MAT 347
Problems for Homework 18
March 11, 2020

1. Let K/M and M/F be field extensions. Prove or give a counterexample:
   (a) If K/F is normal, then K/M is normal.
   (b) If K/F is normal, then M/F is normal.
   (c) If K/M and M/F are normal, then K/F is normal.

2. Prove that every field extension of degree 2 is normal.

   Warning: Do not exclude characteristic 2!

   Note: Are you having a déjà vu? If these first two questions make you think of similar results for groups, there is a good reason for it.

3. The goal of this problem is to prove that a finite extension generated by separable elements is separable. (See Thm. 6.15(i) in the notes.) Note that even if you get stuck on some parts, you can still answer the other parts.

   For the first few questions, let us fix a finite, normal field extension K/F. Given intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq K$, we define $\operatorname{Emb}(M_2/M_1)$ to be the set of M_1–homomorphisms $\varphi : M_2 \to K$. (Recall that this means that φ is a field homomorphism such that $\varphi|_{M_1} = \mathrm{id}$, the identity of M_1.)

   (a) Assume $M_2 = M_1(\alpha)$. Prove that $|\operatorname{Emb}(M_2/M_1)|$ equals the number of distinct roots of $m_{\alpha,M_1}(X)$ in K. Conclude that $|\operatorname{Emb}(M_2/M_1)| \leq [M_2 : M_1]$, with equality iff α is separable over M_1.

   (b) For any intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq K$, prove that
   $$|\operatorname{Emb}(M_3/M_1)| = |\operatorname{Emb}(M_3/M_2)|\,|\operatorname{Emb}(M_2/M_1)|.$$
   (*Hint:* consider $\operatorname{Emb}(M_2/M_1) = \{\sigma_1, \ldots, \sigma_k\}$ and $\operatorname{Emb}(M_3/M_2) = \{\tau_1, \ldots, \tau_\ell\}$. First show that each $\sigma_i : M_2 \to K$ can be extended to some $\widetilde{\sigma}_i : K \to K$ in $\operatorname{Emb}(K/M_1)$, and then show that the $\widetilde{\sigma}_i \circ \tau_j$ are the distinct elements of $\operatorname{Emb}(M_3/M_1)$.)

   (c) Assume that $F \subseteq M \subseteq K$ is any intermediate field. If M/F is not separable, prove that $|\operatorname{Emb}(M/F)| < [M : F]$.

(d) Assume that $F \subseteq M \subseteq K$ is any intermediate field. If $M = F(\alpha_1, \ldots, \alpha_n)$, where each α_i is separable over F, prove that $|\operatorname{Emb}(M/F)| = [M : F]$. Conclude that M/F is separable.

We now deduce some applications.

(e) Prove that the splitting field of a separable polynomial is a separable extension. (*Hint:* use (d).)

(f) Let K/F be a finite, separable extension. Prove that its normal closure is a finite, normal, separable extension. (*Hint:* use (e).)

(g) Let E/F be any finite extension (not necessarily normal). Assume that $E = F(\alpha_1, \ldots, \alpha_n)$ and that α_i is separable over F for all i. Prove that E/F is separable. (*Hint:* use (d), by making a suitable choice of K.)

For practice (not collected)

4. Calculate the degree of $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[5]{2})/\mathbb{Q}$. What about $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2})/\mathbb{Q}$? (*Hint:* try to use the tower law.)

5. Give an example of a finite field extension that is separable but not normal (resp. normal but not separable). What about a finite extension that is neither normal nor separable?

6. Suppose that $K = F(\alpha_1, \ldots, \alpha_n)$. Let L be a splitting field of

$$f(X) := m_{\alpha_1,F}(X) \cdots m_{\alpha_n,F}(X)$$

over K. Show that L is also a splitting field of $f(X)$ over F.

7. Suppose we have a tower of finite field extensions $L/N/K/F$, where N/F is a normal closure of K/F. Show that any F-homomorphism $\phi : K \to L$ satisfies $\phi(K) \subset N$.

8. Suppose that $L/K/F$ are finite field extensions. Prove that L/F is separable iff both L/K and K/F are separable. (*Hint:* for the difficult direction use Problem 3.)