MAT 347
Problems for Homework 17
March 4, 2020

1. (a) Suppose K/F is a field extension of degree p, a prime. Show that $K = F(\alpha)$, for some $\alpha \in K$.

   (b) Suppose that $[K : F] = 8$ and that $f(x) \in F[x]$ has degree 3. If $f(x)$ has a root in K, show that $f(x)$ already has a root in F.

2. Determine $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ as a group, with details. (Hint: first show that the extension has degree 4. Eisenstein might help.)

3. We want to compute the Galois group of the (non-algebraic!) extension $\mathbb{C}(X)/\mathbb{C}$. Here, X is a formal variable. Specifically, we want to prove that $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{C})/Z(\mathrm{GL}_2(\mathbb{C}))$. This quotient is called $\mathrm{PGL}_2(\mathbb{C})$.

   Consider the group of invertible 2×2 matrices with entries in $\mathbb{C}$, namely $\mathrm{GL}_2(\mathbb{C})$. For each $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, define a function $\phi_A : \mathbb{C}(X) \to \mathbb{C}(X)$ by

   $$\phi_A(r(X)) = r\left(\frac{aX + c}{bX + d}\right).$$

   Here $r(X)$ is a formal rational expression in X (i.e. a ratio of two polynomials).

   (a) Prove that $\phi_A \in \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C})$.

   (b) Prove that the map $\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C})$ which sends A to ϕ_A is a group homomorphism.

   (c) Prove that $\ker(\Phi) = Z(\mathrm{GL}_2(\mathbb{C})) = \{cI : c \in \mathbb{C}^\times\}$, where I is the identity matrix.

   (d) Prove that Φ is surjective; in other words, all $\mathbb{C}$-automorphisms ϕ of $\mathbb{C}(X)$ are of the above form. (*Hint:* You need to justify that ϕ sends X to a simple rational function as above! Use Problem 13.2/18 from last homework.)

   (e) Let $H = \Phi(T)$, where $T = \left\{\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) : c \in \mathbb{C}\right\}$. Show that H is a subgroup of $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C})$. Find the invariant subfield of H. Is $\widehat{G}(\widehat{I}(H)) = H$?