MAT 347
Problems for Homework 20
March 26, 2020

1. Let K be a normal closure of $\mathbb{Q}(\sqrt{1+\sqrt{2}})$ over $\mathbb{Q}$. Determine the intermediate fields in $K/\mathbb{Q}$ that are of degree 4 over $\mathbb{Q}$, and for each one describe the corresponding subgroup of the Galois group. You may assume that $[\mathbb{Q}(\sqrt{1+\sqrt{2}}) : \mathbb{Q}] = 4$.

2. Suppose that K/F is a Galois extension with Galois group G. Suppose that $K = F(\alpha)$. Suppose that M is an intermediate field, and that $H := \widehat{G}(M)$ is the corresponding subgroup of G.

   (a) Show that the minimal polynomial $m_{\alpha,M}(X)$ of α over M is $\prod_{h\in H}(X - h(\alpha))$.

   (b) Show that coefficients of $m_{\alpha,M}(X)$ generate M over F. (Hint: let M' be the subfield generated by the coefficients of $m_{\alpha,M}(X)$ over F. Prove that $M' = M$ by e.g. considering $[M'(\alpha) : M']$.)

   (c) Let $K := \mathbb{F}_2(\alpha)$, where $\alpha^6 + \alpha + 1 = 0$ (you may assume this polynomial is irreducible over $\mathbb{F}_2$). Use part (b) to determine a primitive element for each intermediate field strictly between $\mathbb{F}_2$ and K (i.e. don't worry about the extreme cases $\mathbb{F}_2$ and K, which aren't so interesting). Express it in terms of α.

   (d) Find the minimal polynomial over $\mathbb{F}_2$ for each primitive element β you found in part (c). (Hint: it might be easiest to consider linear relations among the first few powers of β.)

3. Let p be any prime number.

   (a) Briefly recall why $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is Galois with Galois group $(\mathbb{Z}/p)^\times$. Write down the isomorphism. (Here, ζ_p is a non-trivial p-th root of unity.)

   (b) Find all intermediate fields in $\mathbb{Q}(\zeta_{11})$ and find a primitive element for each intermediate field strictly between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{11})$. (Hint: don't forget the previous problem.)

   (c) Find a non-square integer d such that $\mathbb{Q}(\sqrt{d})$ is contained in $\mathbb{Q}(\zeta_{11})$. (Hint: use your answer to part (b) and don't forget the previous problem.)

   (d) Show that $\mathbb{Q}(\sqrt[4]{3}) \not\subset \mathbb{Q}(\zeta_p)$ for all primes p.