma_f = \theta^{-1}(\Delta_Y)$ is closed, where 
$$\theta : X \times Y \to Y \times Y, \quad (x, y) \mapsto (f(x), y).$$

(⇐:) We have 
$$\begin{array}{ccc}
\Gamma_f & \hookrightarrow & X \times Y \\
\eta & \downarrow & \\
X & \hookrightarrow & \\
\end{array}$$

with $\Gamma_f \hookrightarrow X \times Y$ the closed immersion.

\[ \eta \text{ bijective} \implies \dim \Gamma_f = \dim X \quad \text{and} \quad 1 = [k(\Gamma_f) : k(X)]_s = [k(\Gamma_f) : k(X)] \]

as $\eta$ is separable. Hence $\eta$ is birational and bijective with $X$ smooth, meaning that $\eta$ is an isomorphism by Theorem \[93\] and

$$f : X \xrightarrow{\eta^{-1}} \Gamma_f \to Y$$

is a morphism.

Now, for simplicity, assume that $G$ is connected, which implies that $X, V, \phi^{-1}(V)$ are irreducible. (For the general case, see Springer.) Suppose that $f \circ \phi$ is regular. It follows from the lemma that $\Gamma_{f \circ \phi} \subset \phi^{-1}(V) \times A^1$ is closed, surjecting onto $\Gamma_f$ via $\phi \times \text{id}$. By Corollary \[89\], $\phi : G \to X$ is “universally open” and so

$$V \times A^1 - \Gamma_f = (\phi \times \text{id})(\phi^{-1}(V) \times A^1 - \Gamma_{f \circ \phi})$$

is open: $\Gamma_f$ is closed. (The point is that $\Gamma_{f \circ \phi}$ is a union of fibers of $\phi \times \text{id}$.)

Also, $\Gamma_{f \circ \phi} \cong \phi^{-1}(V)$ is irreducible, implying that $\Gamma_f$ is irreducible, and

$$\begin{array}{ccc}
\Gamma_{f \circ \phi} & \xrightarrow{\sim} & \phi^{-1}(V) \\
\downarrow & & \downarrow \\
\Gamma_f & \xrightarrow{\text{pr}_1} & V
\end{array}$$

and

$$d\phi \text{ surjective} \implies d(\text{pr}_1) \text{ surjective} \implies \Gamma_f \to V \text{ separable and } V \text{ smooth.}$$

By Lemma \[108\] $f$ is a morphism.

Corollary 109. (i) $\dim(G/H) = \dim G - \dim H$

(ii) $0 \to \text{Lie } H \to \text{Lie } G \xrightarrow{d\pi} T_e(G/H) \to 0$

is exact.
Proof.
(i): $G/H$ is a homogeneous with stabilisers equal to $H$.
(ii): Implied by Corollary \[105\].

**Lemma 110.** Let $H_1 \subset G_1$, $H_2 \subset G_2$ be closed subgroups. The natural map

$$(G_1 \times G_2)/(H_1 \times H_2) \to G_1/H_1 \times G_2/H_2$$

is an isomorphism.

**Proof.** This is a bijective map of homogeneous $G_1 \times G_2$ spaces, which is bijective on tangent spaces by the above. The rest follows from Corollary \[91\].

### 4.2 Quotient algebraic groups.

**Proposition 111.** Suppose that $N \trianglelefteq G$ is a closed normal subgroup. Then $G/N$ is an algebraic group that is affine (and $\pi : G \to G/N$ is a morphism of algebraic groups).

**Proof.** Inversion $G/N \to G/N$ is a morphism, along with multiplication $G/N \times G/N \to G/N$ by Lemma \[110\], which gives that $G/N$ is an algebraic group.

By Corollary \[104\], there exists a $G$-representation $\rho : G \to \text{GL}(V)$ and a line $L \subset V$ such that $N = \text{Stab}_G(L)$ and $\text{Lie } N = \text{Stab}_g(L)$. For $\chi \in X^*(N) = \text{Hom}(N, \mathbb{G}_m)$, let $V_\chi$ be the $\chi$-eigenspace of $V$. (Note that $L \subset V_\chi$ for some $\chi$.) Let $V' = \sum_{\chi \in X^*(N)} V_\chi = \bigoplus_{\chi} V_\chi$ (by linear independence of characters). As $N \trianglelefteq G$, $G$ permutes the $V_\chi$. Define

$$W = \{ f \in \text{End}(V) \mid f(V_\chi) \subset V_\chi \ \forall \chi \} \subset \text{End}(V).$$

Let $\sigma : G \to \text{GL}(W)$ by

$$\sigma(g)f := \rho(g)f\rho(g)^{-1}$$

which is an algebraic representation.

**Claim.** $\sigma$ induces a closed immersion $G/N \hookrightarrow \text{GL}(W)$.

It is enough to show that $\ker \sigma = N$ and $\ker(d\sigma) = \text{Lie } N$.

$$g \in \ker \sigma \iff \rho(g)f = f\rho(g)$$

$$\iff \rho(g) \text{ acts as a scalar on each } V_\chi$$

$$\iff \rho(g)L = L \ \text{as } L \subset V_\chi \text{ for some } \chi$$

$$\iff g \in N$$

The converse is trivial: $\ker \sigma = N$.

By Proposition \[79\], $\phi_f : G \to W, \ g \mapsto \sigma(g)f$ has derivative

$$d\phi_f : g \to W, \ X \mapsto d\sigma(X)f.$$
Check that $d\sigma(X)f = d\rho(X)f - f d\rho(X)$. We have

\[
d\sigma(X) = 0 \iff d\rho(X)f = f d\rho(X) \quad \text{for all } f \in W
\]

\[
\iff d\rho(X) \text{ acts as a scalar on each } V_X
\]

\[
\implies X \in \text{Lie } N \text{ (as above)}.
\]

\[\square\]

**Corollary 112.** Suppose $\phi : G \to H$ is a morphism of algebraic groups with $\phi(N) = 1$, $N \trianglelefteq G$ closed. Then we have a unique factorisation in the category of algebraic groups,

\[
\begin{array}{c}
G \\
\downarrow \phi
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
G/N \\
\downarrow \phi
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
H
\end{array}
\]

In particular, we get that $G/\ker \phi \to \text{im } \phi$ is bijective and is an isomorphism when in characteristic 0.

(Note that in characteristic $p$, $\mathbb{G}_m \xrightarrow{p} \mathbb{G}_m$ is bijective and not an isomorphism.)

**Remark 113.**

\[1 \to N \to G \to G/N \to 1\]

is exact by Corollary 109.

**Exercise.** If $N \subset H \subset G$ are closed subgroups with $N \trianglelefteq G$, then the natural map $H/N \to G/N$ is a closed immersion (so we can think of $H/N$ as a closed subgroup of $G/N$) and we have a canonical isomorphism $(G/N)/(H/N) \xrightarrow{\sim} G/H$ of homogeneous $G$-spaces.

**Exercise.** Assume that char $k = 0$. Suppose $N, H \subset G$ are closed subgroups such that $H$ normalises $N$. Show that $H N$ is a closed subgroup of $G$ and that we have a canonical isomorphism $H N / N \cong H / (H \cap N)$ of algebraic groups. Find a counterexample when char $k > 0$.

**Exercise.** Suppose $H$ is a closed subgroup of an algebraic group $G$. Show that if both $H$ and $G/H$ are connected, then $G$ is connected. (Use, for example, Exercise 5.5.9(1) in Springer.) Variant: Show that if $\varphi : G \to H$ is a homomorphism such that $\ker \varphi$ and $\text{im } \varphi$ are connected, then $G$ is connected. (Hint: show that $\varphi(G^0) = \text{im } \varphi$.)

**Exercise.** Assume that char $k = 0$. Suppose $\phi : G \to H$ is a surjective morphism of algebraic groups. If $H_1 \subset H_2 \subset H$ are closed subgroups, show that the map $\phi$ induces a canonical isomorphism $\phi^{-1}(H_2)/\phi^{-1}(H_1) \cong H_2/H_1$. Find a counterexample when char $k > 0$.

**Example.** The group $\text{PGL}_2$:

Let $Z = \{(x \ x) \mid x \in \mathbb{G}_m\}$. $\text{GL}_2/Z$ is affine and the composition

\[
\text{SL}_2 \hookrightarrow \text{GL}_2 \twoheadrightarrow \text{GL}_2/Z
\]
is surjective, inducing the inclusion of Hopf algebras

\[ k[\text{GL}_2]^Z = k[\text{GL}_2/Z] \hookrightarrow k[\text{SL}_2]. \]

Check that the image is generated by the elements \( \frac{T_i T_j}{\det}, 1 \leq i, j \leq 4 \). (See Springer Exercise 2.1.5(3).)
5. Parabolic and Borel subgroups.

5.1 Complete varieties.

Recall: A variety $X$ is **complete** if for all varieties $Z$, $X \times Z \overset{pr_2}{\to} Z$ is a closed map. In the category of locally compact Hausdorff topological spaces, the analogous property is equivalent to compactness.

**Proposition 114.** Let $X$ be complete.

(i) $Y \subset X$ closed $\implies Y$ complete.

(ii) $Y$ complete $\implies X \times Y$ complete

(iii) $\phi : X \to Y$ morphisms $\implies \phi(X) \subset Y$ is closed and complete, which implies that if $X \subset Z$ is a subvariety, then $X$ is closed in $Z$.

(iv) $X$ irreducible $\implies \mathcal{O}_X(X) = k$

(v) $X$ affine $\implies X$ finite

**Proof.** An exercise (or one can look in Springer).

**Theorem 115.** $X$ projective $\implies X$ complete

**Note:** The converse is not true.

**Lemma 116.** Let $X, Y$ be homogeneous $G$-spaces with $\phi : X \to Y$ a bijective $G$-map. Then $X$ is complete $\iff Y$ is complete.

Note that such a map is an isomorphism if the characteristic of $k$ is 0.

**Proof.** For all varieties $Z$, then projection $X \times Z \to Z$ can be factored as

$$X \times Z \overset{\phi \times \text{id}}{\to} Y \times Z \overset{pr_2}{\to} Z$$

$\phi \times \text{id}$ is bijective and open (by Corollary 89) and is thus a homeomorphism: $Y$ being complete implies that in $X$. Applying the same reasoning to $\phi^{-1} : Y \to X$ gives the converse.

**Definition 117.** A closed subgroup $P \subset G$ is **parabolic** if $G/P$ is complete.
Remark 118. For a closed subgroup $P \subset G$, $G/P$ is quasi-projective by Theorem 107 and so

\[ G/P \text{ projective } \iff G/P \text{ complete } \iff P \text{ parabolic}. \]

The implication of $G/P$ being complete implying that $G/P$ being projective follows from Proposition 114 (iii) applying to the embedding of $G/P$ into some projective space.

Proposition 119. If $Q \subset P$ and $P \subset G$ are parabolic, then $Q \subset G$ is parabolic.

Proof. For all varieties $Z$ we need to show that $G/Q \times Z \xrightarrow{pr_2} Z$ is closed. Fix a closed subset $C \subset G/Q \times Z$. Letting $\pi : G \to G/P$ denote the natural projection, set $D = (\pi \times \text{id}_Z)^{-1}(C) \subset G \times Z$, which is closed. For all $q \in Q$, note that $(g, z) \in D \implies (gq, z) \in D$. It is enough to show that $pr_2(D) \subset Z$ is closed.

Let

\[ \theta : P \times G \times Z \to G \times Z, \quad (p, g, z) \mapsto (gp, z) \]

Then $\theta^{-1}(D)$ is closed for all $q \in Q$

\[ (p, g, z) \in \theta^{-1}(D) \implies (pq, g, z) \in \theta^{-1}(D) \tag{*} \]

Let $\alpha : P \times G \times Z \to P/Q \times G \times Z$ be the natural map.

\[ P \times G \times Z \xrightarrow{\alpha} P/Q \times G \times Z \]

\[ \begin{array}{ccc}
\downarrow \text{pr}_{23} & & \downarrow \text{pr}_{23} \\
G \times Z & & \end{array} \]

By Corollary 89 $\alpha$ is open. By passing to complements, $(*)$ implies that $\alpha(\theta^{-1}(D))$ is closed. $P/Q$ being complete implies that

\[ \text{pr}_{23}(\theta^{-1}(D)) = \{(gp^{-1}, z) \mid (g, z) \in D, p \in P\} \]

is closed. Now,

\[ G \times Z \xrightarrow{\beta} G/P \times Z \]

\[ \begin{array}{ccc}
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
G \times Z & & \end{array} \]

Similarly $\beta$ is open, and so $\beta(\text{pr}_{23}(\theta^{-1}(D)))$ is closed. $G/P$ being complete implies

\[ \text{pr}_2(\beta(\text{pr}_{23}(\theta^{-1}(D)))) = \text{pr}_2(\text{pr}_{23}(\theta^{-1}(D))) = \text{pr}_2(D) = \text{pr}_2(C) \]

is closed. \hfill \square

5.2 Borel subgroups.

Theorem 120 (Borel’s fixed point theorem). Let $G$ be a connected, solvable algebraic group and $X$ a (nonempty) complete $G$-space. Then $X$ has a fixed point.
Proof. We show this by inducting on the dimension of $G$. When $\dim G = 0 \iff G = \{e\}$ the theorem trivially holds. Now, let $\dim G > 0$ and suppose that the theorem holds for dimensions less than $\dim G$. Let $N = [G,G] \leq G$, which is a connected normal subgroup by Proposition 19 and is a proper subgroup as $G$ is solvable. Since $N$ is connected and solvable, by induction
\[
X^N = \{x \in X \mid nx = x \ \forall n \in N\} \neq \emptyset
\]
Since $X^N \subset X$ is closed (both topologically and under the action of $G$, as $N$ is normal), by Proposition 114 $X^N$ is complete; so, without loss of generality suppose that $N$ acts trivially on $X$. Pick a closed orbit $Gx \subset X$, which exists by Proposition 24 and is complete. Since $G/\text{Stab}_G(x)$ is a bijective map of homogeneous $G$-spaces, $G/\text{Stab}_G(x)$ is complete by Proposition 116.

Remark 122. Both of the above results need $G$ connected. It’s easy to find counterexamples with $G$ finite otherwise.

Definition 123. A Borel subgroup of $G$ is a maximal connected solvable closed subgroup $B$ of $G$.

Remarks 124.
- Any $G$ has a Borel subgroup since if $B_1 \subset B_2$ is irreducible $\implies \dim B_1 < \dim B_2$.
- $B_n \subset \text{GL}_n$ is a Borel by Lie-Kolchin.

Theorem 125.

(i) A closed subgroup $P \subset G$ is parabolic $\iff P$ contains a Borel subgroup.

(ii) Any two Borel subgroups are conjugate.
In particular, a Borel subgroup is precisely a minimal – or, equivalently, a connected, solvable – parabolic.

**Remark 126.** We will soon see that any parabolic subgroup is connected (Theorem 152).

**Proof.** For simplicity, assume that $G$ is connected.

(i) ($\Rightarrow$): Suppose that $B$ is a Borel and $P$ is parabolic. $B$ acts on $G/P$. By the Borel fixed point theorem, there is a coset $gP$ such that $Bg \subset gP \implies g^{-1}Bg \subset P$. $g^{-1}Bg$ is Borel.

(i) ($\Leftarrow$): Let $B$ be a Borel. We first show that $B$ is parabolic, inducting on $\dim G$. Pick a closed immersion $G \hookrightarrow \text{GL}(V)$. $G$ acts on $PV$. Let $Gx$ be a closed – hence complete – orbit. Since $G/\text{Stab}_G(x) \to Gx$ is a bijective map of homogeneous spaces, $P := \text{Stab}_G(x)$ is parabolic. By above, $B \subset gPg^{-1}$, for some $g \in G$. Without loss of generality, $B \subset P$. If $P \neq G$, then $B$ is Borel in $P$. Since $P \subset G$ is parabolic and $B \subset P$ is parabolic by induction, it follows that $B \subset G$ is parabolic, by Proposition 119. Suppose $P = G$. $G$ stabilises some line $V_1 \subset V$, which gives a morphism $G \to \text{GL}(V/V_1)$. By induction on $\dim V$, we either obtain a proper parabolic subgroup, in which case we are done by the above, or $G$ stabilises some flag $0 \subset V_1 \subset \cdots V_n = V$, giving that $G \hookrightarrow B_n \implies G$ is solvable $= \implies G = B$ is parabolic.

(ii). Let $B_1, B_2$ be Borel subgroups, which are parabolic by (i). By (i), there is $g \in G$ such that $gB_1g^{-1} \subset B_2 \implies \dim B_1 \leq \dim B_2$. Similarly,

\[ \dim B_2 \leq \dim B_1 \implies \dim B_1 = \dim B_2 \implies gB_1g^{-1} = B_2 \]

\[ \Box \]

**Corollary 127.** Let $\phi : G \to G'$ be a surjective morphism of algebraic groups.

(i) If $B \subset G$ is Borel, then $\phi(B) \subset G'$ is Borel.

(ii) If $P \subset G$ is parabolic, then $\phi(P) \subset G'$ is parabolic.

**Proof.** It is enough to prove (i). Since $B \to \phi(B)$, $\phi(B)$ is connected and solvable. Since $G/B$ is complete and $G/B \to G'/\phi(B)$ it follows that $G'/\phi(B)$ is complete and $\phi(B)$ is parabolic. Now, $\phi(B)$ is connected, solvable, and contains a Borel: $\phi(B)$ is Borel by the maximality in the definition of a Borel subgroup.

\[ \Box \]

**Corollary 128.** If $G$ is connected and $B \subset G$ a Borel, then $Z^0_G \subset Z_B \subset Z_G$.

**Remark 129.** We will soon see that $Z_B = Z_G$ (see Prop. 149).

**Proof.**

$Z^0_G$ connected, solvable $\implies Z^0_G \subset gBg^{-1}$, for some $g \in G$

$\implies Z^0_G = g^{-1}Z^0_Gg \subset B$

$\implies Z^0_G \subset Z_B$

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Now, fix \( b \in \mathbb{Z}_B \) and define the morphism \( \phi : G/B \to G \) of varieties by \( gB \mapsto gb^{-1} \). \( \phi(G/B) \) is complete and closed – hence affine – and irreducible, hence a point:

\[
\phi(G/B) = \{b\} \implies \forall g \in G, gb^{-1} = b \implies b \in \mathbb{Z}_G \implies \mathbb{Z}_B \subset \mathbb{Z}_G
\]

### Proposition 130

**Let** \( G \) **be** a connected group and \( B \subset G \) a Borel. If \( B \) is nilpotent, then \( G \) is solvable; that is, \( B \) nilpotent \( \implies B = G \).

**Proof.** If \( B = 1 \), then \( G = G/B \) is complete, connected, and affine, hence \( G/B = 1 \), so \( G = B \). If \( B \neq 1 \): \( B \) being nilpotent means that

\[
B \supseteq C B \supseteq \cdots \supseteq C^n B = 1
\]

for some \( n > 0 \) (where \( C^i B = [B, C^{i-1} B] \) is connected and closed). Let \( N = C^{n-1} B \), so that

\[
1 = [B, N] \implies N \subset \mathbb{Z}_B \subset \mathbb{Z}_G \quad \text{(above corollary)} \implies N \unlhd G
\]

Hence we have the morphism \( B/N \to G/N \) of algebraic groups, which is a closed immersion by the exercise after Theorems 87, 88. Also, \( B/N \) is a Borel of \( G/N \), by the corollary above, and \( B/N \) is nilpotent.

Inducting on \( \dim G \), we get that \( G/N \) is solvable, which implies that \( G \) is solvable. \( \square \)

### 5.3 Structure of solvable groups.

**Proposition 131.** Let \( G \) **be** connected and nilpotent. Then \( G_s, G_u \) are (connected) closed normal subgroups and \( G_s \times G_u \xrightarrow{\text{mult.}} G \) is an isomorphism of algebraic groups. Moreover, \( G_s \) is a central torus.

**Remark 132.** This generalises Proposition 37 from the commutative case (at least when \( G \) is connected).

**Proof.** Without loss of generality, \( G \subset \text{GL}(V) \) is a closed subgroup. By Proposition 101, \( G_s \subset \mathbb{Z}_G \). The eigenspaces of elements \( G_s \) coincide; let \( V = \bigoplus_{\lambda \in \mathbb{C}^*} V_{\lambda} \) be a simultaneous eigenspace decomposition. Since \( G_s \) is central, \( G \) preserves each \( V_{\lambda} \). By Lie-Kolchin (Proposition 121), we can choose a basis for each \( V_{\lambda} \) such that the \( G \)-action is upper-triangular. Therefore, \( G \subset B_n \), and \( G_s = G \cap D_n, G_u = G \cap U_n \) are closed subgroups, \( G_u \) being normal. We can now show that \( G_s \times G_u \xrightarrow{\sim} G \) as in the proof of Proposition 37. Moreover, \( G_s \) is a torus, being connected and commutative. \( \square \)

**Proposition 133.** Let \( G \) **be** connected and solvable.

1. \( [G, G] \) is a connected, normal closed subgroup and is unipotent.
2. \( G_u \) is a connected, normal closed subgroup and \( G/G_u \) is a torus.
\(\text{Proof.}\)

(i).

\[
\text{Lie-Kolchin } \implies G \hookrightarrow B_n \\
\implies [G, G] \hookrightarrow [B_n, B_n] \subset U_n \\
\implies [G, G] \text{ unipotent}
\]

We already know that it is connected, closed, and normal.

(ii). \(G_u = G \cap U_n\) is a closed subgroup. \(G_u \supseteq [G, G]\) implies that \(G_u \leq G\) and that \(G/G_u\) is commutative. For \([g] \in G/G_u\), \([g] = [g_s] = [g]\): all elements of \(G/G_u\) are semisimple. Since \(G/G_u\) is furthermore connected, it follows that \(G/G_u\) is a torus. It now remains to show that \(G_u\) is connected.

\[
1 \to G_u/[G, G] \to G/[G, G] \to G/G_u \to 1
\]

is exact (by the exercise on exact sequences). By Proposition 37, \(G/[G, G] \cong (G/[G, G])_s \times (G/[G, G])_u\)

Hence \((G/[G, G])_u = G_u/[G, G]\), which is connected by the above. Since \([G, G]\) is also connected, it follows from Springer 5.5.9(1) (exercise) that \(G_u\) is connected.

\[\square\]

**Lemma 134.** Let \(G\) be connected and solvable with \(G_u \neq 1\). Then there exists a closed subgroup \(N \subset Z_{G_u}\) such that \(N \cong G_a\) and \(N \leq G\).

\[\text{Proof.}\] Since \(G_u\) is unipotent, it is nilpotent. Let \(n > 0\) be such that

\[
G_u \supseteq C G_u \supseteq \cdots \supseteq C^n G_u = 1.
\]

The \(C^i G_u\) are connected closed subgroups and are normal as \(G_u\) is normal. Let \(N = C^{n-1} G_u\). Then

\[
1 = [G_u, N] \implies N \subset Z_{G_u},
\]

in particular \(N\) is commutative. If \(\text{char } k = p > 0\), let \(N \hookrightarrow U_m\), for some \(m\), and let \(r\) be minimal such that \(p^r \geq m\) so that \(N^{p^r} = 1\). Then (perhaps for a different \(r > 0\)),

\[
N \supseteq N^p \supseteq \cdots \supseteq N^{p^r} = 1.
\]

The \(N^{p^i}\) are connected, closed, and normal in \(G\). Replace \(N\) by \(N^{p^{r-1}}\). Then WLOG \(N\) is a connected elementary unipotent group and hence is isomorphic to \(G_a^r\) for some \(r\), by Corollary 59.

\(G\) act on \(N\) by conjugation, with \(G_u\) acting trivially. This induces an action \(G/G_u \times N \to N\) (use Lemma 110). \(G/G_u\) acts on \(k[N]\) in a locally algebraic manner, preserving the non-zero subspace \(\text{Hom}(N, G_a) = A(N)\). Since \(G/G_u\) is a torus, there is a nonzero \(f \in \text{Hom}(N, G_a)\) that is a simultaneous eigenvector. So, \((\ker f)^0 \subset N\) has dimension \(r-1\) and is still normal in \(G\). Induct on \(r\).

\[\square\]

**Definitions 135.** A maximal torus of \(G\) is a closed subgroup that is a torus and is a maximal such subgroup with respect to inclusion; they exist by dimension considerations. A temporary definition: a torus \(T\) of a connected solvable group is **Maximal** (versus maximal) if \(\dim T = \dim(G/G_u)\).

(Recall that \(G/G_u\) is a torus.) It is easy to see that Maximal \(\implies\) maximal. We shall soon see that the converse is true as well, after a corollary to the following theorem (so that we can then dispense with the capital \(M\)):
Theorem 136. Let $G$ be connected and solvable.

(i) Any semisimple element lies in a Maximal torus. (In particular, Maximal tori exist.)

(ii) $Z_G(s)$ is connected for all semisimple $s$.

(iii) Any two Maximal tori are conjugate in $G$.

(iv) If $T$ is a Maximal torus, then $G \cong G_u \times T$ (i.e., $G_u \leq G$ and $G_u \times T \xrightarrow{\text{mult.}} G$ is an isomorphism of varieties).

Proof.

(iv): Let $T$ be Maximal and consider $\phi : T \to G/G_u$. Since $\ker \phi = T \cap G_u = 1$ (Jordan decomposition), we have that $\dim \phi(T) = \dim T = \dim G/G_u = \dim \ker \phi \implies \phi(T) = G/G_u$.

$\phi$ is surjective and so $G = TG_u$. Thus multiplication $T \times G_u \to G$ is a bijective map of homogeneous $T \times G_u$-spaces. To see that it is an isomorphism, (if $p > 0$) we need an isomorphism – just an injection by dimension considerations – on Lie algebras, which is equivalent to $\text{Lie } T \cap \text{Lie } G_u = 0$, as is to be shown.

Now, pick a closed immersion $G \hookrightarrow \text{GL}(V)$. Picking a basis for $V$ such that $G_u \subset U_n$ gives that

$$\text{Lie } G_u \subset \text{Lie } U_n = \begin{pmatrix} 0 & * & * \\ & & \\ & & \\ 0 & & \end{pmatrix}$$

consists of nilpotent elements. Picking a basis for $V$ such that $T \subset D_n$ gives that

$$\text{Lie } T \subset \text{Lie } D_n = \text{diag}(*, \ldots, *)$$

consist of semisimple elements. Thus, $\text{Lie } T \cap \text{Lie } G_u = 0$.

(i)–(iii):

If $G_u = 1$, then $G$ is a torus and there is nothing to show. Suppose that $\dim G_u > 0$.

Case 1. $\dim G_u = 1$;

$G_u$ is connected, unipotent and so $G_u \cong G_a$ by Theorem 60. Let $\phi : G_a \to G_u$ be an isomorphism. $G$ acts on $G_u$ by conjugation with $G_u$ acting trivially. We have

$$\text{Aut } G_u \cong \text{Aut } G_a \cong \text{G}_m \text{ (exercise)}.$$

Hence

$$g\phi(x)g^{-1} = \phi(\alpha(g)x)$$

for all $g \in G$, $x \in G_a$, for some character $\alpha : G/G_u \to \text{G}_m$.

$\alpha = 1$: $G_u \subset Z_G$.

$[G, G] \subset G_u$ (Proposition 133) $\implies [G, [G, G]] = 1$, so $G$ is nilpotent

$\implies G \cong G_u \times G_a$ (Proposition 131).
and so $G$ is commutative and $G_s$ is the unique maximal torus. (i)–(iii) are immediate.

\[ \alpha \neq 1: \text{Given } s \in G_s, \text{ let } Z = Z_G(s). \]

\[ G/G_u \text{ commutative } \implies \text{cl}_G(s) \text{ maps to } [s] \in G/G_u \]
\[ \implies \text{cl}_G(s) \subset sG_u \]
\[ \implies \dim \text{cl}_G(s) \leq 1 \]
\[ \implies \dim Z = \dim G - \dim \text{cl}_G(s) \geq \dim G - 1 \]

\[ \alpha(s) \neq 1: \text{For all } x \neq 0 \]
\[ s\phi(x)s^{-1} = \phi(\alpha(s)x) \neq \phi(x) \]

which implies that $Z \cap G_u = 1$, further giving $\dim Z = \dim G - 1$ and

\[ Z_u = 1 \implies Z^0 \text{ is a torus} – \text{which is Maximal} – \text{by Proposition 133 (it is connected, solvable and } Z^0_u = 1) \]
\[ \implies G = Z^0G_u, \text{ by (iv)} \]

If $z \in Z$, then $z = z_0u$ for some $z_0 \in Z^0$ and $u \in G_u$. But

\[ u = z_0^{-1}z \in Z \cap G_u = 1 \implies z = z_0 \in Z^0. \]

Therefore, $Z = Z^0$, giving (iii), and $s \in Z$, giving (i).

\[ \alpha(s) = 1: \text{For all } x \neq 0 \]
\[ s\phi(x)s^{-1} = \phi(\alpha(s)x) = \phi(x) \]

and so $G_u \subset Z$. By the Jordan decomposition, since $s$ commutes with $G_u$, $sG_u \cap G_s = \{s\}$, which means that

\[ \text{cl}_G(s) = \{s\} \implies s \in Z_G \implies Z = G. \]

(ii) follows.

Note that since $\alpha \neq 1$ there is $g = g_s g_u$ such that $\alpha(g_s) = \alpha(g) \neq 1$ and so $Z(g_s)$ is a Maximal torus by the previous case. Hence, since $Z_G(s) = G$, we have $s \in Z_G(g_s)$: (i) follows.

Now it remains to prove (iii) in the general case in which $\alpha \neq 1$. Let $s$ be such that $T, T'$ be Maximal tori. With the identification $T \sim G/G_u$ (see (iv)), let $s \in T$ be such that $\alpha(s) \neq 1$. Then $Z_G(s)$ is Maximal (by the above) and

\[ T \subset Z_G(s) \implies T = Z_G(s) \text{ by dimension considerations.} \]

Likewise, with the identification $T' \sim G/G_u$, pick $s' \in T'$ with $[s] = [s']$ in $G/G_u$ so that $T' = Z_G(s')$. $s' = su$ for some $u = G_u$. The conjugacy class of $s$ (resp. $s'$) – which has dimension 1 by the above – is contained in $sG_u = s'G_u$, which is irreducible of dimension 1:

\[ \text{cl}_G(s) = sG_u = s'G_u = \text{cl}_G(s') \]

since the conjugacy classes are closed (Corollary 98). Therefore, $s'$ is conjugate to $s$ and thus $T, T'$ are conjugate.
Case 2. \( \dim G_u \geq 1 \): Induct on the dimension of \( G \).

Lemma 134 implies that there exists a closed, normal subgroup \( N \subset Z_{G_u} \) isomorphic to \( G_u \). Set \( \overline{G} = G/N \) and \( \overline{G}_u = G_u/N \), so \( \overline{G}/\overline{G}_u \cong G/G_u \). Let \( \pi : G \rightarrow \overline{G} \) be the natural surjection.

(i): If \( s \in G \), define \( \overline{s} = \pi(s) \in \overline{G}_s := \pi(G_s) \). By induction, there is a Maximal torus \( \overline{T} \) in \( \overline{G} \) containing \( \overline{s} \). Let \( H = \pi^{-1}(\overline{T}) \), which is connected since \( N \) and \( \overline{T} \) are connected (exercise, see homework 3). Also, \( H_u = N \) (consider the map \( H \rightarrow \overline{T} \) with kernel \( N \)) has dimension 1. Case 1 implies that there is a torus \( T \ni s \) in \( H \) (Maximal in \( H \)) of dimension \( \dim H/H_u = \dim \overline{T} = \dim G/G_u \); hence, \( T \) is Maximal in \( G \), containing \( s \).

(iii): Let \( T, T' \) be Maximal tori. Then \( \pi(T) = \pi(T') \) are Maximal tori in \( \overline{G} \) and by induction are conjugate: there is \( g \in G \) such that

\[
\pi(T) = \pi(gT'g^{-1}) \implies T, gT'g^{-1} \in \pi^{-1}(\pi(T)) =: H.
\]

As above \( H_u \) is 1-dimensional and so \( T, gT'g^{-1} \) – being Maximal tori in \( H \) – are conjugate in \( H \) and hence in \( G \).

(ii): Again, for \( s \in G \), set \( \overline{s} = \pi(s) \). \( Z_{\overline{G}}(\overline{s}) \) is connected by induction. \( H := \pi^{-1}(Z_{\overline{G}}(\overline{s})) \) is connected since \( N \) and \( Z_{\overline{G}}(\overline{s}) \) are connected (exercise, see homework 3). Since \( \pi(Z_G(s)) \subset Z_{\overline{G}}(\overline{s}) \), we have \( Z_G(s) = Z_H(s) \). If \( H \neq G \), \( Z_H(s) \) is connected by induction and we are done. If \( H = G \), then \( Z_{\overline{G}}(\overline{s}) = G \). Hence,

\[
\text{cl}(\overline{s}) = \{\overline{s}\} \implies \text{cl}(s) \subset \pi^{-1}(\overline{s}) = sN
\]

and so the conjugacy class of \( s \) (recall that it is closed) has dimension at most 1. We can now proceed as in Case 1 to conclude. (Sketch: fix an isomorphism \( \phi : N \rightarrow G_u \). There is a \( \beta \in G_n \) such that \( s\phi(x)s^{-1} = \phi(\beta x) \) for all \( x \in G_a \). If \( \beta \neq 1 \) we deduce \( Z \cap N = 1 \), so \( \dim Z = \dim G - 1 \) and \( G = Z^0 N \). We deduce \( Z = Z^0 \) as above. If \( \beta = 1 \), then \( N \leq Z \), so \( sN \cap G_a = \{s\} \), which implies \( \text{cl}(s) = \{s\} \) and hence \( Z = G \).)

\[\square\]

Remark 137. (i), (iii) above carry over to all connected \( G \), as we shall see soon. However, (ii) can fail in general. (For example, take \( G = \text{PGL}_2 \) in characteristic \( \neq 2 \) and \( s = [\text{diag}(1, -1)] \).

Example. \( D_n \) is a maximal torus of \( B_n \) and \( B_n \cong U_n \times D_n \).

Example. If \( G \) is connected nilpotent it is clear by Proposition 131 that \( G_a \) is the unique maximal torus and the unique Maximal torus.

Lemma 138. If \( \phi : H \rightarrow G \) is an injective homomorphism, then \( \dim H \leq \dim G \).

Proof. Since \( \dim \ker \phi = 0 \), \( \dim H = \dim \phi(H) \leq \dim G \).

\[\square\]

Proposition 139. Let \( G \) be connected and solvable with \( H \subset G \) a closed diagonalisable subgroup.

(i) \( H \) is contained in a Maximal torus.

(ii) \( Z_G(H) \) is connected.

(iii) \( Z_G(H) = N_G(H) \)
Proof. We shall induct on \( \dim G \).

If \( H \subset Z_G \): Let \( T \) be a Maximal torus. For \( h \in H \), for some \( g \in G \),
\[
h \in gT g^{-1} \implies h = g^{-1} h g \in T \implies H \subset T
\]
Also, \( Z_G(H) = N_G(H) = G \).

If \( H \not\subset Z_G \): let \( s \in H - Z_G \). Then \( H \subset Z := Z_G(s) \neq G \) and so \( Z \) is connected by induction. Also by induction, \( s \in T \) for some Maximal torus \( T \); hence \( T \subset Z \). We have injective morphisms
\[
T \to Z/Z_u \to G/G_u \implies \dim T \leq \dim(Z/Z_u) \leq \dim(G/G_u)
\]
But \( T \) is maximal, and so all of the dimensions must coincide: \( T \) is a Maximal torus of \( Z \). By induction \( H \subset gT g^{-1} \) for some \( g \in Z \), implying (i). Also, \( Z_G(H) = Z_Z(H) \) is connected by induction, giving (ii). For (iii), if \( n \in N_G(H), h \in H \), then
\[
[n, h] \in H \cap [G, G] \subset H \cap G_u = 1 \implies n \in Z_G(H) \implies N_G(H) \subset Z_G(H)
\]

Corollary 140. Let \( G \) be connected and solvable, and let \( T \subset G \) be a torus. Then
\[
T \text{ is maximal } \iff T \text{ is Maximal}
\]
Proof. If \( T \) is Maximal and \( T \subset T' \) for some torus \( T' \), then \( T \to T' \to G/G_u \) are injective morphisms, giving
\[
\dim(G/G_u) = \dim T \leq \dim T' \leq \dim(G/G_u)
\]
Hence, \( T = T' \) and \( T \) is maximal. If \( T \) is not Maximal, then \( T \subset T' \) for some Maximal \( T' \) by the above proposition, so \( T \) is not maximal.

5.4 Cartan subgroups.

Remark 141. From now on, \( G \) denotes a connected algebraic group.

Theorem 142. Any two maximal tori in \( G \) are conjugate.

Proof. Let \( T, T' \) be maximal. Since both are connected and solvable they are each contained in Borels: \( T \subset B, T' \subset B' \). There is a \( g \in G \) such that \( gBg^{-1} = B' \). \( gTg^{-1} \) and \( T' \) are two maximal tori in \( B \) and so, by Proposition 136, for some \( b \in B \), \( bgTg^{-1}b^{-1} = T' \).

Corollary 143. A maximal torus in a Borel subgroup of \( G \) is a maximal torus in \( G \).

Proof. Let \( B \) be a Borel subgroup. By the previous proof, any maximal torus of \( G \) is conjugate to a maximal torus of \( B \).
Definition 144. A Cartan subgroup of \( G \) is \( \mathcal{Z}_G(T)^0 \), for a maximal torus \( T \). All Cartan subgroups are conjugate. (We will see in Proposition 150 that \( \mathcal{Z}_G(T) \) is connected.)

Examples.

• \( G = \text{GL}_n \), \( T = D_n \), \( \mathcal{Z}_G(T) = T = D_n \)

• If \( G \) is nilpotent, then the unique maximal torus \( G_s \) is central, so \( G \) is the unique Cartan subgroup.

Proposition 145. Let \( T \subset G \) be a maximal torus. \( C := \mathcal{Z}_G(T)^0 \) is nilpotent and \( T \) is its (unique) maximal torus.

Proof. \( T \subset C \) and so \( T \) is a maximal torus of \( C \). Moreover, \( T \subset \mathcal{Z}_C \). Now \( T \) lies in a Borel subgroup \( B \) of \( C \) and \( T \subset \mathcal{Z}_B \), so by Theorem 136 we have \( B = T \times B_n \), so \( B \) is nilpotent. By Proposition 130, \( C = B \), so \( C \) is nilpotent. Finally \( T \) is the unique maximal torus of \( C \) by Proposition 131.

Lemma 146. Let \( S \subset G \) be a torus. There exists \( s \in S \) such that \( \mathcal{Z}_G(S) = \mathcal{Z}_G(s) \).

Proof. Let \( G \hookrightarrow \text{GL}_n \) be a closed immersion. Since \( S \) is a collection of commuting, diagonalisable elements, without loss of generality, \( S \hookrightarrow D_n \). It is enough to show that \( \mathcal{Z}_{\text{GL}_n}(S) = \mathcal{Z}_{\text{GL}_n}(s) \), for some \( s \in S \). Let \( \chi_i \in X^*(D_n) \) be given by \( \text{diag}(x_1, \ldots, x_n) \mapsto x_i \). It is easy to show that
\[
\mathcal{Z}_G(S) = \{ (x_{ij}) \in \text{GL}_n \mid \forall i, j \quad x_{ij} = 0 \text{ if } \chi_i|_S \neq \chi_j|_S \}.
\]
The set
\[
\bigcap_{\chi_i|_S \neq \chi_j|_S} \{ s \in S \mid \chi_i(s) \neq \chi_j(s) \}
\]
is nonempty and open, and thus is dense; any \( s \) from the set will do.

Lemma 147. For a closed, connected subgroup \( H \subset G \), let \( X = \bigcup_{x \in G} xHx^{-1} \subset G \).

(i) \( X \) contains a nonempty open subset of \( \overline{X} \).

(ii) \( H \) parabolic \( \implies \) \( X \) closed

(iii) If \( (N_G(H) : H) < \infty \) and there is \( y \in G \) lying in only finitely many conjugates of \( H \), then \( \overline{X} = G \).

Proof.

(i): \( Y := \{ (x, y) \mid x^{-1}yx \in H \} = \{ (x, y) \mid y \in xHx^{-1} \} \subset G \times G \)
is a closed subset. Note that
\[
\text{pr}_2(Y) = \{ y \in G \mid y \in xHx^{-1} \text{ for some } x \} = X
\]
By Chevalley, \( X \) contains a nonempty open subset of \( \overline{X} \).
(ii): Let $P$ be parabolic.

$$G \times G \xrightarrow{\pi \times \text{id}} G/H \times G \xrightarrow{\text{pr}_2} G$$

Note that $\pi \times \text{id}$ is open (Corollary 89) and that

$$(x, y) \in Y \iff \forall h \in H \ (xh, y) \in Y.$$ By the usual argument, $(\pi \times \text{id})(Y)$ is closed. Since $G/P$ is complete,

$$\text{pr}_2'(\pi \times \text{id})(Y)) = \text{pr}_2(Y) = X$$

is closed.

(iii): We have an isomorphism

$$Y \cong G \times H, \ (x, y) \mapsto (x, x^{-1}yx)$$

and so $Y$ is irreducible (as $H, G$ are connected). Consider the diagram

$$G \xleftarrow{\text{pr}_1} Y \xrightarrow{\text{pr}_2} G.$$ 

$$\text{pr}_1^{-1}(x) = \{(x, xhx^{-1}) \mid h \in H\} \cong H \implies \text{all fibers of } \text{pr}_1 \text{ have dimension } \dim H$$

$$\implies \dim Y = \dim G + \dim H \quad \text{(Theorem 87).}$$

Moreover,

$$\text{pr}_2^{-1}(y) = \{(x, y) \mid y \in xHx^{-1}\} \cong \{x \mid y \in xHx^{-1}\}$$

Pick $y \in G$ lying in finitely many conjugates of $H$: $x_1Hx_1^{-1}, \ldots, x_nHx_n^{-1}$. Then

$$\text{pr}_2^{-1}(y) = \bigcup_{i=1}^n x_iN_G(H)$$

which is a finite union of $H$ cosets by hypothesis ($(N_G(H) : H) < \infty$). This implies that

$$\dim \text{pr}_2^{-1}(y) = \dim H \implies \text{pr}_2 : Y \to \overline{\text{pr}_2(Y)} \text{ is a dominant map with minimal fibre dimension } \leq \dim H$$

$$\implies \dim Y - \dim \overline{\text{pr}_2(Y)} \leq \dim H$$

$$\implies \dim \overline{\text{pr}_2(Y)} \geq \dim Y - \dim H = \dim G$$

$$\implies \overline{\text{pr}_2(Y)} = G$$

Theorem 148.

(i) Every $g \in G$ is contained in a Borel subgroup.

(ii) Every $s \in G_s$ is contained in a maximal torus.
Proof.
(i): Pick a maximal torus $T \subset G$. Let $C = Z_G(T)^0$ be the associated Cartan subgroup. Because $C$ is connected and nilpotent (Proposition 145), there is a Borel $B \supset C$.

$$T = C_s \quad \text{Proposition 145} \implies N_G(C) = N_G(T) \quad \text("\supset" is obvious)$$
$$\implies (N_G(C) : C) = (N_G(T) : Z_G(T)^0) < \infty \quad \text{Corollary 55}$$

By Lemma 146 there is $t \in T$ such that $Z_G(t)^0 = Z_G(T)^0 = C$. $t$ is contained in a unique conjugate, i.e.,

$$t \in xCx^{-1} \implies xC^{-1} = C$$

by the following.

$$t \in xCx^{-1} \implies x^{-1}tx \in C, \quad \text{which is a semisimple element}$$
$$\implies x^{-1}tx \in C_s = T \subset Z_G(C)$$
$$\implies C \subset Z_G(x^{-1}tx)^0 = x^{-1}Z_G(t)^0 = x^{-1}Cx$$
$$\implies C = x^{-1}Cx \quad \text{(compare dimensions)}$$

Hence, we can apply Lemma 147 (iii) with $H = C$ to get

$$G = \bigcup_x xC^{-1} \subset \bigcup_x xB^{-1} = \bigcup_x Bx^{-1}$$

with the last equality following from Lemma 147 (ii) (this time with $H = B$). Hence, $G = \bigcup xBx^{-1}$, giving (i) of the theorem.

(ii):

$$s \in G_s \implies s \in B, \quad \text{for some Borel } B \text{ by (i)}$$
$$\implies s \in T, \quad \text{for some maximal torus } T \text{ of } B \text{ by Theorem 136 (i).}$$

(A maximal torus in $B$ is a maximal torus in $G$ by Theorem 142.)

**Corollary 149.** If $B \subset G$ is a Borel then $Z_B = Z_G$.

*Proof.* The inclusion $Z_B \subset Z_G$ follows Corollary 128. For the reverse inclusion, if $z \in Z_G$, we have $z \in gBg^{-1}$ for some $g$ by the above Theorem, and so $z = g^{-1}zg \in B$. \qed

**Proposition 150.** Let $S \subset G$ be a torus.

(i) $Z_G(S)$ is connected.

(ii) If $B \subset G$ is a Borel containing $S$, then $Z_G(S) \cap B$ is a Borel in $Z_G(S)$, and all Borels of $Z_G(S)$ arise this way.

*Proof.*

(i): Let $g \in Z_G(S)$ and $B$ a Borel containing $g$. Define

$$X = \{xB \mid g \in xBx^{-1}\} \subset G/B$$
which is nonempty by Theorem 148. Consider the diagram
\[ G/B \xrightarrow{\pi} G \xrightarrow{\alpha} G \]
in which \( \pi \) is the natural surjection and \( \alpha : x \mapsto x^{-1}gx \). We have \( X = \pi(\alpha^{-1}(B)) \). Since \( \pi^{-1}(B) \) is a union of fibres of \( \pi \) and is closed, and \( \pi \) is open, we have that \( X \) is closed. \( X \) is thus complete, being a closed subset of the complete \( G/B \).

\( S \) acts on \( X \subset G/B \), as for all \( s \in S \)
\[ xBx^{-1} \ni g \implies sxBx^{-1}s^{-1} \ni g \ (\text{since} \ g = s^{-1}gs). \]
By the Borel Fixed Point Theorem 120, \( S \) has some fixed point \( xB \in X \), so
\[ SxB = xB \implies Sx \subset xB \implies S \subset xBx^{-1}. \]

Hence, since \( g \) also lies in \( xBx^{-1} \), we have
\[ g \in Z_{xBx^{-1}}(S) \subset Z_G(S)^0 \]
where \( Z_{xBx^{-1}}(S) \) is connected by Proposition 139. Thus, \( Z_G(S) \subset Z_G(S)^0 \): equality.

(ii): Let \( B \) be a Borel containing \( S \) and set \( Z = Z_G(S) \). \( Z \cap B = Z_B(S) \) is connected by Proposition 139 and is also solvable. Therefore, \( Z \cap B \) is a Borel of \( Z \) if and only if it is parabolic, i.e., if \( Z/Z \cap B \) is complete. By the bijective map
\[ Z/(Z \cap B) \rightarrow ZB/B \]
of homogeneous \( Z \)-spaces, we see that suffices to show that
\[ ZB/B \subset G/B \text{ is closed } \iff Y := ZB \subset G \text{ is closed (by the definition of the quotient topology)} \]
\( Z \) being irreducible implies that
\[ Y = \text{im} (Z \times B \xrightarrow{\text{mult}} G) \text{ is irreducible } \implies Y \text{ irreducible.} \]

Let \( \pi : B \rightarrow B/B_u \) be the natural surjection and define
\[ \phi : Y \times S \rightarrow B/B_u, \ (y,s) \mapsto \pi(y^{-1}sy). \]
(To make sure that this definition makes sense, i.e., that \( y^{-1}sy \in B \), first check it when \( y \in Y = ZB \).) For fixed \( y \),
\[ \phi_y : S \rightarrow B/B_u, \ s \mapsto \phi(y,s) = \pi(y^{-1}sy) \]
is a homomorphism. Therefore, by rigidity (Theorem 54), for all \( y \in Y, \phi_e = \phi_y \); for all \( s \in S \)
\[ \pi(y^{-1}sy) = \pi(s). \]
If \( T \supset S \) is a maximal torus, by the conjugacy of maximal tori in \( B \), we have
\[ u y^{-1} S u^{-1} = T \]
for some \( u \in B_u \). But then, by the above,
\[ \pi(uy^{-1}yu^{-1}) = \pi(y^{-1}sy) = \pi(s) \text{ for all } s \in S \]
while \( \pi_{|T} : T \to B/B_u \) is injective (an isomorphism even) (Jordan decomposition). Therefore,

\[
uy^{-1}syu^{-1} = s \implies yu^{-1} \in Z_G(S) = Z \implies y \in ZB = Y
\]

and thus \( Y \) is closed: \( Z \cap B \subset Z \) is Borel. Moreover, any other Borel of \( Z \) is

\[
z(Z \cap B)z^{-1} = Z \cap (zBz^{-1}),
\]

\( zBz^{-1} \) containing \( S \).

\[\square\]

**Corollary 151.**

(i) The Cartan subgroups are the \( Z_G(T) \), for maximal tori \( T \).

(ii) If a Borel \( B \) contains a maximal torus \( T \), then it contains \( Z_G(T) \).

**Proof.**

(i) follows immediately from the above. For (ii), we have that \( Z_G(T) \) is a Borel of \( Z_G(T) \). But \( Z_G(T) \) is nilpotent (Proposition 145) and so \( Z_G(T) \cap B = Z_G(T) \).

\[\square\]

### 5.5 Conjugacy of parabolic and Borel subgroups.

**Theorem 152.**

(i) If \( B \subset G \) is Borel, then \( N_G(B) = B \).

(ii) If \( P \subset G \) is parabolic, then \( N_G(P) = P \) and \( P \) is connected.

**Proof.**

(i): Induct on the dimension of \( G \). If \( G \) is solvable, then \( B = G \) and we are done; suppose otherwise. Let \( H = N_G(B) \) and \( x \in H \). We want to show that \( x \in B \). Pick a maximal torus \( T \subset B \). Then \( xTx^{-1} \subset B \) is another maximal torus, and so \( T, xTx^{-1} \) are \( B \)-conjugate. Without loss of generality — changing \( x \) modulo \( B \) if necessary — suppose that \( T = xTx^{-1} \). Consider

\[
\phi : T \to T, \quad t \mapsto [x, t] = (xtx^{-1})t^{-1}.
\]

Check that \( \phi \) is a homomorphism. (Use that \( T \) is commutative.)

Case 1. \( \text{im } \phi \neq T \):

Let \( S = (\ker \phi)^0 \), which is a torus and is nontrivial since \( \text{im } \phi \neq T \). \( x \) lies in \( Z = Z_G(S) \) and normalises \( Z \cap B \) (which is a Borel of \( Z \) by Proposition 150). If \( Z \neq G \), then \( x \in Z \cap B \subset B \) by induction. Otherwise, if \( Z = G \), then \( S \subset Z_G \) and \( B/S \subset G/S \) is a Borel by Corollary 127, hence,

\[
[x] \text{ normalises } B/S \implies [x] \in B/S \text{ by induction } \implies x \in B.
\]

Case 2. \( \text{im } \phi = T \):

If \( \text{im } \phi = T \), then

\[
T \subset [x, T] \subset [H, H].
\]
By Corollary 104 there is a $G$-representation $V$ and a line $kv \subset V$ such that $H = \text{Stab}_G(kv)$. Say $h v = \chi(h) v$ for some character $\chi : H \to G_m$. $\chi(T) = \{e\}$ since $T \subset [H, H]$ and $\chi(B_u) = \{e\}$ by Jordan decomposition. Thus, as $B = TB_u$ (Theorem 136), $B$ fixes $v$. By the universal property of quotients, we have a morphism

$$G/B \to V, \quad gB \mapsto gv.$$ 

However, the image of the morphism must be a point, as $V$ is affine, while $G/B$ is complete and connected; hence, $G$ fixes $v$ and $H = G$, i.e., $B \triangleleft G$. Therefore, $G/B$ is affine, complete, and connected, and we must have $G = B$. (In particular, $x \in B$.)

(ii): By Theorem 125, $P \supset B$ for some Borel $B$ of $G$. Suppose $n \in N_G(P)$. Then $nBn^{-1}, B$ are both contained in $P$ and are Borels of $P^0$. Therefore, there must be $g \in P^0$ such that

$$nBn^{-1} = gB^{-1} \implies g^{-1}n \in N_G(B) = B \quad \text{by (i)} \implies n \in gB \subset P^0.$$ 

Hence,

$$P \subset N_G(P) \subset P^0 \subset P.$$ 

\begin{proof}
 Fix a Borel $B$. Any parabolic subgroup is conjugate to a unique parabolic containing $B$.
\end{proof}

\begin{remark}
 For a fixed $B$, the parabolics containing $B$ are called \textbf{standard parabolic subgroups}.
\end{remark}

\begin{example}
 If $G = GL_n$ and $B = B_n$, then the standard parabolic subgroups are the subgroups, for integers $n_i \geq 1$ with $n = \sum_i n_i$, consisting of matrices

$$
\begin{pmatrix}
A_{n_1} & * & * & * \\
* & A_{n_2} & * & * \\
& * & \ddots & * \\
& & & A_{n_m}
\end{pmatrix}
$$

where $A_{n_i} \in GL_{n_i}$.
\end{example}

\begin{proof}[Proof of proposition]
 Let $P$ be a parabolic. $P$ contains some Borel $gBg^{-1}$, so $B \subset g^{-1}Pg$. This takes care of existence. For uniqueness, let $P, Q \supset B$ be two conjugate parabolics; say, $P = gQg^{-1}$.

$$gBg^{-1}, B \subset Q \quad \text{Borels} \quad \implies g^{-1}Bg = qBq^{-1} \text{ for some } q \in Q \implies gq \in N_G(B) = B \implies g \in Bq^{-1} \subset Q \implies P = Q$$

\end{proof}

\begin{proposition}
 If $T$ is a maximal torus and $B$ is a Borel containing $T$, then we have a bijection

$$N_G(T)/Z_G(T) \cong \{\text{Borels containing } T\}$$

$$[u] \mapsto nBn^{-1}$$

\end{proposition}
**Exercise.** If $G = GL_n$, $B = B_n$, and $T = D_n$, we have that $Z_G(T) = T$, $N_G(T) = \text{permutation matrices}$, and that $N_G(T)/Z_G(T) \cong S_n$. When $n = 2$, the two Borels containing $T$ are $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

**Proof of proposition.**
If $B' \supset T$ is a Borel, then

$$B' = gBg^{-1} \text{ for some } g \implies g^{-1}Tg, T \subset B \text{ are maximal tori}$$

$$\implies g^{-1}Tg = bTb^{-1} \text{ for some } b \in B$$

$$\implies n := gb \in N_G(T)$$

$$\implies B' = gBg^{-1} = nBn^{-1}.$$ 

Also,

$$nBn^{-1} = B \iff n \in N_G(B) \cap N_G(T) = B \cap N_G(T) = N_B(T) \supseteq Z_B(T) \supseteq Z_G(T).$$

**Remark 156.** Given a Borel $B \subset G$, we have a bijection

$$G/B \sim \{\text{Borels of } G\}$$

$$gB \mapsto gBg^{-1}$$

The projective variety $G/B$ is called the **flag variety** of $G$ (independent of $B$ up to isomorphism).

**Example.** When $G = GL_n$, $B = B_n$

$$G/B \sim \{\text{flags } 0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = k^n\}$$

$$gB \mapsto g\left(\begin{array}{cccc}
* & & & \\
0 & * & & \\
& 0 & * & \\
& & \cdots & \\
& & & * \\
& & & 0 \\
& & & \\
& & & 0
\end{array}\right) \subsetneq \cdots \subsetneq \left(\begin{array}{cc}
* & \\
0 & *
\end{array}\right) = k^n.$$
6. Reductive groups.

6.1 Semisimple and reductive groups.

Definitions 157. The radical $R_G$ of $G$ is the unique maximal connected, closed, solvable, normal subgroup of $G$. Concretely,

$$R_G = \left( \bigcap_{B \text{ Borel}} B \right)^0$$

(Recall that any two Borels are conjugate.) The unipotent radical of $G$ is the unique maximal connected, closed, unipotent, normal subgroup of $G$:

$$R_u G = (R_G)_u = \left( \bigcap_{B \text{ Borel}} B_u \right)^0$$

$G$ is semisimple if $R_G = 1$ and is reductive if $R_u G = 1$.

Remarks 158.

- $G$ semisimple $\implies$ $G$ reductive
- $G/R_G$ is semisimple and $G/R_u G$ is reductive. (Exercise!)
- If $G$ is connected and solvable, then $G = R_G$ and $G/R_u G = G/G_u$ is a torus. Hence a connected, solvable $G$ is reductive $\iff$ $G$ is a torus.

Example.

- $GL_n$ is reductive. Indeed,

$$R(GL_n) \subset \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \cap \left( \begin{array}{cc} * & 0 \\ * & * \end{array} \right) = D_n \implies R_u(GL_n) = 1$$

Similarly, $SL_n$ is reductive.

- $GL_n$ is not semisimple, as $\{\text{diag}(x,x,\ldots,x) \mid x \in k^\times\} \trianglelefteq GL_n$. $SL_n$ is semisimple by Proposition 159 (iii) below.

Proposition 159. $G$ is connected, reductive.

- (i) $R_G = Z_G^0$, a central torus.
- (ii) $R_G \cap D_G$ is finite.
- (iii) $D_G$ is semisimple.

Remark 160. In fact, $R_G \cdot D_G = G$, so $G = D_G$ when $G$ is semisimple. Hence, by (ii) above, $R_G \times D_G \rightarrow G$ is surjective with finite kernel.
Proof.
(i). $1 = R_uG = (RG)_u \implies RG$ is a torus, by Proposition 133. Hence, by rigidity (Corollary 55) $N_G(RG)^0 = Z_G(RG)^0$. Moreover, since $RG \trianglelefteq G$

$$G = N_G(RG)^0 = Z_G(RG)^0 \implies G = Z_G(RG) \implies RG \subset Z_G^0$$

The reverse inclusion is clear.

(ii). $S := RG$ is a torus. Embed $G \hookrightarrow \text{GL}(V)$. $V$ decomposes as $V = \bigoplus_{\chi \in X(S)} V_\chi$.

$S$ is central $\implies G$ stabilises each $V_\chi$ $\implies G \hookrightarrow \prod_\chi \text{GL}(V_\chi)$

It follows that $DG \hookrightarrow \prod_\chi \text{SL}(V_\chi)$ and $RG$ acts by scalars on each $V_\chi$. Since the scalars in $\text{SL}_n$ are given by the $n$-th roots of unity, the result follows.

(iii).

$$DG \trianglelefteq G \implies R(DG) \subset RG$$

$$\implies R(DG) \subset RG \cap DG,$$ which is finite

$$\implies R(DG) = 1$$

\[\square\]

Definition 161. For a maximal torus $T \subset G$,

$$I(T) := \left( \bigcap_{B \supseteq T} B \right)^0$$

which is a connected, closed, solvable subgroup with maximal torus $T$: $I(T) = I(T)_u \times T$ (see Theorem 136).

Claim:

$$I(T)_u = \left( \bigcap_{B_u \supseteq T} B_u \right)^0$$

Proof.

“$\subset$”: For all Borels $B \supset T$

$$I(T) \subset B \implies I(T)_u \subset B_u \implies I(T)_u \subset \bigcap_{B_u \supseteq T} B_u \implies I(T)_u \subset \left( \bigcap_{B_u \supseteq T} B_u \right)^0$$

as $I(T)_u$ is connected.

“$\supset$”: $\left( \bigcap_{B_u \supseteq T} B_u \right)^0 \subset I(T)$ and consists of unipotent elements. \[\square\]

Remark 162.

$$I(T) \supset \left( \bigcap B \right)^0 = RG \implies I(T)_u \supset R_uG$$

In fact, the converse is true and equality holds.
Theorem 163 (Chevalley). \( I(T)_u = R_u G \). Hence, \( G \) reductive \( \iff \) \( I(T)_u = 1 \) \( \iff \) \( I(T) = T \)

Corollary 164. Let \( G \) be connected, reductive.

(i) \( S \subset G \) subtorus \( \implies \) \( Z_G(S) \) connected, reductive.

(ii) \( T \) maximal torus \( \implies \) \( Z_G(T) = T \).

(iii) \( Z_G \) is the intersection of all maximal tori. (In particular, \( Z_G \subset T \) for all maximal tori \( T \)).

Proof of corollary.

(i): \( Z_G(S) \) is connected by Proposition 150. Let \( T \supset S \) be a maximal torus, so that \( T \subset Z_G(S) =: Z \). Again by Proposition 150

\[
\{ \text{Borels of } Z \text{ containing } T \} = \{ Z \cap B \mid B \supset T \text{ Borel of } G \}
\]

\[
\implies I_Z(T) = \left( \bigcap_{B \supset T} (Z \cap B) \right)^0 \subset I(T)_{\text{[163]}} T
\]

\[
\implies I_Z(T) = T
\]

\[
\implies Z \text{ is reductive, by the theorem}
\]

(ii): \( Z_G(T) \) is reductive by (i) and solvable (as it is a Cartan subgroup, which is nilpotent by Proposition 145). Hence, \( Z_G(T) \) is a torus: \( T = Z_G(T) \), by maximality, since \( T \subset Z_G(T) \).

(iii): \( T \) maximal \( \implies \) \( T = Z_G(T) \supset Z_G \). For the converse, let \( H = \bigcap T_{\text{max}} \), which is a closed, normal subgroup of \( G \) (normal because all maximal tori are conjugate). Since \( H \) is commutative and \( H = H_s \), \( H \) is diagonalisable, and by Corollary 55

\[
G = N_G(H)^0 = Z_G(H)^0 \implies G = Z_G(H) \implies H \subset Z_G
\]

\[\square\]

We will now build up several results in order to prove Theorem 163 following D. Luna’s proof from 1999.

Proposition 165. Suppose \( V \) is a \( G_m \)-representation. \( G_m \) acts on \( PV \). If \( v \in V \setminus \{0\} \), write \([v]\) for its image in \( PV \). Then either, \( G_m \cdot [v] = [v] \), i.e., \( v \) is a \( G_m \)-eigenvector, or \( G_m \cdot [v] \) contains two distinct \( G_m \)-fixed points.

Precise version of the proposition: Write \( V = \bigoplus_{n \in \mathbb{Z} = X^*(G_m)} V_n \), where

\[
V_n = \{ v \in V \mid t \cdot v = t^n v \ \forall t \in G_m \}, \text{ i.e., “} v \text{ has weight } n \text{”}
\]

For \( v \in V \), write \( v = \sum_{n \in \mathbb{Z}} v_n \) with \( v_n \in V_n \). Then

\[
[v_r], [v_s] \in G_m \cdot [v]
\]

where \( r = \min \{ n \mid v_n \neq 0 \} \) and \( s = \max \{ n \mid v_n \neq 0 \} \). Clearly, \([v_r], [v_s]\) are \( G_m \)-fixed. In fact, if \( G_m \cdot [v] \neq [v] \), then

\[
G_m \cdot [v] = \{ G_m \cdot [v] \} \cup \{ [v_r] \} \cup \{ [v_s] \}
\]

\[1\text{See for example P. Polo’s M2 course notes (§21 in Séance 5/12/06) at } \text{www.math.jussieu.fr/~polo/M2}\]
Proof. Pick a basis \(e_0, e_1, \ldots, e_n\) of \(V\) such that \(e_i \in V_{m_i}\). Without loss of generality \(m_0 \leq m_1 \leq \cdots \leq m_n\). Write \(v = \sum_i \lambda_i e_i, \lambda_i \in k\). The orbit map \(f : \mathbb{G}_m \to \mathbb{P}V\) is given by mapping \(t\) to
\[
t \cdot [v] = (t^{m_0} \lambda_0 : t^{m_1} \lambda_1 : \cdots : t^{m_n} \lambda_n) = (0 : \cdots : 0 : \lambda_u : \cdots : t^{m_i-r} \lambda_i : \cdots : t^{s-r} \lambda_v : 0 : \cdots : 0)
\]
where \(u = \min\{i \mid \lambda_i \neq 0\}\) and \(v = \max\{i \mid \lambda_i \neq 0\}\), so that \(m_u = r\) and \(m_v = s\).

Define \(\tilde{f} : \mathbb{P}1 \to \mathbb{P}V\) by
\[
(T_0 : T_1) \mapsto (0 : \cdots : 0 : T_1^{s-r} \lambda_u : \cdots : T_0^{m_i-r} T_1^{s-m_i} \lambda_i : \cdots : T_0^{s-r} \lambda_v : 0 : \cdots : 0)
\]
Check that this a morphism and that \(\tilde{f}|_{\mathbb{G}_m} = f\). (In fact, \(\tilde{f}\) is the unique extension of \(f\), since \(\mathbb{P}V\) is separated and \(\mathbb{G}_m\) is dense.) We have
\[
\tilde{f}(\mathbb{P}1) = \tilde{f}(\mathbb{G}_m) \subset \overline{\tilde{f}(\mathbb{G}_m)} = \mathbb{G}_m \cdot [v]
\]
and
\[
\tilde{f}(0 : 1) = (0 : \cdots : 0 : \cdots : 0) = [v_u] \quad \text{and} \quad \tilde{f}(1 : 0) = \cdots = [v_s]
\]
(In fact, we actually have \(\tilde{f}(\mathbb{P}1) = \mathbb{G}_m \cdot [v]\), using the fact that \(\mathbb{P}1\) is complete). \(\square\)

Informally, above, we have
\[
[v_u] = \lim_{t \to 0} t \cdot [v] \in (\mathbb{P}V)^{\mathbb{G}_m}
\]
\[
[v_s] = \lim_{t \to \infty} t \cdot [v] \in (\mathbb{P}V)^{\mathbb{G}_m}
\]

Lemma 166. Let \(M\) be a free abelian group, and \(M_1, \ldots, M_r \subset M\) subgroups such that each \(M/M_i\) is torsion-free. Then
\[
M \neq M_1 \cup \cdots \cup M_r
\]
Proof. Since \(M/M_i\) is torsion-free, it is free abelian, and
\[
0 \to M_i \to M \to M/M_i \to 0
\]
splits, giving that \(M_i\) is a (proper) direct summand of \(M\). Thus, \(M_i \otimes C \subset M \otimes C\); hence
\[
M \otimes C \neq \bigcup_{i=1}^r M_i \otimes C
\]
as the former is irreducible and the latter are proper closed subsets. \(\square\)

Lemma 167. Let \(T\) be a torus and \(V\) and algebraic representation of \(T\), so that \(T\) acts on \(\mathbb{P}V\). Then, there is a cocharacter \(\lambda : \mathbb{G}_m \to T\) such that \((\mathbb{P}V)^T = (\mathbb{P}V)^{\lambda(\mathbb{G}_m)}\).
Proof. Let \(\chi_1, \ldots, \chi_r \in X^*(T)\) be distinct such that \(V = \bigoplus_{i=1}^r V_{\chi_i}\) and \(V_{\chi_i} \neq 0\) for all \(i\). Then
\[
[v] \in (\mathbb{P}V)^T \iff v \in V_{\chi_i} \text{ for some } i
\]
So it is enough to show that there is a cocharacter \(\lambda\) such that
\[
\forall i \neq j \quad \chi_i \circ \lambda \neq \chi_j \circ \lambda \iff (\chi_i - \chi_j) \circ \lambda \neq 0
\]
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Recall from Proposition 33 we have that
\[ X^*(T) \times X_s(T) \to X^*(G_m) \cong \mathbb{Z}, \quad (\chi, \lambda) \mapsto \chi \circ \lambda \]
is a perfect pairing.

Let \( M = X_s(T) \), which is free abelian, and for all \( i \neq j \)
\[ M_{ij} := \{ \lambda \in X_s(T) \mid \langle \chi_i - \chi_j, \lambda \rangle = 0 \} \neq M \quad (\text{as } \chi_i \neq \chi_j) \]
For \( n > 0 \), if \( n\lambda \in M_{ij} \), then \( \lambda \in M_{ij} \), and so \( M/M_{ij} \) is torsion-free. By the above lemma, \( M \neq \bigcup_{i \neq j} M_{ij} \), so there is a \( \lambda \in M \) such that
\[ \forall i \neq j \quad 0 \neq (\chi_i - \chi_j, \lambda) = (\chi_i - \chi_j) \circ \lambda \]

\[ \square \]

**Theorem 168** (Konstant-Rosenlicht). Suppose that \( G \) is unipotent and \( X \) is an affine \( G \)-space. Then all orbits are closed.

**Proof.** Let \( Y \subset X \) be an orbit. Without loss of generality, we replace \( X \) by \( \overline{Y} \) (which is affine). Since \( Y \) is locally closed and dense, it is open. Let \( Z = X - Y \), which is closed. \( G \) acts (locally-algebraic) on \( k[X] \), preserving \( I_X(Z) \subset k[X] \). \( I_X(Z) \neq 0 \), as \( Z \neq X \). By Theorem 40, since \( G \) is unipotent, it has a nonzero fixed point, say, \( f \) in \( I_X(Z) \). \( f \) is \( G \)-invariant and hence is constant on \( Y \). But then
\[ Y \text{ is dense} \implies f \text{ is constant } (\neq 0) \implies k[X] = I_X(Z) \implies Z = \emptyset \implies Y = X \text{ is closed} \]

\[ \square \]

Now, we want to prove Theorem 163. Fix a Borel \( B \subset G \) and set \( X = G/B \), a homogeneous \( G \)-space. Note that
\[ X^T = \{ gB \mid Tg \subset gB \iff T \subset gBg^{-1} \} \leftrightarrow \{ \text{Borel subgroups containing } T \} \]
Furthermore, by Proposition 155, \( X^T \) in bijection with \( N_G(T)/Z_G(T) \) and hence is finite. Thus \( N_G(T)/Z_G(T) \) acts simply transitively on \( X^T \). For \( p \in X^T \), define
\[ X(p) = \{ x \in X \mid p \in \overline{T x} \} \]

**Proposition 169** (Luna). For \( p \in X^T \), \( X(p) \) is open (in \( X \)), affine, and \( I(T) \)-stable.

**Proof.** By Corollary 104 there exists a \( G \)-representation \( V \) and a line \( L \subset V \) such that \( B = \text{Stab}_G(L) \) and \( \text{Lie } B = \text{Stab}_0(L) \). This gives a map of \( G \)-spaces
\[ i : X = G/B \to PV, \quad g \mapsto gL. \]
\( i \) and \( di \) are injective (Corollary 105); hence, \( i \) is a closed immersion (Corollary 105). Without loss of generality, \( X \subset PV \) is a closed \( G \)-stable subvariety – and, replacing \( V \) by the \( G \)-stable \( \langle G \cdot L \rangle \),
we may also suppose that $X$ is not contained in any $PV' \subset PV$ for any subspace $V' \subset V$.

By Lemma 167, there is a cocharacter $\lambda : G_m \to T$ such that $X^T = X^{G_m}$, considering $X$ and $PV$ as $G_m$-spaces via $\lambda$. For $p \in X^T$, write $p = [v_p]$ for some $v_p \in V_{m(p)}$, $m(p) \in Z$ (weight). Pick $p_0 \in X^T$ such that $m_0 := m(0)$ is minimal. Set $e_0 = v_{p_0}$ and extend $e_0$ to a basis $e_0, e_1, \ldots, e_n$ of $V$ such that $\lambda(t)e_i = t^me_i$. Without loss of generality, $m_1 \leq \cdots \leq m_n$. Let $e_0^*, \ldots, e_n^* \in V^*$ denote the dual basis.

Claim 1. $m_0 < m_1$:

Suppose that $m_0 > m_1$. There is $[v] \in X$ such that $e_1^*(v) \neq 0$ (otherwise $X \subset P(\ker e_1^*) \subset PV$).

Then, by Proposition 165

$$[v_{m_1}] = \lim_{t \to 0} \lambda(t)[v] \in (PV)^{G_m} \cap X = X^T$$

(with the inclusion following from the fact that $X$ is complete). This contradicts the minimality of $m_0$, so we must have $m_0 \geq m_1$.

Suppose that $m_0 = m_1$. Define

$$Z = \{ z \mid \text{there is some point of the form } (1 : z : \cdots) \text{ in } X \}$$

If $(1 : z : \cdots) \in X$, then by Proposition 165 as $m_0 = m_1$,

$$(1 : z : \cdots)' = \lim_{t \to 0} \lambda(t)(1 : z : \cdots) \in X^T.$$  

Since $X^T$ is finite, so too is $Z$. Writing $Z = \{ z_1, \ldots, z_r \}$, we have

$$X \subset P(\ker e_0^*) \cup \bigcup_{i=1}^r P(\ker(e_i^* - z_i e_0^*)).$$  

Since $X$ is irreducible, it is contained in one of these subspaces, which is a contradiction.

Therefore, $m_0 < m_1$.

Claim 2. $X(\lambda, p_0) := \{ x \in X | e_0^*(x) \neq 0 \}$ is open in $X$, affine, and $T$-stable. Also, $X(\lambda, p_0) = X(p_0)$, and it is $I(T)$-stable:

$X(\lambda, p_0) = X \cap (e_0^* \neq 0)$ is open in $X$ and affine (as $(e_0^* \neq 0)$ is open and affine in $PV$). It is $T$-stable, as $e_0^*$ is an eigenvector for $T$ (as $e_0$ is an eigenvector for $T$).

If $x \in X(\lambda, p_0)$, as $m_0 < m_i$ for all $i \neq 0$ (Claim 1),

$$\lim_{t \to 0} \lambda(t)x = [e_0] = p_0.$$

Hence, $p_0 \in G_m \cdot x \subset TX$, so $x \in X(p_0)$. Let $x \in X(p_0)$ and suppose that $e_0^*(x) = 0$. Then

$$p_0 \in TX \subset X - X(\lambda, p_0)$$

with $X - X(\lambda, p_0) T$-stable and closed. This is a contradiction and so we must have $x \in X(\lambda, p_0)$. Hence, $X(\lambda, p_0) = X(p_0)$. 

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To show that the set is $I(T)$-stable, we need to show that from the of $G$ on $P(V^*)$ (which arises from the action on $V^*$), we have
\[ e_0^+ = \{ \ell \in V^* \mid \langle \ell, e_0 \rangle = 0 \} \]
First, let us address a third claim.

**Claim 3.** (i) Each $G$-orbit in $P(V^*)$ intersects the open subset $P(V^*) - P(e_0^+)$ and (ii) $G \cdot [e_0^+]$ is closed in $P(V^*)$: (i): Pick $v \in V^* - \{0\}$. If $G \ell \subset e_0^+$, then for all $g \in G$
\[ 0 = \langle g\ell, e_0 \rangle = \langle \ell, g^{-1}e_0 \rangle. \]
But $Ge_0$ spans $V$ (otherwise, $X = Ge_0 \subset P(V') \subset PV$, which is a contradiction) and so
\[ \langle \ell, V \rangle = 0 \implies \ell = 0 \]
which is another contradiction. Hence, $G[\ell] \not\subset P(e_0^+)$. (ii): $e_i^*$ has weight $-m_i$ under the $G_m$-action and
\[ -m_n \leq \cdots \leq -m_1 < -m_0. \]
Hence by Proposition 165 if $x \in P(V^*) - P(e_0^+)$ then $[e_0^*] \in G_m \cdot x$. So, for all $x \in P(V^*)$, by (i),
\[ [e_0^*] \in Gx \implies G[e_0^*] \subset Gx. \]
If $Gx$ is a closed orbit (which exists), we deduce that it is equal to $G[e_0^+]$.

Let us return to Claim 2, that $X(\lambda, p_0)$ is $I(T)$-stable. Recall that $I(T) = \bigcap_{B' \supset T} B' \big)^0$. Define $P = \text{Stab}_G([e_0^*])$. Since $G/P \rightarrow G[e_0^*]$ is bijective map of $G$-spaces and the latter space is complete (Claim 3), it follows that $P$ is parabolic. Hence, there is a parabolic $B'$ of $G$ contained in $P$. Moreover, since $e_0^*$ is a $T$-eigenvector, $T \subset P$. There is a maximal torus of $B'$ conjugate to $T$ in $P$, so without loss of generality suppose that $T \subset B' \subset P$. It follows that $I(T)$ ($\subset B'$) stabilises $[e_0^*]$ and hence also stabilises the set
\[ X(\lambda, p_0) = \{ x \in X \mid e_0^*(x) \neq 0 \}, \]
completing Claim 2.

Now, $N_G(T)$ acts transitively on $X_T$ by above. If $p \in X_T$, then $p = n p_0$ for some $n \in N_G(T)$; hence $X(p) = nX(p_0)$ is open, affine, and stable under $nI(T)n^{-1} = I(T)$ (equality following from the fact that $n$ permutes the Borels containing $T$).

**Corollary 170.** $\dim X \leq 1 + \dim (X - X(p_0))$

**Proof.** Either $X = X(p_0)$ or otherwise. If equality holds, then $X$ is complete, affine, and connected, and is thus a point. In this case, $\dim X = 0$ and the inequality is true. Suppose that $X \neq X(p_0)$ (i.e., $X(\lambda, p_0)$). Pick $y \in X - X(\lambda, p_0)$. Then $e_0^*(y) = 0$, and $e_i^*(y) \neq 0$ for some $i > 0$. Let
\[ U = \{ x \in X \mid e_i^*(x) \neq 0 \} \subset X, \]

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which is nonempty and open. Define the morphism
\[ f : U \to \mathbb{A}^1, \quad x \mapsto \frac{e_0^*(x)}{e_1^*(x)} \]
\[ f^{-1}(0) \subset X - X(\lambda, p_0). \]
By Corollary 89,
\[ \dim(X - X(\lambda, p_0)) \geq \dim U - \dim f(U) \geq \dim U - 1 = \dim X - 1 \]
\[ \square \]

**Proposition 171** (Luna). \( I(T)_u \) acts trivially on \( X = G/B \).

**Proof.** \( J := I(T)_u \). If \( x \in X \), then \( T \overline{x} \) contains a \( T \)-fixed point by the Borel Fixed Point Theorem; hence
\[ X = \bigcup_{x \in X^T} X(p). \]
Fix \( x \in X \). \( J \) being connected, solvable implies that \( \overline{x} \) contains a \( J \)-fixed point \( y \). By the above, we see that \( y \in X(p) \) for some \( p \in X^T \). If
\[ Jx \cap (X - X(p)) \neq \emptyset, \]
with \( X - X(p) \) closed and \( J \)-stable by Proposition 169, then
\[ y \in \overline{x} \subset X - X(p) \]
which is a contradiction. Hence, \( Jx \subset X(p) \), \( X(p) \) being affine by Proposition 169, and \( J \) being unipotent implies that \( Jx \subset X(p) \) is closed by Kostant-Rosenlicht 168. But
\[ y \in X(p) \cap \overline{x} = Jx \quad (Jx \text{ is closed}) \quad \implies \quad Jx = Jy = y, \quad \text{as } y \text{ is } J\text{-fixed} \]
\[ \implies \quad x = y \text{ is } J\text{-fixed} \]
\[ \implies \quad J \text{ acts trivially on } X. \]
\[ \square \]

**Proof of Theorem 163.**
Let \( J = I(T)_u \) again. We want to show that \( J = R_u G \) and we already know that \( J \supset R_u G \). For the reverse inclusion, we have that for all \( g \in G \),
\[ J(gB) = gB \quad \text{(Theorem 171)} \quad \implies \quad Jg \subset gB \]
\[ \implies \quad J \subset gBg^{-1} \]
\[ \implies \quad J \subset (gBg^{-1})_u, \quad \text{as } J \text{ is unipotent} \]
\[ \implies \quad J \subset \left( \bigcap_{g \in J} (gBg^{-1})_u \right)_0 = R_u G, \quad \text{as } J \text{ is connected} \]
\[ \square \]
6.2 Overview of the rest.

Plan for the rest of the course: Given connected, reductive $G$ (and a maximal torus $T$) we want to show the following:

- $\mathfrak{g} = \text{Lie} T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, under the adjoint action of $T$, where $\Phi \subset X^*(T)$ is finite.
- There is a natural bijection $\Phi \rightarrow \Phi^\vee$, where $\Phi^\vee \subset X_*(T)$ is such that $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a root datum (to be defined shortly).
- For all $\alpha \in \Phi$, there is a unique closed subgroup $U_\alpha \subset G$, normalised by $T$, such that $\text{Lie} U_\alpha = \mathfrak{g}_\alpha$.
- $G = \langle T \cup \bigcup_{\alpha \in \Phi} U_\alpha \rangle$.

From now on $G$ denotes a connected, reductive algebraic group. Fix a maximal torus $T$, so that $\mathfrak{g} = \bigoplus_{\lambda \in X^*(T)} \mathfrak{g}_\lambda$ for the adjoint $T$-action. We write $X^*(T)$ additively, so $\mathfrak{g}_0 = \{ X \in \mathfrak{g} | \text{Ad}(t)X = X \text{ for all } t \in T \} = \text{Lie} \mathbb{Z}_G(T)$ Lie $T = t$

Define $\Phi = \Phi(G,T) := \{ \alpha \in X^*(T) - \{0\} | g_\alpha \neq 0 \}$, which is finite. The $\alpha \in \Phi$ are the roots of $G$ (with respect to $T$). Hence,

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Definition 172. The Weyl group of $(G,T)$ is

$W = W(G,T) := N_G(T)/\mathbb{Z}_G(T)$

which is finite by Corollary 55. $W$ acts faithfully on $T$ by conjugation, and hence acts on $X^*(T)$ and $X_*(T)$:

$$w \in W \mapsto \begin{cases} (w^{-1})^*: X^*(T) \rightarrow X^*(T) \\ w*: X_*(T) \rightarrow X_*(T) \end{cases}$$

Explicitly,

$$w\mu = \mu(\hat{w}^{-1}(\cdot)\hat{w}), \text{ for } \mu \in X^*(T)$$

$$w\lambda = \hat{w}\lambda(\cdot)\hat{w}^{-1}, \text{ for } \lambda \in X_*(T)$$

where $\hat{w} \in N_G(T)$ lifts $w$.

Remarks 173.

- The natural perfect pairing $X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ is $W$-invariant: $\langle w\mu, w\lambda \rangle = \langle \mu, \lambda \rangle$.
- $W$ preserves $\Phi \subset X^*(T)$ because $N_G(T)$ permutes the eigenspaces $\mathfrak{g}_\alpha$. (Check that $\text{Ad}(\hat{w})\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha\hat{w}}$.)

Example. $G = \text{GL}_n$, $T = D_n$.

$\mathfrak{g} = M_n(k)$ and $T$ acts by conjugation.

$$\mathfrak{g} = \left( \begin{array}{cccc} * & & & \\ * & & & \\ & \ddots & & \\ & & * & \\ & & & * \end{array} \right) \oplus \bigoplus_{i,j} \left( \begin{array}{c} * \\ \end{array} \right) \bigoplus_{i \neq j} \left( \begin{array}{c} * \\ \end{array} \right)$$

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where in the summands on the right \( \ast \) appears in the \((i,j)\)-th entry. On the \((i,j)\)-th summand, 

\[
\text{diag}(x_1, \ldots, x_n) \in T
\]

acts as multiplication by \(x_i x_j^{-1}\). Letting \(\epsilon_i \in X^*(T)\) denote \(\text{diag}(x_1, \ldots, x_n) \mapsto x_i\), we get that \(\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j\}\). Also, \(W = \tilde{N}_G(T)/T \cong S_n\) acts by permuting the \(\epsilon_i\).

**Lemma 174.** If \(\phi : H \to H'\) is a surjective morphism of algebraic groups and \(T \subset H\) is a maximal torus, then \(\phi(T) \subset H'\) is a maximal torus.

**Proof.** Pick a Borel \(B \supset T\), so that 

\[
B = B_u \rtimes T
\]

\(\phi(B) = \phi(B_u)\phi(T)\). \(\phi(B)\) is a Borel of \(H'\) by Corollary 127. \(\phi(T)\) is a torus, as it is connected, commutative, and consists of semisimple elements. \(\phi(B_u) \subset \phi(B)_u\) is unipotent (Jordan decomposition). Finally,

\[
\phi(T) \to \phi(B)/\phi(B)_u \text{ bijective (Jordan decomposition)} \implies \dim \phi(T) = \frac{\dim \phi(B)}{\dim(B)_u}
\]

\[
\implies \phi(T) \subset \phi(B) \text{ maximal torus}
\]

\[
\implies \phi(T) \subset H' \text{ maximal torus}
\]

\(\square\)

**Lemma 175.** If \(S \subset T\) be a subtorus, then

\[
Z_G(S) \supseteq T \iff S \subset (\ker \alpha)^0 \text{ for some } \alpha \in \Phi
\]

**Proof.** We always have \(Z_G(S) \supset T\). Note that

\[
\text{Lie } Z_G(S) = Z_g(S) = \{X \in g \mid \text{Ad}(s)(X) = X \text{ for all } s \in S\} = t \oplus \bigoplus_{\alpha \in \Phi, \alpha|_S = 1} g_{\alpha}
\]

\[
\text{\ "\supseteq\" } \iff \text{Lie } Z_G(S) \supseteq t, \text{ by dimension considerations}
\]

\[
\iff t \oplus \bigoplus_{\alpha \in \Phi, \alpha|_S = 1} g_{\alpha} \supseteq t
\]

\[
\iff S \subset \ker \alpha, \text{ for some } \alpha \in \Phi
\]

\(\square\)

For \(\alpha \in \Phi\), define \(T_\alpha := (\ker \alpha)^0\), which is a torus of dimension \(\dim T - 1\), as \(\text{im } \alpha = G_m\). Define 

\[
G_\alpha := Z_G(T_\alpha), \text{ which is connected, reductive by Corollary 164}
\]

Note that

\[
T_\alpha \subset Z^0_{G_\alpha} = R(G_\alpha)
\]

Let \(\pi\) denote the natural surjection \(G_\alpha \to G_\alpha/R(G_\alpha)\). By Lemma 174, \(\pi(T)\) is a maximal torus of \(G_\alpha/R(G_\alpha)\).

\[
T_\alpha \subset R(G_\alpha) \implies T/T_\alpha \to \pi(T) \implies \dim \pi(T) \leq 1
\]

If \(\dim \pi(T) = 0\), then

\[
T \subset R(G_\alpha) \subset Z_{G_\alpha} \implies G_\alpha \subset Z_G(T) = T
\]

which is a contradiction by Lemma 175. Hence, \(\dim \pi(T) = 1\).
Definitions 176.

the **rank** of $G = \text{rk } G := \dim T$, where $T$ is a maximal torus

the **semisimple rank** of $G = \text{ss-rk } G := \text{rk}(G/\text{RG})$

Hence, $\text{ss-rk } G = 1$. Note that since all maximal tori are conjugate, rank is well-defined, and that $\text{ss-rk } G \leq \text{rk } G$ by Lemma 174.

**Example.** $G = \text{GL}_n$, $\alpha = \epsilon_i - \epsilon_{i+1}$. We have

$$T_\alpha = \{\text{diag}(x_1, \ldots, x_n) \mid x_i = x_{i+1}\}$$

and

$$G_\alpha = D_{i-1} \times \text{GL}_2 \times D_{n-i-1}. $$

$G_\alpha/\text{RG}_\alpha \cong \text{PGL}_2$ and $\mathcal{D}G_\alpha \cong \text{SL}_2$.

### 6.3 Reductive groups of rank 1.

**Proposition 177.** Suppose that $G$ is not solvable and $\text{rk } G = 1$. Pick a maximal torus $T$ and a Borel $B$ containing $T$. Let $U = Bu$.

(i) $\#W = 2$, $\dim G/B = 1$, and $G = B \sqcup \text{Un}B$, where $n \in N_G(T) - T$.

(ii) $\dim G = 3$ and $G = \mathcal{D}G$ is semisimple.

(iii) $\Phi = \{\alpha, -\alpha\}$ for some $\alpha \neq 0$, and $\dim \mathfrak{g}_{\pm \alpha} = 1$.

(iv) $\psi : U \times B \rightarrow \text{Un}B$, $(u, b) \mapsto unb$, is an isomorphism of varieties.

(v) $G \cong \text{SL}_2$ or $\text{PGL}_2$.

**Remark 178.** In either case, $G/B \cong \mathbb{P}^1$. For example,

$$\text{SL}_2 / \left( \begin{array}{cc} & * \\ * & \end{array} \right) \cong \mathbb{P}^1, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto (a : c)$$

**Proof of proposition.**

(i):

$$W \hookrightarrow \text{Aut}(X^*(T)) \cong \text{Aut}(\mathbb{Z}) = \{\pm 1\} \implies \#W \leq 2$$

If $W = 1$, then $B$ is the only Borel containing $T$, and so by Theorem 163

$$B = I(T) = T \implies B \text{ nilpotent} \implies G \text{ solvable}$$

which contradicts our hypothesis; hence, $\#W = 2$.

Set $X := G/B$. $\dim X > 0$ since $B \neq G$. By Proposition 155 we have $\#X^T = \#W = 2$. By Corollary 170

$$\dim X \leq 1 + \dim(X - X(p_0))$$

Since $X - X(p_0)$ is $T$-stable and closed (Proposition 169), it can contain at most one $T$-fixed point (as $\#X^T = 2, p_0 \in X(p_0)$). By Proposition 165, $T$ acts trivially and so $X - X(p_0)$ is finite:

$$\dim X \leq 1.$$
Now,
\[ \#W = 2 \implies B, nBn^{-1} \text{ are the two Borels containing } T \]
\[ \implies X^T = \{x, nx\}, \text{ where } x := B \in G/B \]
We want to show that \( X = \{x\} \sqcup Unx \), which will imply that \( G = B \sqcup UnB \). Note that \( x \) is \( U \)-fixed, so \( \{x\} \) and \( Unx \) are disjoint (as \( x \neq nx \)). Also, \( Unx \) is \( T \)-stable, as
\[ TUnx = UTnx = UnTx = Unx, \]
and \( Unx \neq \{nx\} \), as otherwise
\[ \{nx\} = Unx = Bnx \implies \{x\} = n^{-1}Bnx \implies n^{-1}Bn \subset \text{Stab}_G(x) = B \implies \text{contradiction} \]
Hence, \( Unx = X \), by dimension considerations, so \( Unx \subset X \) is open, \( X - Unx \) is finite (as \( \dim X = 1 \)), and \( X - Unx \) is \( T \)-stable. \( T \) is connected and so \( U - Unx \subset X^T = \{x, nx\} \implies X - Unx = \{x\} \)

(ii):
\[ 1 = \dim Unx \]
\[ = \dim U - \dim(U \cap nUn^{-1}), \text{ as } Unx \text{ is a } U \text{-orbit} \]
\[ = \dim U, \text{ as } U \cap nUn^{-1} = \text{Stab}_U(nx) \text{ is finite by Theorem 163} \]

Hence,
\[ \dim B = \dim T + \dim U = 1 + 1 = 2 \]
\[ \dim G = \dim B + \dim(G/B) = 2 + 1 = 3 \]

\( DG \) is semisimple by Proposition 159 and is not solvable (as \( G \) is not). \( \text{rk } DG \leq \text{rk } G = 1 \). If \( \text{rk } DG = 0 \), then a Borel of \( DG \) is unipotent, which by Proposition 130 implies that \( DG \) is solvable: contradiction. (Or, \( T_1 = \{1\} \) is a maximal torus and \( T_1 = \mathcal{Z}_{DG}(T_1) = DG \): contradiction.) Hence, \( \text{rk } DG = 1 \), so \( \dim DG = 3 \) by the above: \( DG = G \).

(iii): \( g = t \oplus \bigoplus_{\alpha \in \Phi} g_\alpha \). Since \( \dim g = 3 \) and \( \dim t = 1 \), we have \( \#\Phi = 2 \). Moreover, \( \Phi \) is \( W \)-stable and \( [n] \in W \) acts by \(-1\) on \( X^*(T) \), and so \( \Phi = \{\alpha, \alpha\} \) for some \( \alpha: \dim g_{\pm \alpha} = 1 \). From \( B = U \times T \) we have \( \text{Lie } B = t \oplus \text{Lie } U \) and \( \text{Lie } U = g_\alpha \) or \( g_{-\alpha} \), as \( \text{Lie } U \) is a \( T \)-stable subspace of \( g \) of dimension 1. Without loss of generality, \( \text{Lie } U - g_\alpha \). Likewise,
\[ nBn^{-1} = nUn^{-1} \times T \implies \text{Lie } (nBn^{-1}) = t \oplus \text{Lie } (nUn^{-1}) \]
Since \( \text{Lie } (nUn^{-1}) = \text{Ad}(n)(\text{Lie } U) \) and \( [n] \in W \) acts as \(-1\) on \( X^*(T) \), \( \text{Lie } (nUn^{-1}) = g_{-\alpha} \).

(iv). This is a surjective map of homogeneous \( U \times B \) spaces.
\[ \text{unb} = n \implies u \in U \cap nBn^{-1} = U \cap nUn^{-1}, \text{ which is finite by Theorem 163} \]
\[ \implies U \cap nUn^{-1} = 1, \]
(as \( T \), being connected, acts trivially by conjugation \( \implies U \cap nUn^{-1} \subset Z_G(T) = T \))
\[ \implies \psi \text{ is injective, hence bijective} \]

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\[ d\phi \text{ bijective} \iff d\left( U \times B \rightarrow UnBn^{-1} \right) \text{ injective} \]
\[ \iff d(U \times (nBn^{-1}) \overset{\text{mult.}}{\rightarrow} UnBn^{-1}) \text{ injective} \]
\[ \iff 0 = \text{Lie } U \cap \text{Lie } (nBn^{-1}) = g_\alpha \cap (t \oplus g_{-\alpha}) \]

(v). See Springer 7.2.4. \qedhere

6.4 Reductive groups of semisimple rank 1.

**Lemma 179.** If \( \phi : H \rightarrow K \) is a morphism of algebraic groups, then
\[ d\phi(Ad(h) \cdot X) = Ad(\phi(h)) \cdot d\phi X \]

**Proof.** Exercise. (Easy!) \qedhere

**Proposition 180.** Suppose that ss-rk \( G = 1 \). Set \( \overline{G} := G/RG \) and \( \overline{T} := \text{image of } T \text{ in } \overline{G} \) (\( T \) being a maximal torus). Note that \( X^*(\overline{T}) \subset X^*(T) \) as \( T \rightarrow \overline{T} \).

(i) There is \( \alpha \in X^*(\overline{T}) \) such that \( g = t \oplus g_\alpha \oplus g_{-\alpha} \), and \( \dim g_{\pm\alpha} = 1 \).

(ii) \( \mathcal{DG} \cong \text{SL}_2 \) or \( \text{PGL}_2 \)

(iii) \( \#W = 2 \), so there are precisely two Borels containing \( T \), and, if \( B \) is one, then
\[ \text{Lie } B = t \oplus g_{\pm\alpha} \text{ and } \text{Lie } B_u = g_{\pm\alpha} \]

(iv) If \( T_1 \) denotes the unique maximal torus of \( \mathcal{DG} \) contained in \( T \), then \( \exists! \alpha^\vee \in X_*(T_1) \subset X_*(T) \) such that \( \langle \alpha, \alpha^\vee \rangle = 2 \). Moreover, letting \( W = \{1, s_\alpha\} \), we have
\[ s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha \text{ for all } \mu \in X^*(T) \]
\[ s_\alpha \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \text{ for all } \lambda \in X_*(T) \]

**Proof.**

(i): \( \overline{G} \) is semisimple of rank 1.

We have
\[ 0 \rightarrow \text{Lie } RG \rightarrow \text{Lie } G \rightarrow \text{Lie } \overline{G} \rightarrow 0 \]

From Lemma 179, restricting actions, we have that the morphisms \( T \rightarrow \overline{T} \) and \( \text{Lie } G \rightarrow \text{Lie } \overline{G} \) are compatible with the action of \( T \) on \( \text{Lie } G \) and \( \overline{T} \) on \( \text{Lie } \overline{G} \). \( T \) acts trivially on \( \text{Lie } RG \) (as \( RG \subset T \)). Thus,
\[ \Phi = \Phi(\overline{G}, \overline{T}) = \{\alpha, -\alpha\} \subset X^*(\overline{T}) \subset X^*(T) \]

and \( \dim g_{\pm\alpha} = 1 \).
(ii): \( DG \) is semisimple by Proposition 159. If \( T_1 \subset DG \) is a maximal torus with image \( \overline{T}_1 \) in \( \overline{G} \), then
\[
\dim T_1 = \dim \overline{T}_1 + \dim(T_1 \cap RG) \leq 1
\]
the inequality being due to the fact that \( T_1 \cap RG \subset DG \cap RG \) is finite by Proposition 159. If \( \dim T_1 = 0 \), then the Borel of \( DG \) is unipotent, implying that \( DG \) is solvable, which gives that \( G \) is solvable, a contradiction. Hence, \( \text{rk} \ DG = 1 \). By Proposition 177, \( DG \cong \text{SL}_2 \) or \( \text{PGL}_2 \).

(iii): First a lemma.

**Lemma 181.** Suppose that \( \pi : G \to G' \) with \( \ker \pi \) connected and solvable. Then \( \pi(T) \) is a maximal torus of \( G' \) and we have a bijection
\[
\{ \text{Borels of } G \text{ containing } T \} \overset{\pi}{\longleftrightarrow} \{ \text{Borels of } G' \text{ containing } \pi(T) \}
\]
Moreover, \( G' \) is reductive.

**Proof of lemma.** In the proposed bijection, \( \pi \to \) is well-defined by Corollary 127. For the inverse, note that \( G/\pi^{-1}(B') \to G'/B' \) is bijective, which gives that \( \pi^{-1}(B') \) is parabolic as well as connected and solvable (\( \ker \pi \) and \( B' \) are connected and solvable).

\( \pi^{-1}(RG') \) is a connected, solvable, normal subgroup of the torus \( RG \). \( RG' = \pi(\pi^{-1}(RG')) \) is then a torus and so \( G' \) is reductive.

By the Lemma, \( \#W = \#W(G, T) = 2 \). Pick a Borel \( B \supset T \), so that \( \overline{B} \supset T \) is a Borel.

\[
1 \to RG \to B \to \overline{B} \to 1
\]
being exact implies that
\[
0 \to \text{Lie } RG \to \text{Lie } B \to \text{Lie } \overline{B} \to 0
\]
is also exact. \( T \) again acts trivially on \( \text{Lie } RG \).

\[
\text{Lie } \overline{B} = \text{Lie } T \oplus \mathfrak{g}_{\pm \alpha} \implies \text{Lie } B = t \oplus \mathfrak{g}_{\pm \alpha}.
\]

Also,
\[
\text{Lie } B = t \oplus \text{Lie } B_u \implies \text{Lie } B_u = \mathfrak{g}_{\pm \alpha}
\]
(iv) \( T_1 \) exists, as \( DG \trianglelefteq G \) (exercise). It is unique, as \( T_1 = (T \cap DG)^0 \). (Another exercise: \( T_1 = T \cap DG \). Use that \( DG \) is reductive.) Let \( y \) be a generator of \( X_*(T) \cong \mathbb{Z} \). We have the containment
\[
\text{Lie } DG \subset \mathfrak{g} = t \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}
\]
with \( T_1 \) acting in the former and \( T \) on the latter. \( DG \) being reductive implies – by Proposition 177 –
\[
\Phi(DG, T_1) = \{ \pm \alpha |_{T_1} \}.
\]

\( DG \cong \text{SL}_2 \):

\[
T_1 = \left\{ \begin{pmatrix} x & 1 \\ -1 & x^{-1} \end{pmatrix} \mid x \in k^\times \right\} \subset \text{SL}_2.
\]
By the adjoint action (conjugation), $T_1$ acts on
\[
\text{Lie SL}_2 = M_2(k)_{\text{trace 0}} = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
Its roots are
\[
\alpha : \begin{pmatrix} x \\ x^{-1} \end{pmatrix} \mapsto x^2, \quad -\alpha : \begin{pmatrix} x \\ x^{-1} \end{pmatrix} \mapsto x^{-2}.
\]
Moreover, we can take
\[
y = x \mapsto \begin{pmatrix} x \\ x^{-1} \end{pmatrix}
\]
(or its inverse), which gives
\[
\langle \alpha, y \rangle = \pm 2.
\]

$DG \cong PGL_2 \cong GL_2/G_m$: $T_1$ is equal to the image of $D_2$ in $PGL_2$. By the adjoint action, $T_1$ acts on
\[
\text{Lie PGL}_2 = M_2(k)/k = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus k\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
Its roots are
\[
\alpha : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1x_2^{-1}, \quad -\alpha : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_1x_2^{-1})^{-1} = x_1^{-1}x_2.
\]
Moreover, we can take
\[
y = x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix}
\]
(or its inverse), which gives
\[
\langle \alpha, y \rangle = \pm 1.
\]
Therefore, in any case,
\[
\alpha^\vee := \frac{2y}{\langle \alpha, y \rangle} \in X_*(T_1)
\]
and it is the unique cocharacter such that $\langle \alpha, \alpha^\vee \rangle = 2$.

If $\lambda \in X_*(T)$,
\[
s_\alpha \lambda - \lambda : G_m \to T, \quad x \mapsto [n, \lambda(x)] = n\lambda(x)n^{-1}\lambda(x)^{-1},
\]
where $n \in N_G(T)$ is such that $[n] = s_\alpha$. $s_\alpha \lambda - \lambda$ has image in $(T \cap DG)^0 = T_1$; hence
\[
s_\alpha \lambda - \lambda \in X_*(T_1) = \mathbb{Z}y.
\]
Say $s_\alpha \lambda - \lambda = \theta(\lambda)y$. We have
\[
\theta(\lambda)\langle \alpha, y \rangle = \langle \alpha, s_\alpha \lambda - \lambda \rangle = \langle \alpha, s_\alpha \lambda \rangle - \langle \alpha, \lambda \rangle
\]
\[
= \langle s_\alpha(\alpha), \lambda \rangle - \langle \alpha, \lambda \rangle.
\]
At this point we see that $s_{\alpha}(\alpha) = -\alpha$. (Otherwise, $s_{\alpha}(\alpha) = \alpha$, which implies $\theta = 0$, i.e. that $s_{\alpha}$ acts trivially on $X_*(T)$, which is a contradiction.) So we can continue:

$$= \langle -\alpha, \lambda \rangle - \langle \alpha, \lambda \rangle$$

$$= -2\langle \alpha, \lambda \rangle$$

Therefore,

$$\theta(\lambda) = \frac{-2\langle \alpha, \lambda \rangle}{\langle \alpha, y \rangle}$$

and

$$s_{\alpha}\lambda = \lambda + \theta(\lambda)y = \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, y \rangle}y = \lambda - \langle \alpha, \lambda \rangle\alpha^\vee.$$

If $\mu \in X^*(T)$, then for all $\lambda \in X_*(T)$

$$\langle s_{\alpha}\mu, \lambda \rangle = \langle \mu, s_{\alpha}\lambda \rangle = \langle \mu, \lambda \rangle - \langle \alpha, \lambda \rangle\langle \mu, \alpha^\vee \rangle = \langle \mu - \langle \mu, \alpha^\vee \rangle\alpha, \lambda \rangle$$

and so

$$s_{\alpha}\mu = \mu - \langle \mu, \alpha^\vee \rangle\alpha.$$

Lemma 182.

(i) Let $S \subset T$ be a subtorus such that $\dim S = \dim T - 1$. Then

$$\ker(\text{res} : X^*(T) \to X^*(S)) = \mathbb{Z}\mu$$

for some $\mu \in X^*(T)$.

(ii) If $\nu \in X^*(T)$, $m \in \mathbb{Z} - \{0\}$, then $(\ker \nu)^0 = (\ker m\nu)^0$.

(iii) If $\nu_1, \nu_2 \in X^*(T) - \{0\}$, then

$$(\ker \nu_1)^0 = (\ker \nu_2)^0 \iff m\nu_1 = n\nu_2$$

for some $m, n \in \mathbb{Z} - \{0\}$.

Proof.

(i): res is surjective (exercise, cf. the proof of Proposition 47) and

$$X^*(T) \cong \mathbb{Z}^r, \; X^*(S) \cong \mathbb{Z}^{r-1}.$$

(ii):

“$\subset$”: $\nu(t) = 1 \implies \nu(t)^n = 1$.

“$\supset$”: $t \in (\ker m\nu)^0 \implies \nu(t)^n = 1$, so $\nu((\ker m\nu)^0)$ is connected and finite, hence trivial.

(iii):

“$\leftarrow$”: Clear from (ii).

“$\Rightarrow$”: Define $S = (\ker \nu_1)^0 = (\ker \nu_2)^0 \subset T$, as in (i). Clearly, $\text{res}(\nu_1) = \text{res}(\nu_2) = 0$, so $\nu_i \in \mathbb{Z}\mu$.

The result follows.
6.5 Root data.

Definitions 183. A root datum is a quadruple \((X, \Phi, X^\vee, \Phi^\vee)\), where

(i) \(X, X^\vee\) are free abelian groups of finite rank with a perfect bilinear pairing \(\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}\)

(ii) \(\Phi \subset X\) and \(\Phi^\vee \subset X^\vee\) are finite subsets with a bijection \(\Phi \to \Phi^\vee, \alpha \mapsto \alpha^\vee\)

(these pairing and the bijection also being part of the root datum) satisfying the following axioms:

(1) \(\langle \alpha, \alpha^\vee \rangle = 2\) for all \(\alpha \in \Phi\)

(2) \(s_\alpha(\Phi) = \Phi\) and \(s_\alpha(\Phi^\vee) = \Phi^\vee\) for all \(\alpha \in \Phi\)

where the “reflections” are given by

\[ s_\alpha : X \to X \quad \text{and} \quad s_\alpha : X^\vee \to X^\vee \]

\[ x \mapsto x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad y \mapsto y - \langle \alpha, y \rangle \alpha^\vee \]

A root datum is reduced if \(Q_\alpha \cap \Phi = \{ \pm \alpha \}\) for all \(\alpha \in \Phi\).

Remark 184. Note that the axioms imply that \(s_\alpha(\alpha) = -\alpha\), so \(\Phi = -\Phi\), and \(s_\alpha^2 = 1\) (so \(s_\alpha\) is a group isomorphism). Similarly, \(s_\alpha(\alpha^\vee) = -\alpha^\vee\), so \(\Phi^\vee = -\Phi^\vee\), and \(s_\alpha^2 = 1\). Also \(0 \notin \Phi\) and \(0 \notin \Phi^\vee\), and \(\langle s_\alpha(\mu), s_\alpha(\lambda) \rangle = \langle \mu, \lambda \rangle\). (It is less obvious from the axioms, but also true, that \(-\alpha^\vee = -\alpha^\vee\) and hence that \(s_{-\alpha} = s_\alpha\). For more on root data, see SGA3, Exposé XXI.)

Recall that \(g = t \oplus \bigoplus_{\alpha \in \Phi} g_\alpha\), \(T_\alpha = (\ker \alpha)^0\), \(G_\alpha = Z_G(T_\alpha)\).

Theorem 185.

(i) For all \(\alpha \in \Phi\), \(G_\alpha\) is connected, reductive of semisimple rank 1.

- \(\text{Lie } G_\alpha = t \oplus g_\alpha \oplus g_{-\alpha}\)
- \(\dim g_{\pm \alpha} = 1\)
- \(Q_\alpha \cap \Phi = \{ \pm \alpha \}\)

(ii) Let \(s_\alpha\) be the unique nontrivial element of \(W(G_\alpha, T) \subset W(G, T)\). Then there exists a unique \(\alpha^\vee \in X_*(T)\) such that \(\text{im } \alpha^\vee \subset D G_\alpha\) and \(\langle \alpha, \alpha^\vee \rangle = 2\). Moreover,

\[ s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \text{for all } \mu \in X^*(T), \]

\[ s_\alpha \lambda = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee, \quad \text{for all } \lambda \in X_*(T). \]

(iii) Let \(\Phi^\vee = \{ \alpha^\vee \mid \alpha \in \Phi \}\). Then \((X^*(T), \Phi, X_*(T), \Phi^\vee)\) is a reduced root datum.

(iv) \(W(G, T) = \langle s_\alpha \mid \alpha \in \Phi \rangle\).

Proof.

(i). We saw above that \(G_\alpha\) is connected, reductive of semisimple rank 1.

\[ \text{Lie } G_\alpha = \text{Lie } Z_G(T_\alpha) \cong \mathfrak{g}(T_\alpha) = t \oplus \bigoplus_{\beta \in \Phi, \beta |_{T_\alpha} = 1} g_\beta \]
By Proposition \[180\],

\[
\text{Lie} G_\alpha = t \oplus g_\alpha \oplus g_{-\alpha}
\]

with \( \dim g_{\pm \alpha} = 1 \). Hence, for all \( \beta \in \Phi \),

\[
\beta|_{T_\alpha} = 1 \iff \beta \in \{\pm \alpha\}
\]

\[
\iff (\ker \alpha)^0 \subset (\ker \beta)^0
\]

\[
\iff (\ker \alpha)^0 = (\ker \beta)^0 \quad \text{(dimension considerations)}
\]

\[
\iff \beta \in Q_\alpha \quad \text{(Lemma \[182\])}
\]

(ii): Follows from Proposition \[180\]

(iii):

\( \alpha \mapsto \alpha^\vee \) is bijective ( \( \iff \) injective):

If \( \alpha^\vee = \beta^\vee \), then

\[
s_\alpha s_\beta(x) = (x - \langle x, \beta^\vee \rangle \beta) - \langle (x - \langle x, \beta^\vee \rangle \beta), \alpha^\vee \rangle \alpha
\]

\[
= x - \langle x, \alpha^\vee \rangle (\alpha + \beta) + \langle x, \alpha^\vee \rangle \beta, \beta^\vee \rangle \alpha
\]

\[
= x - \langle x, \alpha^\vee \rangle (\alpha + \beta) + 2\langle x, \alpha^\vee \rangle \alpha
\]

\[
= x + \langle x, \alpha^\vee \rangle (\alpha - \beta)
\]

Therefore, if \( \langle \alpha - \beta, \alpha^\vee \rangle = 0 \), then for some \( n \)

\[
(s_\alpha s_\beta)^n = 1 \implies \forall x, \quad x = (s_\alpha s_\beta)^n(x) = x + n\langle x, \alpha^\vee \rangle (\alpha - \beta)
\]

\[
\implies \forall x, \quad 0 = n\langle x, \alpha^\vee \rangle (\alpha - \beta)
\]

\[
\implies \alpha = \beta
\]

\[s_\alpha \Phi = \Phi;\]

The action of \( s_\alpha \in W \) on \( X^*(T) \) (and \( X_*(T) \)) agrees with the action of \( s_\alpha \) (and \( s_\alpha^\vee \)) in the definition of a root datum by (ii). Also, as noted above, \( W = N_G(T)/T \) preserves \( \Phi. \)

\[s_\alpha^\vee \Phi^\vee = \Phi^\vee;\]

For \( w = [n] \in W, \ (n \in N_G(T)), \ \beta \in \Phi \)

\[
w\beta(\cdot) = \beta(n^{-1}(\cdot)n) \implies \ker(w\beta) = n(\ker \beta)n^{-1} \implies T_{w\beta} = nT_\beta n^{-1}, G_{w\beta} = nG_\beta n^{-1}
\]

From

\[
\text{im} (w(\beta^\vee)) = \text{im} (n\beta^\vee n^{-1}) \subset nDG_\beta n^{-1} = DG_{w\beta}
\]

and

\[
\langle w\beta, w(\beta^\vee) \rangle = \langle \beta, \beta^\vee \rangle = 2
\]

by (ii), we have that \( (w\beta)^\vee = w(\beta^\vee) \). (iii) follows.

(iv): Induct on \( \dim G. \) Let \( w = [n] \in W, \ n \in N_G(T) \). As in the proof of Theorem \[152\] consider the homomorphism

\[
\phi : T \to T, \quad t \mapsto [t, n] = ntn^{-1} t^{-1}.
\]
If $\mathcal{G}(S) = G$, then $S \subset \mathcal{Z}_G$. Define $\overline{G} = G/S$, which is reductive by Lemma 181 and $\overline{T} = T/S$, which is a maximal torus of $\overline{G}$. By induction, the (iv) holds for $\overline{G}$.

It is an easy check that we have

$$N_G(T)/T = W(G, T) \twoheadrightarrow W(\overline{G}, \overline{T}) = N_{\overline{G}}(\overline{T})/\overline{T}$$

restricting to

$$W(G_\alpha, T) \twoheadrightarrow W(\overline{G}_\alpha, \overline{T}), \ s_\alpha \mapsto s_\alpha.$$

Therefore, (iv) follows for $\overline{G}$.

For $G$, we define $\tilde{\Phi}$ as the set of roots $\alpha$ such that $\alpha|_S = 1$. This is a root system in $G$. Theorem 185 does not depend (up to isomorphism) on the choice of $T$, as any two maximal tori are conjugate.

Facts:
1. Isomorphism Theorem: Two connected reductive groups are isomorphic \( \iff \) their root data are isomorphic.

2. Existence Theorem: Given a reduced root datum, there exists a reductive group that has the root datum.

(See Springer §9, §10.)

**Theorem 187.**

(i) For all \( \alpha \in \Phi \) there is a unique connected closed \( T \)-stable unipotent subgroup \( U_\alpha \subset G \) such that \( \text{Lie } U_\alpha = \mathfrak{g}_\alpha \). There exists an isomorphism \( u_\alpha : G_\alpha \rightarrow U_\alpha \) (unique up to scalar) such that

\[
 tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \quad \text{for all } x \in G_\alpha, t \in T.
\]

(ii) \( G = \langle T, U_\alpha (\alpha \in \Phi) \rangle \) (i.e., \( G \) is the smallest subgroup containing \( T \) and all of the \( U_\alpha \))

(iii) \( Z_G = \bigcap_{\alpha \in \Phi} \ker \alpha \)

**Proof.**

(i): Let \( B_\alpha \) denote the Borel subgroup of \( G_\alpha \) containing \( T \) with \( \text{Lie } B_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \) (Proposition 180, Theorem 185). Then \( U_\alpha := (B_\alpha)_u \) satisfies all assumptions by Proposition 180. Also, \( \dim U_\alpha = \dim \mathfrak{g}_\alpha = 1 \) and \( U_\alpha \cong G_\alpha \) by Theorem 60. Let \( u_\alpha : G_\alpha \rightarrow U_\alpha \) denote any isomorphism; any other differs by a scalar as \( \text{Aut } G_\alpha \cong \mathbb{G}_m \). So \( tu_\alpha(x)t^{-1} = u_\alpha(\chi(t)x) \) for some \( \chi(t) \in k^\times \). Via \( u_\alpha \), identify \( U_\alpha \overset{\partial(t)}{\rightarrow} U_\alpha \) with \( G_\alpha \overset{\chi(t)}{\rightarrow} G_\alpha \), the derivative of the latter is \( k \overset{\alpha(t)}{\rightarrow} k \). However, by Theorem 78 we must have \( \alpha(t) = \chi(t) - \text{ and thus } \alpha = \chi \).

It remain to show that \( U_\alpha \) is unique. If \( U'_\alpha \) is another connected, closed, \( T \)-stable, and unipotent with \( \text{Lie } U'_\alpha = \mathfrak{g}_\alpha \), by the same argument as above we get an isomorphism \( u'_\alpha : G_\alpha \rightarrow U'_\alpha \) such that \( tu'_\alpha(x)t^{-1} = u'_\alpha(\alpha(t)x) \). Hence, \( U'_\alpha \subset G_\alpha \) (as \( \alpha(T_\alpha) = 1 \)).

\[
 T \text{ normalises } U'_\alpha \implies TU'_\alpha \text{ is closed, connected, and solvable (exercise)}
\]

\[
 \implies TU'_\alpha \text{ is contained in a Borel containing } T
\]

\[
 \implies TU'_\alpha \subset B_\alpha, \quad \text{as } \text{Lie } U'_\alpha = \mathfrak{g}_\alpha
\]

\[
 \implies U'_\alpha = (TU'_\alpha)_u \subset (B_\alpha)_u = U_\alpha
\]

\[
 \implies U'_\alpha = U_\alpha \text{ (dimension)}
\]

(ii): By Corollary 21, \( \langle T, U_\alpha (\alpha \in \Phi) \rangle \) is connected, closed. Its Lie algebra contains \( \mathfrak{t} \) and all of the \( \mathfrak{g}_\alpha \), hence coincides with \( \mathfrak{g} \). Thus

\[
 \dim(T, U_\alpha (\alpha \in \Phi)) = \dim \mathfrak{g} = \dim G \implies \langle T, U_\alpha (\alpha \in \Phi) \rangle = G
\]

(iii): \( Z_G \subset T \) by Corollary 164 By (i), \( t \in T \) commutes with \( U_\alpha \iff \alpha(t) = 1 \), which implies that \( Z_G \subset \bigcap_{\alpha} \ker \alpha \). The reverse inclusion follows by (ii).
Appendix. An example: the symplectic group

Set \( G = \text{Sp}_{2n} = \{ g \in \text{GL}_{2n} | g^t J g = J \} \), where \( J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \).

**Fact.** \( G \) is connected. (See, for example, Springer 2.2.9(1) or Borel 23.3.2)

Define

\[
T = G \cap D_{2n} = \{ \text{diag}(x_1, \ldots, x_{2n}) | \text{diag}(x_1, \ldots, x_{2n}) \cdot \text{diag}(x_{n+1}, \ldots, x_{2n}, x_1, \ldots, x_n) = I \}
\]

\[
= \{ \text{diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) \}
\]

\[
\cong G^n
\]

Clearly \( Z_G(T) = T \), implying that \( T \) is a maximal torus and \( \text{rk} \ G = n \). Write \( \epsilon_i, 1 \leq i \leq n \), for the morphisms

\[
T \to G_m, \quad \text{diag}(x_1, \ldots, x_n^{-1}) \mapsto x_i,
\]

which form a basis of \( X^*(T) \).

**Lemma 188.** If \( \rho : G \to \text{GL}(V) \) is a faithful (or just injective) \( G \)-representation that is semisimple, then \( G \) is reductive.

**Proof.**

\( U := R_u G \) is a connected, unipotent, normal subgroup of \( G \). Write \( V = \bigoplus_{i=1}^r V_i \) with \( V_i \) irreducible (\( V \) is semisimple). \( V_i^U \neq 0 \), as \( U \) is unipotent (Proposition 40), and \( V_i^U \subset V_i \), is \( G \)-stable, as \( U \leq G \): \( V_i^U = V_i \). Hence, \( U \) acts trivially on \( V \), and is thus trivial, since \( \rho \) is injective.

We will show that the natural faithful representation \( G \hookrightarrow \text{GL}_{2n} \) is irreducible and hence \( G \) is reductive. Let \( V = k^{2n} \) denote the underlying vector space with standard basis \( (e_i)_{i=1}^{2n} \). We have \( V = \bigoplus_{i=1}^{2n} ke_i \) and, for all \( t \in T \),

\[
t e_i = \begin{cases} 
\epsilon_i(t)e_i, & i \leq n \\
\epsilon_{i-n}(t)^{-1}e_i, & i > n 
\end{cases}
\]

Any \( G \)-subrepresentation of \( V \) is a direct sum of \( T \)-eigenspaces; hence, it is enough to show that \( N_G(T) \) acts transitively on the \( ke_i \), which is equivalent to it acting transitively on \( \{ \pm \epsilon_1, \ldots, \pm \epsilon_n \} \subset X^*(T) \).

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Footnote:

For another elementary proof, see my post here: [http://mathoverflow.net/questions/98881/connectedness-of-the-linear-algebraic-group-so-n](http://mathoverflow.net/questions/98881/connectedness-of-the-linear-algebraic-group-so-n)
A calculation shows that the elements
\[ g_i := \text{diag}(I_{i-1}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_{n-2}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_{n-i-1}), \quad (1 \leq i < n) \]
lie in \( G \), where \( \text{diag}(A_1, A_2, \ldots) \) denotes a matrix with square blocks \( A_1, A_2, \ldots \) along the diagonal in the given order. As well
\[ g_n := \begin{pmatrix} \text{diag}(I_{n-1}, 0) & E_{nn} \\ -E_{nn} & \text{diag}(I_{n-1}, 0) \end{pmatrix}, \]
lies in \( G \), where \( E_{nn} \in M_n(k) \) has a 1 in the \((n, n)\)-entry and 0’s elsewhere. Note that the \( g_i \in NG(T) \) for all \( i \) and \( g_i : \epsilon_i \mapsto \epsilon_{i+1} \), for \( 1 \leq i < n \), and \( g_n : \epsilon_n \mapsto -\epsilon_n \) (with \( g_i \cdot \epsilon_j = \epsilon_j \) for \( i \neq j \)). Hence, \( NG(T) \) does act transitively on \( \{ \pm \epsilon_i \} \), so \( V \) is irreducible and \( G \) is reductive.

Lie Algebra:
If \( \psi : \text{GL}_{2n} \to \text{GL}_{2n}, g \mapsto g^tJg \), then \( d\psi : M_{2n}(k) \to M_{2n}(k), X \mapsto X^tJ + JX \) (as in the proofs of Propositions 79 and 80). Hence,
\[ g \subset \{ X \in M_{2n}(X) \mid X^tJ + JX \} =: g'. \]
Checking that \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in g' \) if and only if \( B^t = B, C^t = C, \) and \( D = -A^t \), we see that
\[ \dim g' = n^2 + 2 \left( \frac{n+1}{2} \right) = n(2n + 1) \]

Claim: \( \dim G \geq n(2n + 1) \)
Define
\[ \phi : \text{GL}_{2n} \to A^{(2n)}, \quad g \mapsto ((g^tJg)_{ij})_{i<j}. \]
We have \( \phi^{-1}((J_{ij})_{i<j}) = G \), (because \( g^tJg \) is antisymmetric). (This is still okay when \( p = 2 \).) So,
\[ (2n)^2 = \dim \text{GL}_{2n} \leq \dim \phi(\text{GL}_{2n}) + \text{minimal fibre dimension} \leq \left( \frac{2n}{2} \right) + \dim G \]
and
\[ \dim G \geq (2n)^2 - \left( \frac{2n}{2} \right) = n(2n + 1). \]
Hence,
\[ \dim g \leq n(2n + 1) \leq \dim G = \dim g \quad \implies \quad \dim g = n(2n + 1) \]
and so
\[ \dim G = n(2n + 1), \quad \text{and} \quad g = \{ X \in M_{2n}(k) \mid X^tJ + JX = 0 \}. \]

Roots:
Write \( E_{ij} \) for the \((2n) \times (2n)\) matrix with a 1 in the \((i, j)\)-entry and 0’s elsewhere. By the above,
\[ g = \mathfrak{t} \oplus \left( \bigoplus_{i \neq j} k(E_{ij} - E_{ji}) \right) \oplus \left( \bigoplus_{i \leq j} k(E_{ij} + E_{ji}) \right) \oplus \left( \bigoplus_{i \leq j} k(E_{ij} + E_{ji}) \right) \]

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(with $E_{ij} + E_{ji}$ in the latter factors replaced with $E_{ii}$ if $i = j$ and $p = 2$). Correspondingly,

$$\Phi = \{ \epsilon_i - \epsilon_j \mid i \neq j \} \cup \{ \epsilon_i + \epsilon_j \mid i \leq j \} \cup \{ -(\epsilon_i + \epsilon_j) \mid i \leq j \}$$

(A check: $\#\Phi = n(n - 1) + \binom{n+1}{2} + \binom{n+1}{2} + \binom{n+1}{2} + \binom{n+1}{2} = 2n^2 = \dim g - \dim t$.)

**Coroots:**
Let $\epsilon_1^*, \ldots, \epsilon_n^*$ denote the dual basis, so

$$\epsilon_i^*(x) = \text{diag}(1, \ldots, x, \ldots, x^{-1}, \ldots, 1) = \text{diag}(I_{i-1}, x, I_{n-1}, x^{-1}, I_{n-i}).$$

We have

$$G_{\epsilon_i - \epsilon_j} = G \cap (D_{2n} + kE_{ij} + kE_{ji} + kE_{n+i,n+j} + kE_{n+j,n+i})$$

and so $G_{\epsilon_i - \epsilon_j}$ is contained in

$$G \cap \{ I_{2n} + (a - 1)E_{ii} + bE_{ij} + cE_{ji} + (d - 1)E_{jj} + (a' - 1)E_{n+i,n+i} + b'E_{n+i,n+j} + c'E_{n+j,n+i} + (d' - 1)E_{n+j,n+j} \}$$

where $a, b, c, d, a', b', c', d'$ are such that $ad - bc = 1 = a'd' - b'c'$. It follows that

$$(\epsilon_i - \epsilon_j)^\vee = \epsilon_i^* - \epsilon_j^*.$$ 

Similarly, $(\epsilon_i + \epsilon_j)^\vee = \epsilon_i^* + \epsilon_j^*$ and $(-\epsilon_i - \epsilon_j)^\vee = -\epsilon_i^* - \epsilon_j^*$.

$G$ is semisimple: $RG = Z_G^0 = \left( \bigcap_{\Phi \ker \alpha} \right)^0 = 1$.

A Borel subgroup of $G$: We can explicitly compute a Borel subgroup, for example as explained for the even orthogonal group in Homework 4 (2017). (For this it would be more convenient to choose an antidiagonal form $J$ when we define $G$!)