

# A CLASSIFICATION OF IRREDUCIBLE ADMISSIBLE MOD $p$ REPRESENTATIONS OF $p$ -ADIC REDUCTIVE GROUPS

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ABSTRACT. Let  $F$  be a locally compact non-archimedean field,  $p$  its residue characteristic, and  $\mathbf{G}$  a connected reductive group over  $F$ . Let  $C$  be an algebraically closed field of characteristic  $p$ . We give a complete classification of irreducible admissible  $C$ -representations of  $G = \mathbf{G}(F)$ , in terms of supercuspidal  $C$ -representations of the Levi subgroups of  $G$ , and parabolic induction. Thus we push to their natural conclusion the ideas of the third-named author, who treated the case  $\mathbf{G} = \mathrm{GL}_m$ , as further expanded by the first-named author, who treated split groups  $\mathbf{G}$ . As in the split case, we first get a classification in terms of supersingular representations of Levi subgroups, and as a consequence show that supersingularity is the same as supercuspidality.

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## I. INTRODUCTION

**I.1.** The study of congruences between classical modular forms has met considerable success in the past decades. When interpreted in the natural framework of automorphic

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forms and representations, such congruences naturally lead to representations over fields of positive characteristic, rather than complex representations. In our local setting, where the base field is a locally compact non-archimedean field  $F$ , this means studying representations of  $G = \mathbf{G}(F)$ , where  $\mathbf{G}$  is a connected reductive group over  $F$ , on vector spaces over a field  $C$  of positive characteristic  $p$ , which we assume algebraically closed. As  $C$  is fixed throughout, we usually say representation instead of representation on a  $C$ -vector space or  $C$ -representation.

Our representations satisfy natural requirements: they are smooth, in that every vector has open stabilizer in  $G$  (smoothness is always understood for representations of  $G$  or its subgroups), and most of the time they are **admissible**: a representation of  $G$  on a  $C$ -vector space  $W$  is admissible if it is smooth and for every open subgroup  $J$  in  $G$ , the space  $W^J$  of vectors fixed under  $J$  has finite dimension. The overall goal is to understand irreducible admissible representations of  $G$ .

If  $\ell$  is the residue characteristic of  $F$ , the situation depends very much on whether  $\ell$  is equal to  $p$  or not. Indeed,  $G$  has an open subgroup which is a pro- $\ell$  group, and when  $\ell \neq p$  such a group acts semisimply on smooth representations, whereas if  $\ell = p$  it acts semisimply only if it acts trivially! Here we consider only the case  $\ell = p$  (see e.g. [Vig1] for the case  $\ell \neq p$ ).

**I.2.** Thus we assume from now on that  $F$  has residue characteristic  $p$ . In this paper we classify irreducible admissible representations of  $G$  in terms of parabolic induction and supercuspidal representations of Levi subgroups of  $G$ . Such a classification was obtained for  $\mathbf{G} = \mathrm{GL}_2$  in the pioneering work of L. Barthel and R. Livné [BL1, BL2] – see also some recent work [Abd, Che, Ko, KX, Ly2] on situations where, mostly,  $\mathbf{G}$  has relative semisimple rank 1.

New ideas towards the general case were set forth by the third-named author [He1, He2], who gave the classification for  $\mathbf{G} = \mathrm{GL}_n$  over a  $p$ -adic field  $F$ ; his ideas were further expanded by the first-named author [Abe] to treat the case of a split group  $\mathbf{G}$ , still over a  $p$ -adic field  $F$ . T. Ly extended the arguments of [He1, He2] to treat  $\mathbf{G} = \mathrm{GL}_{3/D}$  where  $D$  is a division algebra over  $F$ , allowing  $F$  to have characteristic  $p$ . Here, using the first steps accomplished in [HV1, HV2], we treat general  $\mathbf{G}$  and  $F$ .

**I.3.** To express our classification, we recall parabolic induction. If  $P$  is a parabolic subgroup of  $G$  and  $\tau$  a representation of  $P$  on a  $C$ -vector space  $W$ , we write  $\mathrm{Ind}_P^G \tau$  for the natural representation of  $G$ , by right translation, on the space  $\mathrm{Ind}_P^G W$  of smooth functions  $f : G \rightarrow W$  such that  $f(pg) = \tau(p)f(g)$  for  $p$  in  $P$ ,  $g$  in  $G$ . The functor  $\mathrm{Ind}_P^G$  is exact. In fact we use  $\mathrm{Ind}_P^G \tau$  only when  $\tau$  comes via inflation from a representation  $\sigma$  of the Levi quotient of  $P$ , and we write  $\mathrm{Ind}_P^G \sigma$  instead of  $\mathrm{Ind}_P^G \tau$ . A representation of  $G$  is said to be **supercuspidal** if it is irreducible, admissible, and does not appear as a subquotient of a parabolically induced representation  $\mathrm{Ind}_P^G \sigma$ , where  $P$  is a proper parabolic subgroup of  $G$  and  $\sigma$  an irreducible admissible representation of the Levi quotient of  $P$ .<sup>1</sup>

First we construct irreducible admissible representations of  $G$ . The construction uses the “generalized Steinberg” representations investigated by E. Große-Klönne [GK] and the third-named author [He2] when  $\mathbf{G}$  is split, and by T. Ly [Ly1] in general: for any pair of parabolic subgroups  $Q \subset P$  in  $G$ ,  $\mathrm{St}_Q^P$  is the natural representation of  $P$  in the

<sup>1</sup>See Section VI.1 for why we cannot drop the requirement that  $\sigma$  be irreducible admissible.

quotient of  $\text{Ind}_Q^P 1$  by the sum of the subspaces  $\text{Ind}_{Q'}^P 1$ , for parabolic subgroups  $Q'$  with  $Q \subsetneq Q' \subset P$ ; the representation  $\text{St}_Q^P$  factors through the unipotent radical  $U_P$  of  $P$  and gives the representation  $\text{St}_{Q/U_P}^{P/U_P}$  of its reductive quotient studied in [GK, Ly1], so  $\text{St}_Q^P$  is irreducible and admissible (loc. cit.).

Start with a parabolic subgroup  $P$  of  $G$ , with Levi quotient  $M$ , and a representation  $\sigma$  of  $M$ . Then there is a largest parabolic subgroup  $P(\sigma)$  of  $G$ , containing  $P$ , such that  $\sigma$  inflated to  $P$  extends to  $P(\sigma)$  (see II.7). That extension is unique, we write it  ${}^e\sigma$ ; it is trivial on the unipotent radical of  $P(\sigma)$ . It is irreducible and admissible if  $\sigma$  is. We consider triples  $(P, \sigma, Q)$ : a triple consists of a parabolic subgroup  $P$  of  $G$ , a representation  $\sigma$  of the Levi quotient  $M$  of  $P$ , and a parabolic subgroup  $Q$  of  $G$  with  $P \subset Q \subset P(\sigma)$ ; we say that the triple is **supercuspidal** if  $\sigma$  is a supercuspidal representation of  $M$ . To a triple  $(P, \sigma, Q)$  we associate the representation  $I(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G ({}^e\sigma \otimes \text{St}_Q^{P(\sigma)})$ .

**Theorem 1.** *For a supercuspidal triple  $(P, \sigma, Q)$ ,  $I(P, \sigma, Q)$  is irreducible and admissible.*

**Theorem 2.** *Let  $(P, \sigma, Q)$  and  $(P', \sigma', Q')$  be supercuspidal triples. Then  $I(P, \sigma, Q)$  and  $I(P', \sigma', Q')$  are isomorphic if and only if there is an element  $g$  of  $G$  such that  $P' = gPg^{-1}$ ,  $Q' = gQg^{-1}$  and  $\sigma'$  is equivalent to  $p' \mapsto \sigma(g^{-1}p'g)$ .*

**Theorem 3.** *Any irreducible admissible representation of  $G$  is isomorphic to  $I(P, \sigma, Q)$  for some supercuspidal triple  $(P, \sigma, Q)$ .*

Hopefully the classification expressed by these theorems will be useful in extending the mod  $p$  local Langlands correspondence beyond  $\text{GL}_2(\mathbb{Q}_p)$ .

**I.4.** Using the classification results above, it is possible to describe the irreducible components of  $\text{Ind}_P^G \sigma$  where  $P$  is a parabolic subgroup of  $G$  and  $\sigma$  an irreducible admissible representation of the Levi quotient  $M$  of  $P$ ; in particular we show that  $\text{Ind}_P^G \sigma$  has finite length and that all its irreducible subquotients are admissible and occur with multiplicity one.

Also we have a notion of **supercuspidal support**: if  $(P, \sigma, Q)$  is a supercuspidal triple, then  $\pi = I(P, \sigma, Q)$  occurs as a subquotient of  $\text{Ind}_P^G \sigma$  and if  $\pi$  occurs as a subquotient of  $\text{Ind}_{P'}^G \sigma'$  for a supercuspidal representation  $\sigma'$  of (the Levi quotient of) a parabolic subgroup  $P'$  of  $G$  then  $(P', \sigma')$  is conjugate to  $(P, \sigma)$  in  $G$  as in Theorem 2. It is the conjugacy class of  $(P, \sigma)$  that we call the supercuspidal support of  $\pi$ .

**Remark** The classification and its consequences are rather simpler than for complex representations: intertwining operators do not exist in our context; this “explains” the multiplicity one result above, which does not hold for complex representations [Ze]. By contrast, supercuspidal mod  $p$  representations remain a complete mystery, apart the case of  $\text{GL}_2(\mathbb{Q}_p)$  [Br] and groups closely related to it [Abd, Che, Ko, KX].

The existence of a supercuspidal support for complex irreducible representations is a classical result; for mod  $\ell$  representations with  $\ell \neq p$  it is unknown (even for finite reductive groups of characteristic  $p$  outside the case of general linear groups), except for inner forms of  $\text{GL}_n(F)$  where, as above, it is not proved directly but is a consequence of the classification of irreducible representations.

**I.5.** As in [He2, Abe] our classification is not established directly using supercuspidality. Rather we get a classification in terms of supersingular representations of Levi subgroups

of  $G$  – the term was first used by Barthel and Livné for  $G = \mathrm{GL}_2(F)$  – and deduce Theorems 1 to 3 from it. To define supersingularity, we need to make some choices, and a priori the notion depends on these choices.

So we fix a maximal  $F$ -split torus  $\mathbf{S}$  in  $\mathbf{G}$  and a special point  $\mathbf{x}_0$  in the apartment corresponding to  $\mathbf{S}$  in the semisimple Bruhat-Tits building of  $G$ ; we let  $K$  be the special parahoric subgroup of  $G$  corresponding to  $\mathbf{x}_0$ . We also fix a minimal parabolic subgroup  $B$  of  $G$  with Levi subgroup  $Z$ , the  $F$ -points of the centralizer of  $\mathbf{S}$ , and we write  $U$  for the unipotent radical of  $B$ .

Let  $V$  be an irreducible representation of  $K$  – it has finite dimension. If  $(\pi, W)$  is an admissible representation of  $G$ , then  $\mathrm{Hom}_K(V, W)$  is a finite-dimensional  $C$ -vector space; by Frobenius reciprocity  $\mathrm{Hom}_K(V, W)$  is identified with  $\mathrm{Hom}_G(\mathrm{ind}_K^G V, W)$ , where  $\mathrm{ind}_K^G$  denotes compact induction, so that  $\mathrm{Hom}_K(V, W)$  is a right-module over the intertwining algebra  $\mathcal{H}_G(V) = \mathrm{End}_G(\mathrm{ind}_K^G V)$  of  $V$  in  $G$ . If  $\mathrm{Hom}_K(V, W)$  is not zero we say that  $V$  is a **weight** of  $\pi$ ; in that case the centre<sup>2</sup>  $\mathcal{Z}_G(V)$  of  $\mathcal{H}_G(V)$  has eigenvectors in  $\mathrm{Hom}_K(V, W)$ , and we focus on the corresponding characters of  $\mathcal{Z}_G(V)$ , which we call the (Hecke) **eigenvalues** of  $\mathcal{Z}_G(V)$  in  $\pi$ .

For any parabolic subgroup  $P$  of  $G$  containing  $B$ , with Levi component  $M$  containing  $Z$  and unipotent radical  $N$ , the space of coinvariants  $V_{N \cap K}$  of  $N \cap K$  in  $V$  provides an irreducible representation of  $M \cap K$  and by [He1, He2, HV2] there is a natural injective algebra homomorphism

$$\mathcal{S}_M^G : \mathcal{H}_G(V) \rightarrow \mathcal{H}_M(V_{N \cap K}).$$

It induces a homomorphism between centres  $\mathcal{Z}_G(V) \rightarrow \mathcal{Z}_M(V_{N \cap K})$ . Both homomorphisms are localizations at a central element. A character  $\chi : \mathcal{Z}_G(V) \rightarrow C$  is said to be **supersingular** if, in the above situation, it can be extended to a character of  $\mathcal{Z}_M(V_{N \cap K})$  only when  $P = G$ . A **supersingular** representation of  $G$  is an irreducible admissible representation  $(\pi, W)$  such that for all weights  $V$  of  $\pi$ , all eigenvalues of  $\mathcal{Z}_G(V)$  in  $\pi$  are supersingular<sup>3</sup>.

A triple  $(P, \sigma, Q)$  as in I.3 is a  **$B$ -triple** if  $P$  contains  $B$ ; it is said to be **supersingular** if it is a  $B$ -triple and  $\sigma$  is a supersingular representation of the Levi quotient of  $P$ .

Theorems 1 to 3 are consequences of the following results.

**Theorem 4.** *For each supersingular triple  $(P, \sigma, Q)$ , the representation  $I(P, \sigma, Q)$  is irreducible and admissible. If  $\pi$  is an irreducible admissible representation of  $G$ , there is a supersingular triple  $(P, \sigma, Q)$  such that  $\pi$  is isomorphic to  $I(P, \sigma, Q)$ ; moreover  $P$  and  $Q$  are unique and  $\sigma$  is unique up to isomorphism.*

**Theorem 5.** *Let  $\pi$  be an irreducible admissible representation of  $G$ . Then  $\pi$  is supercuspidal if and only if it is supersingular.*

(For  $\mathbf{G} = \mathrm{GL}_2$  this was discovered by Barthel and Livné.)

Note that Theorem 5 implies, in particular, that the notion of supersingularity does not depend on the choices of  $\mathbf{S}$ ,  $K$ ,  $B$  necessary for the definition – beware that in general two choices of  $K$  will not even be conjugate under the adjoint group of  $G$ .

<sup>2</sup>Note that  $\mathcal{H}_G(V)$  is commutative in many cases, for example when  $G$  is split, but not in general [HV1].

<sup>3</sup>That is consistent with the definition in [He2, Abe]; but the reader should be aware that the definition in [HV2] is slightly different, maybe not equivalent.

**Remarks** 1) We also show that, if  $\pi$  is an irreducible admissible representation of  $G$ , and for some weight  $V$  of  $\pi$  there is an eigenvalue of  $\mathcal{Z}_G(V)$  in  $\pi$  which is supersingular, then  $\pi$  is supersingular/supercuspidal.

2) Let  $(P, \sigma, Q)$  be a supersingular (or supercuspidal)  $B$ -triple. Then  $I(P, \sigma, Q)$  is finite dimensional if and only if  $P = B$  and  $Q = G$ .

**I.6.** As in [He2] and [Abe], a lot of our arguments bear on the relation between parabolic induction in  $G$  and compact induction from  $K$  to  $G$ .

Let  $V$  be an irreducible representation of  $K$ , and let  $P$  be a parabolic subgroup of  $G$  containing  $B$ , with Levi component  $M$  containing  $Z$ , and unipotent radical  $N$ . In [HV2], inspired by [He1], [He2], a canonical intertwiner

$$\mathcal{I} : \text{ind}_K^G V \longrightarrow \text{Ind}_P^G(\text{ind}_{M \cap K}^M V_{N \cap K})$$

was investigated. In fact the morphism  $\mathcal{S}_M^G$  of I.5 is such that for  $f$  in  $\text{ind}_K^G V$  and  $\Phi$  in  $\mathcal{H}_G(V)$  we have

$$\mathcal{I}(\Phi(f)) = \mathcal{S}_M^G(\Phi)(\mathcal{I}(f)),$$

where the action of  $\mathcal{S}_M^G(\Phi)$  on  $\mathcal{I}(f)$  is via its natural action on  $\text{ind}_{M \cap K}^M V_{N \cap K}$ . Under a suitable regularity condition of  $V$  with respect to  $P$  [HV2], cf. III.14 Theorem,  $\mathcal{I}$  induces an isomorphism

$$\chi \otimes \text{ind}_K^G V \xrightarrow{\sim} \text{Ind}_P^G(\chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K})$$

for any character  $\chi$  of  $\mathcal{Z}_G(V)$  which extends to  $\mathcal{Z}_M(V_{N \cap K})$ : such an extension is unique, we still denote it by  $\chi$ ; the first tensor product is over  $\mathcal{Z}_G(V)$ , the second one over  $\mathcal{Z}_M(V_{N \cap K})$ . Here we obtain a generalization of that result, which we now proceed to explain.

We consider an irreducible representation  $V$  of  $K$ , and a character  $\chi : \mathcal{Z}_G(V) \rightarrow C$ . There is a smallest parabolic subgroup  $P$  containing  $B$  – we write  $P = MN$  as above – such that  $\chi$  extends to a character, still written  $\chi$ , of  $\mathcal{Z}_M(V_{N \cap K})$ ; there is a natural parabolic subgroup  $P_e$ , containing  $P$ , such that the representation  $\chi \otimes (\text{ind}_{M \cap K}^M V_{N \cap K})$  of  $M$ , inflated to  $P$ , extends to a representation of  $P_e$  – write  ${}^e(\chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K})$  for that extension. Using similar notation as in I.3, we write  $I_e(P, \chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K}, Q)$  for  $\text{Ind}_{P_e}^G({}^e(\chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K}) \otimes \text{St}_Q^{P_e})$  when  $Q$  is a parabolic subgroup between  $P$  and  $P_e$ .

**Theorem 6.** *With the previous notation,  $\tau = \chi \otimes \text{ind}_K^G V$  has a natural filtration by subrepresentations  $\tau_Q$ , where  $Q$  runs through parabolic subgroups of  $G$  with  $P \subset Q \subset P_e$  and  $\tau_{Q'} \subset \tau_Q$  if  $Q' \subset Q$ . The quotient  $\tau_Q / \sum_{Q' \subsetneq Q} \tau_{Q'}$  is isomorphic to  $I_e(P, \chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K}, Q)$ .*

This last theorem should be compared to the following (the proof, in Chapter V, explains that comparison). Let  $\pi = \text{Ind}_P^G(\chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K})$ . It also has a natural filtration by subrepresentations  $\pi_Q$  for  $Q$  as above, but this time  $\pi_{Q'} \subset \pi_Q$  if  $Q' \supset Q$ , and the quotient  $\pi_Q / \sum_{Q' \supseteq Q} \pi_{Q'}$  is isomorphic to  $I_e(P_e, \chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K}, Q)$ . In particular the filtrations on  $\tau$  and  $\pi$  give rise to the same subquotients, but in reverse order, so to say. (We note that the representation  $\pi_Q$  above corresponds to the representation  $I_Q$  in Chapter V.)

A striking example is when  $V$  is trivial character of  $K$  and  $\chi$  is the “trivial” character of  $\mathcal{Z}_G(V) = \mathcal{H}_G(V)$ : in that case  $P = B = ZU$ ,  $P_e = G$ , and  $\chi \otimes \text{ind}_{Z \cap K}^Z V_{U \cap K}$  is the trivial

character of  $Z$ . In  $\pi = \text{Ind}_B^G 1$ , the trivial character of  $G$  occurs as a subrepresentation and the Steinberg representation  $\text{St}_B^G$  as a quotient, whereas the reverse is true in  $\chi \otimes \text{ind}_K^G 1$ .

Theorem 6 is new even for  $\text{GL}_n$  ( $n > 2$ ). A weaker version of this theorem is proved in [Abe, Proposition 4.7] when  $\mathbf{G}$  is split with simply connected derived subgroup and  $P = B$  (and in [BL2] in the further special case when  $\mathbf{G} = \text{GL}_2$ ). On the way, following the ideas of [Abe], we prove the freeness of  $R_M \otimes_{\mathcal{Z}_G(V)} \text{ind}_K^G V$  as  $R_M$ -module, where  $R_M$  denotes the “supersingular quotient” of  $\mathcal{Z}_M(V_{N \cap K})$ . This may be of independent interest. Again this result was established for  $\mathbf{G} = \text{GL}_2$  in [BL1], but see also the recent paper [GK2].

**I.7.** To prove Theorem 4 we follow the same strategy as in [He2, Abe]. If  $(P, \sigma, Q)$  is a supersingular triple, we need to prove that  $\pi = I(P, \sigma, Q)$  is irreducible; that is done by showing that for any weight  $V$  of  $\pi$  and any eigenvector  $\varphi$  for  $\mathcal{Z}_G(V)$  in  $\text{Hom}_K(V, \pi)$  with corresponding eigenvalue  $\chi$ ,  $\pi$  is generated as a representation of  $G$  by the image of  $\varphi$ . When  $V$  is suitably regular, that is seen as a consequence of the isomorphism  $\chi \otimes \text{ind}_K^G V \simeq \text{Ind}_P^G(\chi \otimes \text{ind}_{M \cap K}^M V_{N \cap K})$  recalled in I.6 above (see III.14). We reduce to that suitably regular case by using a *change of weight theorem*, which gives explicit sufficient conditions on  $V, V'$ , and  $\chi$  for having an isomorphism  $\chi \otimes \text{ind}_K^G V \simeq \chi \otimes \text{ind}_K^G V'$ . (Here  $V'$  is an irreducible representation of  $K$  that is “slightly less regular” than  $V$  and such that  $(V')_{U \cap K} \simeq V_{U \cap K}$ .) We refer the reader to Sections IV.2, III.23 for the precise statement and its use in the proof of Theorem 4.

The main novelty in our methods is our proof of the change of weight theorem. It is also the hardest and most subtle part of our arguments. Previously, for split groups, a version of this theorem was established in [He2, §6] and [Abe, §4] by computing the composition of two intertwining operators and applying the Lusztig–Kato formula. We do not know if this approach can be generalized. Our new proof does not involve Kazhdan–Lusztig polynomials, but rather proceeds by embedding  $\text{ind}_K^G V, \text{ind}_K^G V'$  into the parabolically induced representation  $\mathcal{X} = \text{Ind}_B^G(\text{ind}_{Z \cap K}^Z \psi_V)$  using the intertwiner  $\mathcal{I}$  of I.6, where  $\psi_V : Z \cap K \rightarrow C^\times$  describes the action of  $Z \cap K$  on  $V_{U \cap K} \simeq (V')_{U \cap K}$ . The representation  $\text{ind}_K^G V$  contains a canonical (up to scalar) fixed vector under a pro- $p$  Iwahori subgroup  $I \subset K$  which generates  $\text{ind}_K^G V$  as a representation of  $G$ , and similarly for  $\text{ind}_K^G V'$ . Our proof then studies the action of the pro- $p$ -Iwahori Hecke algebra  $\text{End}_G(\text{ind}_I^G 1)$  on  $\mathcal{X}^I$  to relate the two compact inductions inside  $\mathcal{X}$ . We crucially rely on the description of the pro- $p$ -Iwahori Hecke algebra recently given for general  $G$  by the fourth-named author in [Vig4], in particular the Bernstein relations in this algebra.

We arrive at a *dichotomy* in IV.1 Theorem and IV.2 Corollary, namely our change of weight results depend on whether or not  $\psi_V$  is trivial on a certain subgroup of  $Z \cap K$ . When  $G$  is split, the triviality is always guaranteed, but that is not always so for inner forms of  $\text{GL}_n$  [Ly3, Lemme 3.10.1] and even for unramified unitary groups in 3 variables. This dichotomy may explain why we did not find an easy generalization of the previous proofs for split  $G$ .

**I.8.** Let  $\pi$  be an irreducible admissible representation of  $G$ ,  $P = MN$  a parabolic subgroup of  $G$ , and  $\tau$  an irreducible admissible representation of  $M$  inflated to  $P$ . In a sequel to this article we will apply our classification to tackle natural questions as the computation of the  $N$ -coinvariants or the  $P$ -ordinary part of  $\pi$ , the description of the lattice

of subrepresentations of  $\text{Ind}_P^G \tau$ , the generic irreducibility of the representations  $\text{Ind}_P^G \tau \chi$  where  $\chi$  runs over the unramified characters of  $M$  (this question was raised by J.-F. Dat).

**I.9.** We end this introduction with some comments on the organization of the paper. In Chapter II we fix notation and we examine when a representation of a parabolic subgroup of  $G$ , trivial on its unipotent radical, can be extended to a larger parabolic subgroup. For a triple  $(P, \sigma, Q)$  as in I.3, we construct  $I(P, \sigma, Q)$  and show that it is admissible if  $\sigma$  is. In Chapter III we give most of the proof of Theorem 4. The irreducibility proof was outlined in I.7. The proof that  $\pi = I(P, \sigma, Q)$  determines  $P$ ,  $Q$ , and  $\sigma$  up to isomorphism comes from examining the possible weights and Hecke eigenvalues for  $\pi$  (III.24). Finally to prove that every irreducible admissible representation  $\pi$  of  $G$  has the form  $I(P, \sigma, Q)$  we use the filtration theorem (Theorem 6). The proof of the change of weight theorem is given in Chapters IV; this is the technical heart of our paper. In Chapter V we deduce the filtration theorem from the change of weight theorem. We trust that the reader will see easily that there is no loop in our arguments. Finally Chapter VI gives the proof of Theorems 1, 2, 3, 5 and other consequences of the classification, already stated in I.4. That section can essentially be read independently, taking Theorem 4 for granted.

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## II. EXTENSION TO A LARGER PARABOLIC SUBGROUP

**II.1.** Let us first fix notation, valid throughout the paper. As stated in the introduction, our base field  $F$  is locally compact and non-archimedean, of residue characteristic  $p$ ; its ring of integers is  $\mathcal{O}$ , its residue field  $k$ , and  $q$  is the cardinality of  $k$ ; we write  $|\cdot|$  for the normalized absolute value of  $F$ .

A linear algebraic group over  $F$  will be written with a boldface letter like  $\mathbf{H}$ , and its group of  $F$ -points will be denoted by the corresponding ordinary letter  $H = \mathbf{H}(F)$ <sup>4</sup>.

We fix our connected reductive  $F$ -group  $\mathbf{G}$ <sup>5</sup>, and a maximal  $F$ -split torus  $\mathbf{S}$  in  $\mathbf{G}$ ; we write  $\mathbf{Z}$  for the centralizer of  $\mathbf{S}$  in  $\mathbf{G}$ ,  $\mathcal{N}$  for its normalizer, and  $W_0 = W(\mathbf{G}, \mathbf{S})$  for the Weyl group  $\mathcal{N}/\mathbf{Z}$ ; we recall that  $W_0 = \mathcal{N}/\mathbf{Z}$  [Bo, 21.2 Theorem]. We also fix a minimal  $F$ -parabolic subgroup  $\mathbf{B}$  of  $\mathbf{G}$  with Levi subgroup  $\mathbf{Z}$ , and write  $\mathbf{U}$  for its unipotent radical. As is customary, we say that  $P$  is a parabolic subgroup of  $G$  to mean that  $P = \mathbf{P}(F)$ , where  $\mathbf{P}$  is an  $F$ -parabolic subgroup of  $\mathbf{G}$ . If  $P$  contains  $B$ , we usually write  $P = MN$  to mean that  $M$  is the Levi component of  $P$  containing  $Z$ , and  $N$  the unipotent radical of  $P$ ; we then write  $P_{\text{op}} = MN_{\text{op}}$  for the parabolic subgroup opposite to  $P$  with respect to  $M$ ; in particular  $B_{\text{op}} = ZU_{\text{op}}$ .

We let  $\Phi$  be the set of roots of  $\mathbf{S}$  in  $\mathbf{G}$ , so  $\Phi$  is a subset of the group  $X^*(\mathbf{S})$  of characters of  $\mathbf{S}$ ; we let  $\Phi^+$  be the subset of roots of  $\mathbf{S}$  in  $\mathbf{U}$ , called positive roots, and  $\Delta$  for the set of simple roots of  $\mathbf{S}$  in  $\mathbf{U}$ . If  $X_*(\mathbf{S})$  is the group of cocharacters of  $\mathbf{S}$  we write  $\langle \cdot, \cdot \rangle$  for the natural pairing  $X^*(\mathbf{S}) \times X_*(\mathbf{S}) \rightarrow \mathbb{Z}$ ; for  $\alpha$  in  $\Phi$ , the corresponding coroot [SGA3, exposé XXVI, §7] is written  $\alpha^\vee$  and for  $I \subset \Phi$  we put  $I^\vee = \{\alpha^\vee \mid \alpha \in I\}$ . We choose a positive definite symmetric bilinear form on  $X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ , invariant under  $W_0$ , which

<sup>4</sup>We shall use a similar convention for groups over  $k$ .

<sup>5</sup> $\mathbf{G}$  is fixed, but otherwise arbitrary, so the results we establish for  $\mathbf{G}$  can be applied to other reductive groups over  $F$ .

induces a notion of orthogonality between roots; for roots  $\alpha, \beta$  we have  $\alpha \perp \beta$  if and only if  $\langle \alpha, \beta^\vee \rangle = 0$ .

For  $\alpha$  in  $\Phi$  we write  $U_\alpha = \mathbf{U}_\alpha(F)$ , for the corresponding root subgroup ( $\mathbf{U}_\alpha$  is written  $\mathbf{U}_{(\alpha)}$  in [Bo, §21]), and  $s_\alpha \in W_0$  for the corresponding reflection. For  $I \subset \Delta$  we let  $W_I$  be the subgroup generated by  $\{s_\alpha \mid \alpha \in I\}$ ,  $\mathcal{N}_I$  for the inverse image of  $W_I$  in  $\mathcal{N}$ ,  $P_I$  for the parabolic subgroup  $UN_IU$  (it contains  $B$ ),  $P_I = M_I N_I$  for its Levi decomposition,  $M_I$  containing  $Z$ ; if  $I$  is a singleton  $\{\alpha\}$  we rather write  $P_\alpha = M_\alpha N_\alpha$ . We set  $\Delta_P = I$  if  $P = P_I$ . We note that for  $I, J \subset \Delta$ ,  $P_{I \cap J} = P_I \cap P_J$ ,  $M_{I \cap J} = M_I \cap M_J$ .

**II.2.** As announced in the introduction, we tackle here a preliminary question: if  $P$  is a parabolic subgroup of  $G$  and  $\sigma$  a representation of  $P$  trivial on its unipotent radical  $N$ , when can  $\sigma$  be extended to a larger parabolic subgroup  $Q$  of  $G$ ? Dividing by the unipotent radical of  $Q$ , which is contained in  $N$ , we lose no generality in assuming that  $Q = G$ . If  $\sigma$  extends to  $G$ , then any extension has to be trivial on the normal subgroup  $\langle {}^G N \rangle$  of  $G$  generated by  $N$ , so that  $\sigma$  has to be trivial on  $P \cap \langle {}^G N \rangle$ . So we need to understand what  $\langle {}^G N \rangle$  is. That question, which involves no representation theory, will be dealt with presently.

**II.3.** Of particular importance in our setting will be the subgroup  $G'$  of  $G$  generated by  $U$  and  $U_{\text{op}}$ . Beware that the notation, which will be applied to other reductive groups (like the Levi subgroups of  $\mathbf{G}$ ), is unusual, and that  $G'$  is not generally the group of points over  $F$  of a reductive subgroup of  $\mathbf{G}$ : this occurs already for  $\mathbf{G} = \text{PGL}_2$ . Since  $G$  is generated by  $U$ ,  $U_{\text{op}}$  and  $Z$ , see e.g. [BoT, Proposition 6.25],  $G'$  is normal in  $G$  so is also the subgroup of  $G$  generated by the unipotent radicals of the parabolic subgroups of  $G$ , and we have  $G = ZG'$ . Sometimes we have  $G' = G$ , though.

**Proposition** *Assume that  $\mathbf{G}$  is semisimple, simply connected,  $F$ -simple and isotropic. Then  $G' = G$ , and  $G$  has no non-central proper normal subgroup. Besides,  $Z$  is generated by the  $Z \cap M'_\alpha$ ,  $\alpha$  running through  $\Delta$ .*

**Proof** The first assertion is due to Platonov [PIR, Theorem 7.6] and the second one then follows from work of Tits [PIR, Theorem 7.1]. The final assertion is due to Prasad and Raghunathan [PrR] – actually their result is valid over any field.  $\square$

**Remark** Let  $\mathbf{G}$  be as in the proposition, let  $\alpha \in \Delta$  and  $\mathbf{G}_\alpha$  the subgroup of  $\mathbf{G}$  generated by  $\mathbf{U}_\alpha$  and  $\mathbf{U}_{-\alpha}$ ; since  $\mathbf{G}_\alpha$  satisfies the hypotheses of the proposition, we have  $M'_\alpha = G'_\alpha = G_\alpha$ .

**II.4.** To understand  $G'$  in general we need the simply connected covering  $\mathbf{G}^{\text{sc}}$  of the derived group  $\mathbf{G}^{\text{der}}$  of  $\mathbf{G}$ . Recall that  $\mathbf{G}^{\text{sc}}$  is the direct product of its  $F$ -simple components. We let  $\mathcal{B}$  be an indexing set for the **isotropic**  $F$ -simple components of  $\mathbf{G}^{\text{sc}}$  and for  $b \in \mathcal{B}$  we write  $\tilde{\mathbf{G}}_b$  for the corresponding component. We put  $\mathbf{G}^{\text{is}} = \prod_{b \in \mathcal{B}} \tilde{\mathbf{G}}_b$ , and denote by  $\iota$  the natural homomorphism  $\mathbf{G}^{\text{is}} \rightarrow \mathbf{G}$ , factoring through  $\mathbf{G}^{\text{is}} \rightarrow \mathbf{G}^{\text{sc}} \rightarrow \mathbf{G}^{\text{der}} \rightarrow \mathbf{G}$ .

We need to understand the relation between parabolic subgroups of  $\mathbf{G}$  and parabolic subgroups of  $\mathbf{G}^{\text{is}}$ . The following comes from [Bo, §21, §22], going through the factorization of  $\iota$ .

The connected component of  $\iota^{-1}(\mathbf{S})$  is a maximal  $F$ -split torus  $\tilde{\mathbf{S}}$  of  $\mathbf{G}^{\text{is}}$ , and  $\mathbf{S}$  is the product of  $\iota(\tilde{\mathbf{S}})$  and the maximal  $F$ -split torus in the centre of  $\mathbf{G}$ . The centralizer of  $\tilde{\mathbf{S}}$  in  $\mathbf{G}^{\text{is}}$  is  $\tilde{\mathbf{Z}} = \iota^{-1}(\mathbf{Z})$ , its normalizer  $\tilde{\mathcal{N}} = \iota^{-1}(\mathcal{N})$ , and  $\iota$  induces an isomorphism  $W(\mathbf{G}^{\text{is}}, \tilde{\mathbf{S}}) =$



$W(\mathbf{G}, \mathbf{S})$  (see in particular [Bo, 22.6 Theorem]); in particular  $W_0$  has representatives in  $\iota(G^{\text{is}})$ . As  $\mathbf{G}^{\text{is}}$  is a direct product  $\prod \tilde{\mathbf{G}}_b$  (over  $b \in \mathcal{B}$ ) we have corresponding natural decompositions  $\tilde{\mathbf{S}} = \prod \tilde{\mathbf{S}}_b$ ,  $\tilde{\mathbf{Z}} = \prod \tilde{\mathbf{Z}}_b$ ,  $\tilde{\mathcal{N}} = \prod \tilde{\mathcal{N}}_b$  and  $W(\mathbf{G}^{\text{is}}, \mathbf{S}) = \prod W(\tilde{\mathbf{G}}_b, \tilde{\mathbf{S}}_b)$ . Note that  $\iota(\tilde{\mathbf{G}}_b)$  is normal in  $\mathbf{G}$  for each  $b \in \mathcal{B}$ .

The map  $\mathbf{P} \mapsto \tilde{\mathbf{P}} = \iota^{-1}(\mathbf{P})$  is a bijection between  $F$ -parabolic subgroups of  $\mathbf{G}$  and  $F$ -parabolic subgroups of  $\mathbf{G}^{\text{is}}$ , and  $\iota$  induces an isomorphism (cf. loc. cit.) of the unipotent radical  $\tilde{\mathbf{N}}$  of  $\tilde{\mathbf{P}}$  onto the unipotent radical  $\mathbf{N}$  of  $\mathbf{P}$ . Also,  $\tilde{\mathbf{M}} = \iota^{-1}(\mathbf{M})$  is the Levi component of  $\tilde{\mathbf{P}}$  containing  $\tilde{\mathbf{Z}}$ . In particular  $\tilde{\mathbf{B}} = \iota^{-1}(\mathbf{B})$  is a minimal  $F$ -parabolic subgroup of  $\mathbf{G}^{\text{is}}$ ; it is the direct product of minimal parabolic subgroups  $\tilde{\mathbf{B}}_b$  of  $\tilde{\mathbf{G}}_b$ , and its unipotent radical  $\tilde{\mathbf{U}}$  is the direct product of the  $\tilde{\mathbf{U}}_b$ , with  $\tilde{\mathbf{U}}_b$  the unipotent radical of  $\tilde{\mathbf{B}}_b$ . Via  $\iota$  we get an identification<sup>6</sup> of the roots of  $\mathbf{S}$  in  $\mathbf{U}$  with the roots of  $\tilde{\mathbf{S}}$  in  $\tilde{\mathbf{U}}$ , so that  $\Delta$ , in particular, also appears as the set of simple roots of  $\tilde{\mathbf{S}}$  in  $\tilde{\mathbf{U}}$ ; as such  $\Delta$  is a disjoint union of the sets  $\Delta_b$ ,  $b \in \mathcal{B}$ , where  $\Delta_b$  is the set of roots of  $\tilde{\mathbf{S}}$  (or  $\tilde{\mathbf{S}}_b$ ) in  $\tilde{\mathbf{U}}_b$ ; that partition of  $\Delta$  is the finest partition into mutually orthogonal subsets. Those subsets are the connected components of the Dynkin diagram of  $\mathbf{G}$  (with set of vertices  $\Delta$ ) so we can view  $\mathcal{B}$  as the set of such components.

**Proposition**  $G' = \iota(G^{\text{is}})$ .

Indeed by II.3 Proposition we have  $\tilde{G}'_b = \tilde{G}_b$  for each  $b \in \mathcal{B}$  so  $(G^{\text{is}})' = G^{\text{is}}$ ; since  $\iota$  induces an isomorphism of  $\tilde{U}$  onto  $U$  and  $\tilde{U}_{\text{op}}$  onto  $U_{\text{op}}$ , we get  $G' = \iota(G^{\text{is}})$ .  $\square$

Note that the proposition implies that  $Z \cap G' = \iota(\tilde{Z})$ .

**II.5. Notation** For  $I \subset \Delta$ , set  $\mathcal{B}(I) = \{b \in \mathcal{B} \mid I \cap \Delta_b \neq \emptyset\}$ .

**Proposition** Let  $I \subset \Delta$ . Then the normal subgroup  $\langle {}^G N_I \rangle$  of  $G$  generated by  $N_I$  is  $\iota(\prod_{b \in \mathcal{B}(I)} \tilde{G}_b)$ .

**Proof** We have  $\tilde{N}_I = \prod_{b \in \mathcal{B}} (\tilde{N}_I \cap \tilde{G}_b)$  and  $\tilde{N}_I \cap \tilde{G}_b$  is the unipotent subgroup of  $\tilde{G}_b$  corresponding to  $I \cap \Delta_b \subset \Delta_b$ . For  $b \in \mathcal{B} - \mathcal{B}(I)$ ,  $\tilde{N}_I \cap \tilde{G}_b$  is trivial; for  $b \in \mathcal{B}(I)$ ,  $\tilde{N}_I \cap \tilde{G}_b$  is non-trivial, and provides a non-central subgroup of  $\tilde{G}_b$  so by II.3 Proposition the normal subgroup of  $G^{\text{is}}$  generated by  $\tilde{N}_I \cap \tilde{G}_b$  is  $\tilde{G}_b$ ; the proposition follows.  $\square$

**Corollary**

- (i)  $P_I \cap \langle {}^G N_I \rangle = \iota(\prod_{b \in \mathcal{B}(I)} (\tilde{P}_I \cap \tilde{G}_b))$ ,
- (ii)  $M_I \cap \langle {}^G N_I \rangle = \iota(\prod_{b \in \mathcal{B}(I)} (\tilde{M}_I \cap \tilde{G}_b))$ ,
- (iii)  $M_I \langle {}^G N_I \rangle = G$ ,
- (iv)  $\langle {}^G N_I \rangle$  contains  $N_{I, \text{op}}$ .

<sup>6</sup>More precisely the natural map  $\tilde{\mathbf{S}} \rightarrow \mathbf{S}$  induces a group homomorphism  $X^*(\mathbf{S}) \rightarrow X^*(\tilde{\mathbf{S}})$  through which the roots of  $\mathbf{S}$  in  $\mathbf{U}$  are identified with the roots of  $\tilde{\mathbf{S}}$  in  $\tilde{\mathbf{U}}$ . By [SGA3, Exp. XXVI, 7.4] if  $\alpha$  is a root of  $\mathbf{S}$  in  $\mathbf{U}$  and  $\tilde{\alpha}$  the corresponding root of  $\tilde{\mathbf{S}}$  in  $\tilde{\mathbf{U}}$ , then  $\tilde{\alpha}^\vee$  goes to  $\alpha^\vee$  via the transposed morphism  $X_*(\tilde{\mathbf{S}}) \rightarrow X_*(\mathbf{S})$ . In the sequel we make no distinction between  $\alpha$  and  $\tilde{\alpha}$ ,  $\alpha^\vee$  and  $\tilde{\alpha}^\vee$ .

**Proof** Parts (i) and (ii) are immediate consequences of the previous considerations. Let us prove (iii). From the proposition  $\langle {}^G N_I \rangle$  contains  $\iota(\tilde{G}_b)$  for  $b \in \mathcal{B}(I)$ , but for  $b \in \mathcal{B} - \mathcal{B}(I)$ ,  $M_I$  contains  $\iota(\tilde{G}_b)$ , so finally  $M_I \langle {}^G N_I \rangle$  contains  $\iota(G^{\text{is}}) = G'$ . Since  $M_I$  contains  $Z$  and  $G = ZG'$ , we get (iii). Part (iv) follows from the proposition because  $N_{I, \text{op}}$  is  $\iota(\prod_{b \in \mathcal{B}(I)} (\tilde{N}_{I, \text{op}} \cap \tilde{G}_b))$ .  $\square$

**Remark** For  $b \in \mathcal{B}$ ,  $\tilde{M}_I \cap \tilde{G}_b$  can be also described as the product  $\prod_c \tilde{M}_{\Delta_c}$  over the connected components  $c$  of the Dynkin diagram obtained from that of  $\tilde{G}_b$  by deleting vertices outside  $I$ . (We note that the product is not direct.)

**II.6.** There is another useful characterization of  $M_I \cap \langle {}^G N_I \rangle$ .

**Proposition** *Let  $I \subset \Delta$ . Then  $M_I \cap \langle {}^G N_I \rangle$  is the normal subgroup of  $M_I$  generated by  $Z \cap M'_\alpha$ , for  $\alpha$  running through  $\Delta - I$ .*

**Proof** Let  $\alpha \in \Delta - I$  and let  $b \in \mathcal{B}$  be such that  $\alpha \in \Delta_b$ , so that  $M'_\alpha \subset \iota(\tilde{G}_b)$ . As  $\alpha \notin I$ ,  $b$  belongs to  $\mathcal{B}(I)$  so  $\iota(\tilde{G}_b)$  is included in  $\langle {}^G N_I \rangle$  by II.5 Proposition, and consequently  $Z \cap M'_\alpha \subset M_I \cap \langle {}^G N_I \rangle$ . To prove that  $M_I \cap \langle {}^G N_I \rangle$  is the normal subgroup of  $M_I$  generated by the  $Z \cap M'_\alpha$ ,  $\alpha \in \Delta - I$ , it is enough, by loc. cit., to work within  $\tilde{G}_b$ . So we now assume that  $\mathbf{G} = \mathbf{G}^{\text{is}}$  and  $\mathbf{G}$  is  $F$ -simple. If  $I = \Delta$ ,  $N_I$  is trivial so there is nothing to prove. So let us assume  $I \neq \Delta$ , so that  $\langle {}^G N_I \rangle = G$  by II.3 since  $N_I$  is not trivial. We can apply to  $\mathbf{M}_I$  all the considerations applied to  $\mathbf{G}$  in the current chapter, so we see that  $M_I = Z \prod H_J$  where  $J$  runs through connected components of the Dynkin diagram with set of vertices  $I$  associated to  $M_I$ , and  $\mathbf{H}_J$  is the corresponding semisimple simply connected  $F$ -simple subgroup of  $\mathbf{M}_I$ . Let  $J$  be such a connected component. As the Dynkin diagram attached to  $G$  is by assumption connected, there is  $\alpha$  in  $\Delta - I$  with  $\langle J, \alpha^\vee \rangle \neq 0$ . Choose  $\alpha'$  in  $J$  with  $\langle \alpha', \alpha^\vee \rangle \neq 0$  and  $x \in F^\times$  with  $\alpha'(\alpha^\vee(x))^2 \neq 1$ . We have  $\alpha^\vee(x) \in Z \cap M'_\alpha$ ,  $U_{\alpha'} \subset H_J \subset M_I$ , and the map from  $U_{\alpha'}$  to itself given by  $u \mapsto \alpha^\vee(x)u\alpha^\vee(x)^{-1}u^{-1}$  is onto<sup>7</sup>. The normal subgroup of  $M_I$  generated by  $Z \cap M'_\alpha$  contains  $\alpha^\vee(x)$ <sup>8</sup> and  $u\alpha^\vee(x)^{-1}u^{-1}$  for  $u \in U_{\alpha'}$ , so it contains  $U_{\alpha'}$ . By II.3 Proposition it contains  $H_J$  and in particular  $Z \cap M'_{\alpha''}$  for all  $\alpha'' \in J$ . We conclude that the normal subgroup of  $M_I$  generated by the  $Z \cap M'_\alpha$ ,  $\alpha \in \Delta - I$ , contains  $Z \cap M'_\alpha$  for all  $\alpha \in \Delta$ . By II.3 Proposition it contains  $Z$ ; since we have seen that it contains each  $H_J$ , it is equal to  $M_I = Z \prod H_J$ .  $\square$

**II.7.** Keeping the same notation, we can now derive consequences for representations.

**Proposition** *Let  $I \subset \Delta$ , and let  $\sigma$  be a representation of  $M_I$ . Then the following conditions are equivalent:*

- (i)  $\sigma$  extends to a representation of  $G$  trivial on  $N_I$ ,
- (ii) for each  $b \in \mathcal{B}(I)$ ,  $\sigma$  is trivial on  $\iota(\tilde{M}_I \cap \tilde{G}_b)$ ,
- (iii) for each  $\alpha \in \Delta - I$ ,  $\sigma$  is trivial on  $Z \cap M'_\alpha$ .

*When these conditions are satisfied, there exists a unique extension  ${}^e\sigma$  of  $\sigma$  to  $G$  which is trivial on  $N_I$ , and it is smooth, admissible or irreducible if and only if  $\sigma$  is.*

<sup>7</sup>If  $2\alpha'$  is not a root, then  $\alpha^\vee(x)$  acts on  $U_{\alpha'}$  (a vector group) via multiplication by  $\alpha'(\alpha^\vee(x))$ . If  $2\alpha'$  is a root, then  $\alpha^\vee(x)$  acts on  $U_{2\alpha'}$  via  $\alpha'(\alpha^\vee(x))^2$  and on  $U_{\alpha'}/U_{2\alpha'}$  via  $\alpha'(\alpha^\vee(x))$ .

<sup>8</sup>It follows from II.4, footnote 5, that  $\alpha^\vee(x)$  belongs to  $M'_\alpha$ ; on the other hand it belongs to  $S \subset M_I$ .

**Proof** As already said in II.2, if  $\sigma$  extends to a representation of  $G$  trivial on  $N_I$ , the extension is trivial on  $\langle {}^G N_I \rangle$  so  $\sigma$  is certainly trivial on  $M_I \cap \langle {}^G N_I \rangle$ . Consequently (i) implies (ii) and (iii) by II.5, II.6. Conversely, under assumptions (ii) or (iii),  $\sigma$  is trivial on  $M_I \cap \langle {}^G N_I \rangle$  hence extends, trivially on  $\langle {}^G N_I \rangle$ , to a representation of  $M_I \langle {}^G N_I \rangle$ , which is  $G$  by II.5 Corollary (iii). Besides the extension  ${}^e\sigma$  is necessarily unique. Assume that  $\sigma$  extends to a representation  ${}^e\sigma$  of  $G$  trivial on  $N_I$ . Since  $\sigma$  and  ${}^e\sigma$  have the same image,  $\sigma$  is irreducible if and only if  ${}^e\sigma$  is. As  $P_I$  is a topological subgroup of  $G$ ,  $\sigma$  is smooth if  ${}^e\sigma$  is. Conversely, assume that  $\sigma$  is smooth and let  $x$  be a vector in the space of  $\sigma$ ,  $J$  its stabilizer in  $P_I$ ; by II.5 Corollary (iv),  $N_{I,\text{op}}$  acts trivially on  ${}^e\sigma$  and the stabilizer of  $x$  in  $G$ , which contains  $N_{I,\text{op}}J$ , is open in  $G$ , so  ${}^e\sigma$  is smooth.

As  $P_I$  is a topological subgroup of  $G$ ,  ${}^e\sigma$  is admissible if  $\sigma$  is. Conversely assume  ${}^e\sigma$  is admissible; for each open subgroup  $J$  of  $M_I$ , a vector in  $\sigma$  fixed by  $J$  is also fixed by the subgroup generated by  $J$ ,  $N_I$  and  $N_{I,\text{op}}$  which is open in  $G$ , so  $\sigma$  is admissible.  $\square$

**Remark 1** By II.5 Remark, condition (ii) illustrates that  $\sigma$  can extend to  $G$  (trivially on  $N_I$ ) only for very strong reasons: for any connected component  $\Delta_b$  of the Dynkin diagram of  $G$  meeting  $\Delta - I$ ,  $\sigma$  has to be trivial on  $M'_{\Delta_c}$  for any connected component  $\Delta_c$  of the Dynkin diagram of  $M_I$  included in  $\Delta_b$ . By II.3 Proposition applied to  $M_{\Delta_c}^{\text{is}}$  that last condition is also equivalent to  $\sigma$  being trivial on  $U_\beta$  for some, or any,  $\beta \in \Delta_c$ .

**Notation** Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ , and let  $\sigma$  be a representation of  $M$ . We let  $\Delta(\sigma)$  be the set of  $\alpha \in \Delta - \Delta_P$  such that  $\sigma$  is trivial on  $Z \cap M'_\alpha$ . We let  $P(\sigma)$  be the parabolic subgroup corresponding to  $\Delta(\sigma) \sqcup \Delta_P$ .

**Corollary 1** *Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ , and let  $\sigma$  be a representation of  $M$ . Then the parabolic subgroups of  $G$  containing  $P$  to which  $\sigma$  extends, trivially on  $N$ , are those contained in  $P(\sigma)$ . In that case the extension is unique and is smooth, admissible or irreducible if  $\sigma$  is.*

The corollary is immediate from the proposition applied to Levi components of parabolic subgroups of  $G$  containing  $P$ .

**Remark 2** Since any parabolic subgroup  $P$  of  $G$  is conjugate to one containing  $B$ , it follows, as stated in the introduction, that if  $\sigma$  is a representation of  $P$  trivial on its unipotent radical, there is a maximal parabolic subgroup  $P(\sigma)$  of  $G$  to which  $\sigma$  can be extended, and the extension is smooth, admissible or irreducible if (and only if)  $\sigma$  is.

**Corollary 2** *Keep the assumptions and notation of Corollary 1, and assume further that  $\Delta(\sigma)$  is not orthogonal to  $\Delta_M$ . Then there is a proper parabolic subgroup  $Q$  of  $M$ , containing  $M \cap B$ , such that  $\sigma$  is trivial on the unipotent radical of  $Q$ ; moreover  $\sigma$  is a subrepresentation of  $\text{Ind}_Q^M(\sigma|_Q)$ , and  $\sigma|_Q$  is irreducible or admissible if  $\sigma$  is. In particular,  $\sigma$  cannot be supercuspidal.*

**Proof** We may assume that  $G = P(\sigma)$ . Let  $\alpha \in \Delta(\sigma)$  not orthogonal to  $\Delta_M$ , and let  $b \in \mathcal{B}$  such that  $\alpha \in \Delta_b$ . Then  $\Delta_b \cap \Delta_M \neq \Delta_b$ , so  $\sigma$  is trivial on  $\iota(\tilde{M} \cap \tilde{G}_b)$  by the proposition. As  $\alpha$  is not orthogonal to  $\Delta_M$ ,  $\Delta_b \cap \Delta_M$  is not empty. If  $Q$  is the (proper) parabolic subgroup of  $M$  corresponding to  $\Delta_M - \Delta_b$ , then  $\iota(\tilde{M} \cap \tilde{G}_b)$  contains the unipotent radical  $N_Q$  of  $Q$  and  $\sigma$  is trivial on  $N_Q$ . Then, obviously,  $\sigma$  is a subrepresentation of  $\text{Ind}_Q^M(\sigma|_Q)$  and by the proposition, applied to  $M$  instead of  $G$ , if  $\sigma$  is irreducible or admissible, so is

its restriction to the Levi component of  $Q$ . By the definition of supercuspidality,  $\sigma$  cannot be supercuspidal.  $\square$

**Remark 3** The last assertion of Corollary 2 explains why the case of interest to us is when  $\Delta_M$  and  $\Delta(\sigma)$  are orthogonal – an analogous result will be obtained when  $\sigma$  is assumed supersingular instead of supercuspidal (III.17 Corollary). As a special case, assume that the (relative) Dynkin diagram of  $\mathbf{G}$  is connected, and  $\sigma$  is a supercuspidal representation of  $M$  extending to  $G$ . Then either  $M = G$  or  $M = Z$ ; in the latter case,  $\sigma$  is trivial on  $Z \cap G'$  and finite dimensional.

**Remark 4** For the record, let us state a few useful facts when  $\Delta$  is the disjoint union of two subsets  $I$  and  $J$ , orthogonal to each other. Then  $M'_I$  and  $M'_J$  are normal subgroups of  $G$ , commuting with each other. We have  $G' = M'_I M'_J$ ,  $M_I = Z M'_I$ ,  $M_J = Z M'_J$ ,  $M_I \cap M_J = Z$  and in particular  $M_I \cap M'_\alpha = Z \cap M'_\alpha$  for  $\alpha \in J$ . Also,  $M'_I \cap M'_J$  is finite and central in  $G$ : indeed, decomposing  $\tilde{G}$  as  $\tilde{G}_I \times \tilde{G}_J$ ,  $M'_I \cap M'_J$  is simply the image under  $(g_1, g_2) \mapsto \iota(g_1)$  of  $\text{Ker } \iota \subset \tilde{G}_I \times \tilde{G}_J$ . The inclusion of  $M_I$  in  $G$  induces an isomorphism  $M_I / (M_I \cap M'_J) \simeq G / M'_J$  (and similarly for  $M_J$ ).

**Remark 5** Let  $\alpha \in \Delta$  belong to the component  $\Delta_b$ . The normal subgroup of  $G$  generated by  $Z \cap M'_\alpha$  is  $\iota(\tilde{G}_b)$  because  $Z \cap M'_\alpha$  is not central in  $M'_\alpha$ . If  $\sigma$  is a representation of  $G$  which is trivial on  $Z \cap M'_\alpha$ , it is then trivial on  $\iota(\tilde{G}_b)$  and the conclusions of Corollary 2 hold (with  $M = G$ ).

**II.8.** To go further we need the generalized Steinberg representations already recalled in the introduction.

**Lemma** *Let  $Q$  be a parabolic subgroup of  $G$ . Then lifting functions on  $G$  to functions on  $G^{\text{is}}$  via  $\iota$  gives an isomorphism of  $\text{Ind}_Q^G 1$  with  $\text{Ind}_{\tilde{Q}}^{G^{\text{is}}} 1$ . The representation  $\text{St}_Q^G \circ \iota$  of  $G^{\text{is}}$  is isomorphic to  $\text{St}_{\tilde{Q}}^{G^{\text{is}}}$ ; the restriction of  $\text{St}_Q^G$  to  $G'$  is irreducible and admissible.*

**Proof** We have  $ZG' = G$  and  $Q$  contains  $Z$ , so  $G = QG'$ . Besides  $Q \cap G' = \iota(\tilde{Q})$ . It follows that  $\iota$  induces a bijection of  $\tilde{Q} \backslash G^{\text{is}}$  onto  $Q \backslash G$ ; that bijection is continuous hence is a homeomorphism by Arens' theorem [MZ, p. 65]. The first assertion follows and the others are immediate consequences.  $\square$

Now let  $P = MN$  be a parabolic subgroup of  $G$ , let  $\sigma$  be a representation of  $M$ , inflated to  $P$ . Then by II.7 Corollary 1,  $\sigma$  extends (uniquely) to a representation  ${}^e\sigma$  of  $P(\sigma)$ . For each parabolic subgroup  $Q$  with  $P \subset Q \subset P(\sigma)$  we can form the representation  ${}^e\sigma \otimes \text{St}_Q^{P(\sigma)}$  of  $P(\sigma)$ .

**Proposition**  *$\sigma$  is irreducible (resp. admissible) if and only if  ${}^e\sigma \otimes \text{St}_Q^{P(\sigma)}$  is irreducible (resp. admissible).*

From this, we get (see for instance [Vig3, Lemma 4.7]):

**Corollary**  *$\sigma$  is admissible if and only if  $\text{Ind}_{P(\sigma)}^G ({}^e\sigma \otimes \text{St}_Q^{P(\sigma)})$  is admissible.*

**Proof of the proposition** The unipotent radical of  $P(\sigma)$  acts trivially on both  ${}^e\sigma$  and  $\text{St}_Q^{P(\sigma)}$ . Therefore we may assume  $P(\sigma) = G$ .

By the lemma above  $\text{St}_Q^G \circ \iota$  is the generalized Steinberg representation  $\text{St}_Q^{G^{\text{is}}}$ . For  $b$  in  $\mathcal{B} - \mathcal{B}(\Delta_Q)$ ,  $\Delta_Q \cap \Delta_b = \Delta_b$  so that by construction  $\text{St}_Q^{G^{\text{is}}}$  is trivial on  $\tilde{G}_b$ ; consequently its restriction to  $H = \prod_{b \in \mathcal{B}(\Delta_Q)} \tilde{G}_b$  is irreducible. On the other hand by II.5,  ${}^e\sigma$  is trivial on the normal subgroup  $\iota(H)$ . If  $\sigma$  is irreducible, the irreducibility of  ${}^e\sigma \otimes \text{St}_Q^G$  comes then from Clifford theory as in [Abe, Lemma 5.3]<sup>9</sup>.

Assume that  $\sigma$  is admissible, so  ${}^e\sigma$  is admissible too. As above  $\iota(H)$  acts trivially on  ${}^e\sigma$  and the restriction of  $\text{St}_Q^G$  to  $\iota(H)$  is admissible. If  $L$  is an open subgroup of  $G$ , the vectors in  $\text{St}_Q^G$  fixed under  $L \cap \iota(H)$  form a finite dimensional vector space  $X$ . The vectors fixed by  $L$  in  ${}^e\sigma \otimes \text{St}_Q^G$  are in  ${}^e\sigma \otimes X$ . There is an open subgroup  $L'$  of  $L$  acting trivially on  $X$  and  $({}^e\sigma \otimes X)^{L'} = {}^e\sigma^{L'} \otimes X$  is finite dimensional. Consequently  ${}^e\sigma \otimes \text{St}_Q^G$  is admissible.

Conversely, if  ${}^e\sigma \otimes \text{St}_Q^G$  is irreducible, obviously  $\sigma$  is irreducible. If  ${}^e\sigma \otimes \text{St}_Q^G$  is admissible so is  $\sigma$ . Indeed if  $J$  is an open subgroup of  $G$  then  $({}^e\sigma)^J \otimes (\text{St}_Q^G)^J$  is contained in  $({}^e\sigma \otimes \text{St}_Q^G)^J$ , so if  $J$  is small enough for  $(\text{St}_Q^G)^J$  to be non-zero, we deduce that  $({}^e\sigma)^J$  is finite-dimensional; thus  ${}^e\sigma$  is admissible and so is  $\sigma$  by II.7 Proposition.  $\square$

**Remark** Assume that  $\Delta_M$  is orthogonal to  $\Delta - \Delta_M$ . Let  $\sigma$  be a representation of  $M$  which extends to  $G$  trivially on  $N$ , and let  $Q$  be a parabolic subgroup of  $G$  containing  $P$ .

1) The representation  ${}^e\sigma \otimes \text{St}_Q^G$  of  $G$  determines  $\sigma$  and  $Q$ .

2) Any subquotient  $\pi$  of  ${}^e\sigma \otimes \text{St}_Q^G$  is of the form  ${}^e\sigma_\pi \otimes \text{St}_Q^G$  for some representation  $\sigma_\pi$  of  $M$  which extends to  $G$  trivially on  $N$ .

**Proof** 1) We put  $J = \Delta - \Delta_M$ . As  $Q$  contains  $M$ ,  $\text{St}_Q^G$  is trivial on the normal subgroup  $M'$ , and restricting to  $M_J$  functions on  $G$  gives an isomorphism of  $\text{St}_Q^G$  onto  $\text{St}_{Q \cap M_J}^{M_J}$ . The restriction of  ${}^e\sigma \otimes \text{St}_Q^G$  to  $M'_J$  is a direct sum of irreducible representations  $\text{St}_Q^G|_{M'_J}$ , and that representation determines  $Q$  (II.8 Lemma). Seen as a representation of  $G$ ,  $\text{Hom}_{M'_J}(\text{St}_Q^G, {}^e\sigma \otimes \text{St}_Q^G)$  is isomorphic to  ${}^e\sigma$  (use for example [Abe, Lemma 5.3]), and  ${}^e\sigma$  determines  $\sigma$ .

2) The restriction of  $\pi$  to  $M'_J$  is a sum of copies of the irreducible representation  $\text{St}_Q^G|_{M'_J}$ . By Clifford's theory (loc. cit.),  $\pi$  is isomorphic to  $\text{Hom}_{M'_J}(\text{St}_Q^G, \pi) \otimes \text{St}_Q^G$ . Moreover,  $\text{Hom}_{M'_J}(\text{St}_Q^G, \pi)$  is a representation of  $G$  trivial on  $M'_J$  hence determines a representation  $\sigma_\pi$  of  $M$  via the map  $M \rightarrow G/M'_J$  and  ${}^e\sigma_\pi \simeq \text{Hom}_{M'_J}(\text{St}_Q^G, \pi)$  as a representation of  $G$ .  $\square$

### III. SUPERSINGULARITY AND CLASSIFICATION

**III.1.** This chapter is devoted to the proof of I.5 Theorem 4, and is rather long. It is divided into parts A) to H). In part A) we give some more detail on supersingularity, and in part B) we describe a parametrization for the irreducible representations of  $K$ . The next step in part C) is to determine the weights and eigenvalues of parabolically induced representations. We then proceed to the analysis of the representations  $I(P, \sigma, Q)$ : we first determine  $P(\sigma)$  in part D), and after that we compute the weights and eigenvalues of  $I(P, \sigma, Q)$  for a supersingular triple  $(P, \sigma, Q)$  in part E). The subsequent proof of the irreducibility of  $I(P, \sigma, Q)$  in part F) uses a change of weight theorem proved in

<sup>9</sup>To apply that lemma, note that Schur's lemma is valid for the restriction of  $\text{St}_Q^G$  to  $\iota(H)$ .

Chapter IV. From the knowledge of weights and eigenvalues, we easily deduce in part G) when  $I(P_1, \sigma_1, Q_1)$  is isomorphic to  $I(P_2, \sigma_2, Q_2)$  for supersingular triples  $(P_1, \sigma_1, Q_1)$  and  $(P_2, \sigma_2, Q_2)$ . In part H) we finally prove exhaustion, i.e. that every irreducible admissible representation of  $G$  has the form  $I(P, \sigma, Q)$  for some supersingular triple  $(P, \sigma, Q)$ : that uses a result established only in Chapter V as a further consequence of the change of weight theorem.

**Notation** As  $K$  is fixed throughout, it is convenient to write  $H^0$  for  $H \cap K$ , when  $H$  is a subgroup of  $G$ . We also write  $\bar{H}$  for  $(H \cap K)/(H \cap K(1))$ , where  $K(1)$  is the pro- $p$  radical of  $K$ . We put  $Z(1) = Z \cap K(1)$ ; it is the pro- $p$  radical of  $Z^0$ .

### A) Supersingularity

**III.2.** Consider an irreducible representation  $(\rho, V)$  of  $K$ ; it is finite-dimensional and trivial on  $K(1)$ . The classification of such objects will be recalled in part B).

We view the intertwining algebra  $\mathcal{H}_G(V)$  as a Hecke algebra, the convolution algebra of compactly supported functions  $\Phi : G \rightarrow \text{End}_C(V)$  satisfying

$$\Phi(kgk') = \rho(k)\Phi(g)\rho(k') \quad \text{for } g \text{ in } G, k \text{ and } k' \text{ in } K.$$

The convolution operation is given by

$$(III.2.1) \quad (\Phi * \Psi)(g) = \sum_{h \in G/K} \Phi(h)\Psi(h^{-1}g) \quad \text{for } \Phi, \Psi \text{ in } \mathcal{H}_G(V).$$

The action on  $\text{ind}_K^G V$  is also given by convolution:

$$(III.2.2) \quad (\Phi * f)(g) = \sum_{h \in G/K} \Phi(h)f(h^{-1}g) \quad \text{for } f \in \text{ind}_K^G V, \Phi \in \mathcal{H}_G(V).$$

**III.3.** We need to recall the structure of  $\mathcal{H}_G(V)$  and its centre  $\mathcal{Z}_G(V)$ , as elucidated in [HV1], building on [He1, He2]; note that  $\mathcal{H}_G(V)$  is commutative in the context of [He1, He2, Abe].

Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ . Then the space of coinvariants  $V_{N^0}$  of  $N^0$  in  $V$  affords an irreducible representation of  $M^0$  (which is the special parahoric subgroup of  $M$  corresponding to the special point  $\mathbf{x}_0$ ). For each representation  $\sigma$  of  $M$  on a vector space  $W$ , Frobenius reciprocity and the equalities  $G = KP = PK$ ,  $P^0 = M^0N^0$ , give a canonical isomorphism:

$$(III.3.1) \quad \text{Hom}_G(\text{ind}_K^G V, \text{Ind}_P^G W) \xrightarrow{\sim} \text{Hom}_M(\text{ind}_{M^0}^M V_{N^0}, W)$$

The natural algebra homomorphism  $\mathcal{S}_M^G : \mathcal{H}_G(V) \rightarrow \mathcal{H}_M(V_{N^0})$  of I.5 is given concretely by

$$(III.3.2) \quad [\mathcal{S}_M^G(\Psi)(m)]\bar{v} = \sum_{n \in N^0 \setminus N} \overline{\Psi(nm)(v)} \quad \text{for } m \text{ in } M, v \text{ in } V,$$

where a bar indicates the image in  $V_{N^0}$  of a vector in  $V$  [HV2, Proposition 2.2]. Recall that (III.3.1) is  $\mathcal{H}_G(V)$ -linear if we let  $\mathcal{H}_G(V)$  act on the right-hand side via  $\mathcal{S}_M^G$ . Recall also that  $\mathcal{S}_M^G$  is injective [HV2, Proposition 4.1].

For varying  $P = MN$ , the homomorphisms  $\mathcal{S}_M^G$  satisfy obvious transitivity properties, and  $\mathcal{S}_Z^G$  identifies  $\mathcal{H}_G(V)$  with a subalgebra of  $\mathcal{H}_Z(V_{U^0})$  which we now describe. For a root  $\alpha$  in  $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ , the group homomorphism  $|\alpha| : x \mapsto |\alpha(x)|$  from  $S$  to  $\mathbb{R}_+^\times$  extends

uniquely to a group homomorphism  $Z \rightarrow \mathbb{R}_+^\times$  trivial on  $Z^0$ , and we still write  $|\alpha|$  for that extension. We write  $Z^+$  for the set of  $z$  in  $Z$  such that  $|\alpha|(z) \leq 1$  for all  $\alpha \in \Delta$ . Then by [HV2, Proposition 4.2]  $\mathcal{H}_G(V)$  is identified via  $\mathcal{S}_Z^G$  with the subalgebra of  $\mathcal{H}_Z(V_{U^0})$  consisting of elements supported on  $Z^+$ . By [HV1, 1.8 Theorem],  $\mathcal{Z}_G(V)$  is the subalgebra  $\mathcal{H}_G(V) \cap \mathcal{Z}_Z(V_{U^0})$  of  $\mathcal{Z}_Z(V_{U^0})$  consisting of elements supported on  $Z^+$ .

**III.4.** The group  $Z$  normalizes  $Z^0$  and its pro- $p$  radical  $Z(1)$  and the quotient  $Z/Z^0$  is a finitely generated abelian group. The coinvariant space  $V_{U^0}$  is in fact a line, and  $Z^0$  acts on it via a character  $\psi_V : Z^0 \rightarrow C^\times$  trivial on  $Z(1)$ : see part B), for the difference between the notation  $\psi_V$  here and in [HV2]. For  $z \in Z$ , the coset  $Z^0 z$  supports a non-zero function in  $\mathcal{H}_Z(V_{U^0})$  if and only if  $z$  normalizes  $\psi_V$ , and such a function is in  $\mathcal{Z}_Z(V_{U^0})$  if and only if  $\psi_V(z z' z^{-1} z'^{-1}) = 1$  for all  $z' \in Z$  normalizing  $\psi_V$ .

**Notation** We let  $Z_{\psi_V}$  be the subgroup of  $Z$  defined by this last condition. It contains  $S$  and  $Z^0$ .

For  $z \in Z$  normalizing  $\psi_V$  we write  $\tau_z \in \mathcal{H}_Z(V_{U^0})$  for the function with support  $Z^0 z$  and value  $\text{id}_{V_{U^0}}$  at  $z$ ; we have

$$\tau_z * \tau_{z'} = \tau_{zz'} \text{ for } z, z' \text{ in } Z \text{ normalizing } \psi_V.$$

Identifying  $\mathcal{H}_G(V)$  and  $\mathcal{H}_M(V_{N^0})$  with subalgebras of  $\mathcal{H}_Z(V_{U^0})$  via  $\mathcal{S}_Z^G$  and  $\mathcal{S}_Z^M$ , we can now describe  $\mathcal{H}_M(V_{N^0})$  as the localization of  $\mathcal{H}_G(V)$  at some central element [HV2, Proposition 4.5] (so that  $\mathcal{Z}_M(V_{N^0})$  is the localization of  $\mathcal{Z}_G(V)$  at the same element).

**Proposition** *Let  $M = M_I$  for some  $I \subset \Delta$ , and let  $s \in S$  satisfy  $|\alpha|(s) < 1$  for  $\alpha \in \Delta - I$ ,  $|\alpha|(s) = 1$  for  $\alpha \in I$ . Then  $\mathcal{H}_M(V_{N^0})$  is the localization of  $\mathcal{H}_G(V)$  at  $\tau_s$ , and  $\mathcal{Z}_M(V_{N^0})$  the localization of  $\mathcal{Z}_G(V)$  at  $\tau_s$ .*

**Notation** For each  $\alpha \in \Delta$ , we choose  $z_\alpha$  in  $S$  such that  $|\alpha|(z_\alpha) < 1$  and  $|\alpha'|(z_\alpha) = 1$  for  $\alpha' \in \Delta - \{\alpha\}$ . For a character  $\chi$  of  $\mathcal{Z}_G(V)$ , we let  $\Delta_0(\chi) = \{\alpha \in \Delta \mid \chi(\tau_{z_\alpha}) = 0\}$ .

In the above proposition, we can take  $s = \prod_{\alpha \in \Delta - I} z_\alpha$ ; then  $\tau_s$  is the product  $\tau_s = \prod_{\alpha \in \Delta - I} \tau_{z_\alpha}$  in any order.

**Lemma** *Let  $\chi$  be a character of  $\mathcal{Z}_G(V)$ . Then  $I = \Delta_0(\chi)$  is the smallest subset of  $\Delta$  such that  $\chi$  extends to a character of  $\mathcal{Z}_{M_I}(V_{N_I^0})$ . For  $z$  in  $Z^+ \cap Z_{\psi_V}$  we have  $\chi(\tau_z) \neq 0$  if and only if  $|\alpha|(z) = 1$  for all  $\alpha \in \Delta_0(\chi)$ . In particular,  $\Delta_0(\chi)$  does not depend on  $\{z_\alpha\}$ .*

**Proof** As  $\mathcal{Z}_{M_I}(V_{N_I^0})$  is the localization of  $\mathcal{Z}_G(V)$  at  $\prod_{\alpha \in \Delta - I} \tau_{z_\alpha}$ ,  $\chi$  extends to a character of  $\mathcal{Z}_{M_I}(V_{N_I^0})$  if and only if  $\chi(\tau_{z_\alpha}) \neq 0$  for  $\alpha \in \Delta - I$ . The first assertion follows. Let  $z \in Z^+ \cap Z_{\psi_V}$ ; if for some  $\alpha \in \Delta_0(\chi)$  we have  $|\alpha|(z) < 1$ , then for some positive integer  $r$ ,  $z^r = z_\alpha t$  with  $t \in Z^+ \cap Z_{\psi_V}$ , and  $\chi(\tau_z)^r = \chi(\tau_{z_\alpha})\chi(t) = 0$ , so  $\chi(\tau_z) = 0$ ; if  $|\alpha|(z) = 1$  for all  $\alpha \in \Delta_0(\chi)$  then with  $s = \prod_{\alpha \in \Delta - \Delta_0(\chi)} z_\alpha$  there is a positive integer  $x$  such that  $s^r = zt$  for some  $t \in Z^+ \cap Z_{\psi_V}$  and similarly  $\chi(\tau_z) \neq 0$  since  $\chi(\tau_s) \neq 0$ .  $\square$

We write  $Z_\Delta^\perp$  for the set of  $z \in Z$  with  $|\alpha|(z) = 1$  for all  $\alpha \in \Delta$ . Using the lemma, we can restate the definition of supersingularity (I.5) for a character of  $\mathcal{Z}_G(V)$ .

**Corollary** *For a character  $\chi$  of  $\mathcal{Z}_G(V)$ , the following conditions are equivalent:*

- (i)  $\chi$  is supersingular,
- (ii)  $\Delta_0(\chi) = \Delta$ ,
- (iii)  $\chi(\tau_z) = 0$  for all  $z$  in  $Z^+ \cap Z_{\psi_V}$  not in  $Z_{\Delta}^{\perp}$ .

## B) Irreducible representations of $K$

**III.5.** As recalled above, irreducible representations of  $K$  factor through  $\overline{K} = K/K(1)$ . We first examine  $\overline{K}$ .

Information about  $\overline{K}$  comes from [BT1, BT2], see also [Ti]. The group  $\overline{K}$  is naturally the group of points (over the residue field  $k$  of  $F$ ), of a connected reductive group, which we write  $\mathbf{G}_k$ , so that  $\overline{K} = \mathbf{G}_k(k)^{10}$ . We also have  $\overline{S} = \mathbf{S}_k(k)$ , where  $\mathbf{S}_k$  is a maximal split torus in  $\mathbf{G}_k$ , with a natural identification of  $X^*(\mathbf{S}_k)$  and  $X^*(\mathbf{S})$ ; if  $\mathbf{Z}_k$  is the centralizer of  $\mathbf{S}_k$  in  $\mathbf{G}_k$  then  $\overline{Z} = \mathbf{Z}_k(k)$ , and similarly for the normalizer  $\mathcal{N}_k$  of  $\mathbf{S}_k$  in  $\mathbf{G}_k$ . As  $K$  is a special parahoric subgroup, every element of  $W_0$  has a representative in  $K$  so that  $W_0 = \mathcal{N}^0/Z^0$ , and reduction mod  $K(1)$  yields an identification of  $W_0$  with  $W(\mathbf{G}_k, \mathbf{S}_k) = \overline{\mathcal{N}}/\overline{Z}$ .

Similarly  $\overline{B} = \mathbf{B}_k(k)$  for a minimal parabolic subgroup  $\mathbf{B}_k$  of  $\mathbf{G}_k$  with Levi component  $\mathbf{Z}_k$  (which is a torus since  $k$  is finite) and unipotent radical  $\mathbf{U}_k$  such that  $\overline{U} = \mathbf{U}_k(k)$ .

**III.6.** The root system of  $\mathbf{S}_k$  in  $\mathbf{G}_k$  is a sub-root system of the root system of  $\mathbf{S}$  in  $\mathbf{G}$ , using the above-mentioned identification of  $X^*(\mathbf{S}_k)$  and  $X^*(\mathbf{S})$ . We write  $\Phi_k$  for the set of roots of  $\mathbf{S}_k$  in  $\mathbf{G}_k$ ; we have  $\Phi_k \subset \Phi$ . A reduced root  $\alpha \in \Phi$  belongs to  $\Phi_k$  if  $2\alpha$  is not a root in  $\Phi$ ; if  $\alpha$  and  $2\alpha$  are roots in  $\Phi$ , then  $\alpha$  or  $2\alpha$  or both are in  $\Phi_k$  – all three cases can occur.

So we get a natural bijection  $\alpha \mapsto \overline{\alpha}$  from reduced roots in  $\Phi$  to reduced roots in  $\Phi_k$ , which sends positive roots to positive roots, and the set  $\Delta$  of simple roots in  $\Phi$  to the set  $\Delta_k$  of simple roots in  $\Phi_k$ . When  $\alpha \in \Phi$  is reduced, we have  $\overline{U}_\alpha = \mathbf{U}_{k, \overline{\alpha}}(k)$ . Henceforward we **identify** the reduced roots of  $\Phi_k$  with those of  $\Phi$ , hence  $\Phi_k$  with  $\Phi$ ,  $\Delta_k$  with  $\Delta$ , via  $\alpha \mapsto \overline{\alpha}$ . Then for  $I \subset \Delta$  the parabolic subgroup  $P_I = M_I N_I$  is such that  $\overline{P}_I = \mathbf{P}_{I,k}(k)$ ,  $\overline{M}_I = \mathbf{M}_{I,k}(k)$ ,  $\overline{N}_I = \mathbf{N}_{I,k}(k)$ .

**III.7.** Let  $\mathbf{B}_{\text{op}}$  be the parabolic subgroup of  $\mathbf{G}$  opposite to  $\mathbf{B}^{11}$  (with respect to  $\mathbf{Z}$ ) and  $\mathbf{U}_{\text{op}}$  its unipotent radical; then  $\overline{B}_{\text{op}} = \mathbf{B}_{k, \text{op}}(k)$  where  $\mathbf{B}_{k, \text{op}}$  is the parabolic subgroup of  $\mathbf{G}_k$  opposite to  $\mathbf{B}_k$ . Similarly we have  $\overline{U}_{\text{op}} = \mathbf{U}_{k, \text{op}}(k)$  for their unipotent radicals.

From [BoT, Proposition 6.25] we get that  $\overline{G}$  is generated by the union of  $\overline{Z}$ ,  $\overline{U}$ ,  $\overline{U}_{\text{op}}$ . The subgroup  $\overline{G}'$  of  $\overline{G}$  generated by the union of  $\overline{U}$  and  $\overline{U}_{\text{op}}$  is normal in  $\overline{G}$ ; it is the image in  $\overline{G}$  of  $\mathbf{G}_{k, \text{sc}}(k)$  where  $\mathbf{G}_{k, \text{sc}}$  is the simply connected covering of the derived group of  $\mathbf{G}_k$ . Note<sup>12</sup> that  $G'^0$  certainly contains  $U^0$  and  $(U_{\text{op}})^0$  so that its image in  $\overline{G}$  contains  $\overline{G}'$ . But it can be larger, so we need to distinguish  $\overline{G}'$  and  $\overline{G}'^{13}$ ; the discrepancy is actually quite important in Chapter IV.

**Lemma** (i) *The map  $(U \cap K(1)) \times Z(1) \times (U_{\text{op}} \cap K(1)) \rightarrow K(1)$  given by the product law is bijective, and similarly for any order of the factors.*

<sup>10</sup>We warn the reader that when  $\mathbf{G}$  is semisimple,  $\mathbf{G}_k$  is not necessarily semisimple. If  $\mathbf{H}_k$  is an algebraic group over  $k$ , we put  $H_k = \mathbf{H}_k(k)$ , so that for many algebraic subgroups  $\mathbf{H}$  of  $\mathbf{G}$  in the current chapter, we can use indifferently the notations  $\overline{H}$  or  $H_k$  for  $(H \cap K)/(H \cap K(1))$  – we mostly use  $\overline{H}$ .

<sup>11</sup>When convenient, we put the index op on top.

<sup>12</sup>Recall  $G'$  is the subgroup of  $G$  generated by  $U$  and  $U_{\text{op}}$ .

<sup>13</sup>To avoid confusion, we sometimes write  $G'_k$  rather than  $\overline{G}'$ .



(ii)  $K$  is generated by the union of  $U^0$ ,  $Z^0$  and  $(U_{\text{op}})^0$ .

**Proof** Assertion (i) is due to Bruhat and Tits [BT2, 4.6.8 Corollaire]. Since  $\overline{G}$  is generated by the union of  $\overline{Z}$ ,  $\overline{U}$  and  $\overline{U}_{\text{op}}$ ,  $K$  is generated by the union of  $Z^0$ ,  $U^0$ ,  $(U_{\text{op}})^0$  and the normal subgroup  $K(1)$ ; then (ii) follows from (i).  $\square$

The lemma has a consequence which will be useful later. As in III.4 we write  $Z_{\Delta}^{\perp}$  for the set of  $z \in Z$  such that  $|\alpha|(z) = 1$  for all  $\alpha \in \Delta$ . Equivalently,  $Z_{\Delta}^{\perp} = \text{Ker } v_Z$  in the notation of [HV1, 3.2]. (We have in fact that  $|\alpha|(z) = q^{-\langle \alpha, v_Z(z) \rangle}$  for  $\alpha \in \Delta$  and  $z \in Z$ .)

**Corollary**  $Z_{\Delta}^{\perp}$  is the normalizer of  $K$  in  $Z$ .

**Proof** If  $z \in Z$  normalizes  $K$  it also normalizes  $U_{\alpha}^0$  for all  $\alpha \in \Phi$ . Given the action of  $z$  on the filtration of  $U_{\alpha}$  [Ti], that is equivalent to  $|\alpha|(z) = 1$  for  $\alpha \in \Phi$ . Conversely if  $|\alpha|(z) = 1$  for  $\alpha$  in  $\Delta$  then  $|\alpha|(z) = 1$  for all  $\alpha$  in  $\Phi$  and  $z$  normalizes  $U_{\alpha}^0$  for all  $\alpha \in \Phi$ ; it then normalizes  $U^0$  and  $(U_{\text{op}})^0$ , so it normalizes  $K$ . That proves that  $Z_{\Delta}^{\perp}$  is the normalizer of  $K$  in  $Z$ .  $\square$

**Remark** By the Cartan decomposition the normalizer of  $K$  in  $G$  is  $Z_{\Delta}^{\perp}K$ .

**III.8.** We can now recall (see [HV1, HV2] and the references therein) the parametrization of the irreducible representations of  $\overline{G}$ , up to isomorphism.

If  $(\rho, V)$  is an irreducible representation of  $\overline{G}$ , then  $V^{\overline{U}}$  is a line, on which  $\overline{Z}$  acts via a character, say  $\eta : \overline{Z} \rightarrow C^{\times}$ . Let  $\Delta(\eta)$  be the set of simple roots  $\alpha \in \Delta$  such that  $\eta$  is trivial on  $\overline{Z} \cap M'_{\alpha, k}$  (where  $M_{\alpha, k}$  is the Levi subgroup of  $\overline{G}$  corresponding to  $\{\alpha\}$ ), and as in III.7  $M'_{\alpha, k}$  is the subgroup of  $M_{\alpha, k}$  generated by (the union of)  $\overline{U}_{\alpha}$  and  $\overline{U}_{-\alpha}$ . The stabilizer of the line  $V^{\overline{U}}$  in  $\overline{G}$  is a parabolic subgroup containing  $\overline{B}$  corresponding to a subset  $\Delta_V$  of  $\Delta(\eta)$ , and  $V$  is characterized up to isomorphism by the pair  $(\eta, \Delta_V)$ ; all such pairs do occur. In [HV2],  $(\eta, \Delta_V)$  is called the **standard parameter** of  $V$ .

**III.9.** In this paper, we are interested in coinvariants rather than invariants, so we use different parameters. Let  $V$  be an irreducible representation of  $\overline{G}$  with standard parameter  $(\eta, \Delta_V)$ .

**Lemma** The group  $\overline{Z}$  acts on the line  $V_{\overline{U}}$  via the character  $\eta \circ w_0$  where  $w_0$  is the longest element in  $W_0$ . Moreover the stabilizer of the kernel of  $V \rightarrow V_{\overline{U}}$  is the parabolic subgroup of  $\overline{G}$  corresponding to the subset  $-w_0\Delta_V$  of  $\Delta$ .

**Proof** By [HV2, Proposition 3.14] the projection  $V \rightarrow V_{\overline{U}}$  induces a  $\overline{Z}$ -equivariant isomorphism of  $V^{\overline{U}_{\text{op}}}$  onto  $V_{\overline{U}}$ ; the first assertion comes from [loc. cit. 3.11]. The stabilizer we look at is also the stabilizer of the line  $(V^*)^{\overline{U}}$  in the contragredient representation  $V^*$  of  $V$ ; the second assertion follows from by [loc. cit. 3.12].  $\square$

**Definition** The **parameter** of  $V$  is the pair  $(\psi_V, \Delta(V))$  where  $\overline{Z}$  acts on  $V_{\overline{U}}$  via  $\psi_V$  and the stabilizer in  $\overline{G}$  of the kernel of  $V \rightarrow V_{\overline{U}}$  is  $\overline{P}_{\Delta(V)}$ .

**Remarks** 1) We have  $\psi_V = \eta \circ w_0$  and  $\Delta(V) = -w_0\Delta_V$ .

2) The antistandard parameter of  $V$  [HV2, 3.11] is  $(\psi_V, -\Delta(V))$ .

3)  $V$  is determined up to isomorphism by its parameter. One has  $\Delta(V) \subset \Delta(\psi_V)$ , and all pairs  $(\psi, I)$  with  $I \subset \Delta(\psi)$  occur as parameters.

**III.10. Lemma** *Let  $V$  be an irreducible representation of  $K$ , and let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ .*

- (i)  $V_{\overline{N}}$  is an irreducible representation of  $\overline{M}$  with parameter  $(\psi_V, \Delta_M \cap \Delta(V))$ .
- (ii)  $V$  is  $\overline{P}_{\text{op}}$ -regular in the sense of [HV2, Def. 3.6] if and only if  $\Delta(V) \subset \Delta_M$ .

Here  $\overline{P}_{\text{op}} = \overline{M} \overline{N}_{\text{op}}$  is the parabolic subgroup of  $\overline{G}$  opposite to  $\overline{P}$  (relative to  $\overline{M}$ ).

**Proof** By [loc. cit. 3.11]  $V^{\overline{N}_{\text{op}}}$  is an irreducible representation of  $\overline{M}$  and its antistandard parameters are  $(\psi_V, -(\Delta_M \cap \Delta(V)))$ . On the other hand, the projection  $V \rightarrow V_{\overline{N}}$  induces an  $\overline{M}$ -equivariant isomorphism of  $V^{\overline{N}_{\text{op}}}$  onto  $V_{\overline{N}}$ , so (i) comes from Remark 2) above. By [loc. cit., Def. 3.6]  $V$  is  $\overline{P}_{\text{op}}$ -regular if and only if  $-\Delta(V) \subset -\Delta_M$  i.e.  $\Delta(V) \subset \Delta_M$ , whence (ii).  $\square$

**Remark** Since  $\overline{P}_{\Delta(V)}$  is the stabilizer of the kernel of the projection  $V \rightarrow V_{\overline{U}}$ ,  $V$  is one-dimensional if and only if  $\Delta(V) = \Delta$ . It follows from part (i) of the lemma that  $V_{\overline{N}}$  is one-dimensional if and only if  $\Delta_M \subset \Delta(V)$ . That provides a useful characterization of  $\Delta(V)$ .

**Examples** 1) Consider the case where  $V$  is the trivial representation of  $\overline{G}$ . Then  $\psi_V = 1$  and  $\Delta(V) = \Delta$ . Representations  $V$  with parameter  $(1, I)$  for  $I \subset \Delta$  are particularly important to us (cf. III.18 below).

2) Let  $\eta$  be a character of  $\overline{Z}$ ; then  $\eta$  extends to a character of  $\overline{M}_{\Delta(\eta)}$ : indeed that extension is the irreducible representation of  $\overline{M}_{\Delta(\eta)}$  with parameter  $(\eta, \Delta(\eta))$ .

**III.11.** Consider the simply connected covering  $\mathbf{G}_{k,\text{sc}}$  of the derived group  $\mathbf{G}_{k,\text{der}}$  of  $\mathbf{G}_k$  and write  $j : \mathbf{G}_{k,\text{sc}} \rightarrow \mathbf{G}_k$  for the natural morphism. Put  $G_{k,\text{sc}} = \mathbf{G}_{k,\text{sc}}(k)$ . We can repeat exactly the same considerations as in II.4 in this context of finite reductive groups, and we use the analogous notation – note however that since  $k$  is finite, every  $k$ -simple component of  $\mathbf{G}_{k,\text{sc}}$  is isotropic. In particular  $j$  induces an isomorphism between  $\tilde{\mathbf{U}}_k$  and  $\mathbf{U}_k$ , and  $\Delta_k$  also appears as the set of simple roots of  $\tilde{\mathbf{S}}_k$  in  $\tilde{\mathbf{U}}_k$ .

From III.7, recall that

$$G'_k = j(G_{k,\text{sc}}).$$

**Proposition** *Let  $(\rho, V)$  be an irreducible representation of  $G_k$  with parameter  $(\psi_V, \Delta(V))$ . Then  $(\rho \circ j, V)$  is an irreducible representation of  $G_{k,\text{sc}}$  with parameter  $(\psi_V \circ j|_{\tilde{Z}_k}, \Delta(V))$ .*

Here  $\tilde{Z}_k = \tilde{\mathbf{Z}}_k(k)$  where  $\tilde{\mathbf{Z}}_k$  is the centralizer of  $\tilde{\mathbf{S}}_k$  in  $\mathbf{G}_{k,\text{sc}}$ ; we use similarly abbreviated notation below. By the fact above and the inclusion  $G'_k = \overline{G}' \subset \overline{G}$  (III.7), we get:

**Corollary** *The restriction of  $\rho$  to  $G'_k$ , and a fortiori to  $\overline{G}'$ , is irreducible.*

**Proof of the proposition** Since  $V_{\tilde{U}_k}$ , equal to  $V_{\overline{U}}$ , is one-dimensional, the cosocle of  $\rho \circ j$  is irreducible. Similarly  $V^{\tilde{U}_{k,\text{op}}}$  equal to  $V^{\overline{U}_{\text{op}}}$  is one dimensional, so the socle of  $\rho \circ j$  is irreducible too. As the projection of  $V^{\overline{U}_{\text{op}}}$  to  $V_{\overline{U}}$  is non-zero, the map from the socle of  $\rho \circ j$  to its cosocle is non-zero, and  $\rho \circ j$  is indeed irreducible. Clearly  $\tilde{Z}_k$  acts on  $V_{\tilde{U}_k} = V_{\overline{U}}$  by  $z \mapsto \psi_V \circ j(z)$ , and  $\tilde{P}_{\Delta(V),k} = j^{-1}(\overline{P}_{\Delta(V)})$  stabilizes the kernel of  $V \rightarrow V_{\tilde{U}_k}$ . But for  $I \subset \Delta$ , we have  $\overline{P}_I = \overline{Z}j(\tilde{P}_{I,k})$ , so if  $\tilde{P}_{I,k}$  stabilizes that kernel,  $I \subset \Delta(V)$ .  $\square$

### C) Weights of parabolically induced representations

**III.12.** Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ , and  $(\tau, W)$  a representation of  $M$ . We investigate the weights of  $\text{Ind}_P^G W$  and the corresponding Hecke eigenvalues. From now on, we identify the irreducible representations of  $K$  and those of  $\overline{G} = K/K(1)$ .

In this part C) we let  $(\rho, V)$  be an irreducible representation of  $K$ , with parameter  $(\psi_V, \Delta(V))$ . Recall that if  $(\pi, X)$  is a representation of  $G$ , for example  $X = \text{Ind}_P^G W$ , then  $\text{Hom}_K(V, X)$  is a right  $\mathcal{H}_G(V)$ -module. The formula for the action is

$$(III.12.1) \quad (\varphi\Phi)(v) = \sum_{g \in G/K} g\varphi(\Phi(g^{-1})v) \quad \text{for } v \in V, \varphi \in \text{Hom}_K(V, X),$$

and  $\Phi \in \mathcal{H}_G(V)$ .

**Proposition** (i) *The natural isomorphism*

$$\text{Hom}_K(V, \text{Ind}_P^G W) \xrightarrow{\text{can}} \text{Hom}_{M^0}(V_{N^0}, W)$$

is  $\mathcal{H}_G(V)$ -linear, where  $\mathcal{H}_G(V)$  acts on the right-hand side via  $\mathcal{S}_M^G$ .

(ii)  $V$  is a weight for  $\text{Ind}_P^G W$  if and only if  $V_{N^0}$  is a weight for  $W$ .

(iii) The map  $\mathcal{S}_M^G$  identifies the eigenvalues of  $V$  in  $\text{Ind}_P^G W$  and the eigenvalues of  $V_{N^0}$  in  $W$ .

**Proof** (i) comes from III.3 and (ii) is an immediate consequence. We have seen that  $\mathcal{Z}_M(V_{N^0})$  is the localization of  $\mathcal{Z}_G(V)$  at some element  $\tau_s$ . Clearly  $\tau_s$  acts invertibly on  $\text{Hom}_{M^0}(V_{N^0}, W)$ ; as the canonical isomorphism is  $\mathcal{H}_G(V)$ -linear,  $\tau_s$  also acts invertibly on  $\text{Hom}_K(V, \text{Ind}_P^G W)$ , which gives (iii).  $\square$

A useful consequence of (III.12.1) is the following lemma. Recall that for  $z \in Z^+ \cap Z_{\psi_V}$ ,  $\mathcal{Z}_G(V)$  contains a unique element  $T_z$  such that  $\text{Supp } T_z = KzK$  and  $T_z(z) \in \text{End}_C(V)$  induces the identity on  $V^{\overline{U}^{\text{op}}}$  [HV1, 7.3, 2.9].

**Lemma** Let  $(\pi, X)$  be a representation of  $G$  and  $\varphi \in \text{Hom}_K(V, X)$ . Let  $z \in Z_{\psi_V}$ . Assume  $z \in Z_{\Delta}^1$ , i.e. that  $z$  normalizes  $K$ . Then  $\mathcal{S}_Z^G(T_z) = \tau_z$  and  $(\varphi\tau_z)(v) = z^{-1}\varphi(v)$  for  $v$  in  $V^{\overline{U}^{\text{op}}}$ . If  $\varphi$  is an eigenvector for  $\mathcal{Z}_G(V)$  with eigenvalue  $\chi$ , then  $z^{-1}$  acts on  $\varphi(V^{\overline{U}^{\text{op}}})$  by  $\chi(\tau_z)$ .

**Proof** By assumption  $zK = Kz$ , and the endomorphism  $T_z(z)$  satisfies  $\rho(k)T_z(z) = T_z(z)\rho(z^{-1}kz)$  for  $k \in K$  [loc. cit., 7.3]. As  $z$  normalizes  $U^0$  and  $(U_{\text{op}})^0$ ,  $T_z(z)$  induces endomorphisms of  $V_{\overline{U}}$  and  $V^{\overline{U}^{\text{op}}}$ ; since the natural map  $V^{\overline{U}^{\text{op}}} \rightarrow V_{\overline{U}}$  is an isomorphism,  $T_z(z)$  induces the identity on  $V_{\overline{U}}$ . From (III.3.2) we get  $\mathcal{S}_Z^G(T_z) = \tau_z$ , and (III.12.1) gives

$$(\varphi T_z)(v) = z^{-1}\varphi(T_z(z)v) \quad \text{for } v \in V,$$

hence the result.  $\square$

**III.13.** Let  $\varphi \in \text{Hom}_K(V, \text{Ind}_P^G W)$  and  $\varphi_M \in \text{Hom}_{M^0}(V_{N^0}, W)$  correspond via (III.3.1). Then  $\varphi$  gives rise to a  $G$ -morphism, again written  $\varphi$ , from  $\text{ind}_K^G V$  to  $\text{Ind}_P^G W$ , and similarly we get an  $M$ -morphism  $\varphi_M : \text{ind}_{M^0}^M V_{N^0} \rightarrow W$ .

Consider the following diagram, where horizontal maps are canonical isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0})) & \xrightarrow{\sim \text{can}} & \mathrm{Hom}_M(\mathrm{ind}_{M^0}^M V_{N^0}, \mathrm{ind}_{M^0}^M V_{N^0}) \\ \downarrow \mathrm{Ind}_P^G \varphi_M & & \downarrow \varphi_M \\ \mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G W) & \xrightarrow{\sim \text{can}} & \mathrm{Hom}_M(\mathrm{ind}_{M^0}^M V_{N^0}, W) \end{array}$$

By naturality, the vertical maps obtained by composing with  $\mathrm{Ind}_P^G \varphi_M$  and  $\varphi_M$ , as indicated, make the diagram commutative. The identity map of  $\mathrm{ind}_{M^0}^M V_{N^0}$  yields the canonical intertwiner

$$(III.13.1) \quad \mathcal{I} : \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0})$$

mentioned in I.6. We get:

**Lemma**  $\mathrm{Ind}_P^G \varphi_M \circ \mathcal{I} = \varphi$ .

By [HV2, Proposition 4.1],  $\mathcal{I}$  is injective. As  $\mathcal{I}$  is  $\mathcal{H}_G(V)$ -linear, it factors as follows:

$$\begin{aligned} \mathrm{ind}_K^G V &\longrightarrow \mathcal{Z}_M(V_{N^0}) \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V \xrightarrow{u} \mathcal{H}_M(V_{N^0}) \otimes_{\mathcal{H}_G(V)} \mathrm{ind}_K^G V \\ &\xrightarrow{\Theta} \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0}), \end{aligned}$$

for some canonical map  $\Theta$ . Since  $\mathcal{H}_M(V_{N^0})$  is the localization of  $\mathcal{H}_G(V)$  at some central element, and  $\mathcal{Z}_M(V_{N^0})$  is the localization of  $\mathcal{Z}_G(V)$  at the same element, the map  $u$  is an isomorphism.

**III.14.** The main result of [HV2] is, taking into account III.10 Lemma (ii):

**Theorem** *Let  $(\psi_V, \Delta(V))$  be the parameter of  $V$ . If  $\Delta(V) \subset \Delta_M$  then the map  $\Theta$  of III.13 is an isomorphism.*

We derive some consequences.

**Corollary 1** *Let  $\varphi \in \mathrm{Hom}_K(V, \mathrm{Ind}_P^G W)$  be an eigenvector for  $\mathcal{Z}_G(V)$ . If  $\Delta(V) \subset \Delta_M$  and if  $\varphi_M(V_{N^0})$  generates  $W$  as a representation of  $M$ , then  $\varphi(V)$  generates  $\mathrm{Ind}_P^G W$  as a representation of  $G$ .*

**Proof** By the theorem,  $\Theta$  is surjective. By hypothesis  $\varphi_M : \mathrm{ind}_{M^0}^M V_{N^0} \rightarrow W$  is surjective, so by III.13 Lemma the map induced by  $\varphi$

$$\mathcal{Z}_M(V_{N^0}) \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G W$$

is surjective. But  $\mathcal{Z}_G(V)$  acts on  $\varphi$  via a character which extends to  $\mathcal{Z}_M(V_{N^0})$  (III.12 Proposition (ii)) so we conclude that  $\varphi(\mathrm{ind}_K^G V) = \mathrm{Ind}_P^G W$ , hence the result.  $\square$

**Corollary 2** *Assume that  $(\tau, W)$  is irreducible and admissible. Then  $\mathrm{Ind}_P^G W$  is irreducible if and only if every non-zero subrepresentation of it contains a weight  $V$  with  $\Delta(V) \subset \Delta_M$ .*

**Proof** Since  $W$  has some weight, by III.12 Proposition (i) and III.10 Lemma (i),  $\mathrm{Ind}_P^G W$  has a weight  $V$  with  $\Delta(V) \subset \Delta_M$ . Conversely if a subrepresentation  $X$  of  $\mathrm{Ind}_P^G W$  contains a weight  $V$  with  $\Delta(V) \subset \Delta_M$ , there is an eigenvector  $\varphi \in \mathrm{Hom}_K(V, X)$  for  $\mathcal{Z}_G(V)$ . As  $\tau$  is irreducible,  $\varphi_M(V_{N^0})$  generates  $W$  and by the proposition  $X = \mathrm{Ind}_P^G W$ .  $\square$

### D) Determination of $P(\sigma)$ for supersingular $\sigma$

**III.15.** We want to apply the preceding corollary to prove the irreducibility of  $I(P, \sigma, Q)$  for a supersingular triple  $(P, \sigma, Q)$ . That can only be done in stages. First we determine  $P(\sigma)$  in terms of weights and eigenvalues of  $\sigma$ . In other words, we determine the set  $\Delta(\sigma)$  of  $\alpha \in \Delta - \Delta_M$  such that  $\sigma$  is trivial on  $Z \cap M'_\alpha$  (II.7).

As the generality will be useful in Chapter V, we consider the situation where  $P = MN$  is a parabolic subgroup of  $G$  containing  $B$ , and  $(\sigma, W)$  is a representation of  $M$  satisfying the following hypothesis:

(H) There is an irreducible representation  $(\rho, V)$  of  $M^0$  and some  $\varphi$  in  $\text{Hom}_{M^0}(V, W)$  such that  $\sigma$  is generated by  $\varphi(V)$  as a representation of  $M$ .

Hypothesis (H) is certainly true if  $\sigma$  is irreducible and admissible, and then we can take  $\varphi$  to be an eigenvector for  $\mathcal{Z}_M(V)$ , and the corresponding eigenvalue is supersingular if  $\sigma$  is. As before, write  $(\psi_V, \Delta(V))$  for the parameter of  $V$ .

**Lemma** *Assume Hypothesis (H). Let  $\alpha \in \Delta$ . If  $\sigma$  is trivial on  $Z \cap M'_\alpha$ , then  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ .*

**Proof** If  $\sigma$  is trivial on  $Z \cap M'_\alpha$ , then certainly  $Z \cap M'_\alpha$  acts trivially on  $\varphi(V)$ . As  $\varphi \in \text{Hom}_{M^0}(V, W)$  is injective,  $Z^0 \cap M'_\alpha$  acts trivially on  $V$  hence on  $V_{U^0}$  and  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ .  $\square$

**III.16. Proposition** *Let  $\alpha \in \Delta$ .*

- (i) *If  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  then  $Z \cap M'_\alpha \subset Z_{\psi_V}$ .*
- (ii)  *$|\alpha|(Z \cap M'_\alpha)$  is isomorphic to  $\mathbb{Z}$ .*
- (iii) *Let  $z \in Z \cap M'_\alpha$ . Then  $|\alpha|(z) = 1$  iff and only if  $z \in Z^0 \cap M'_\alpha$ .*

**Notation** By (ii) there is an element  $a_\alpha$  in  $Z \cap M'_\alpha$  with  $|\alpha|(a_\alpha) > 1$ , such that  $|\alpha|(a_\alpha)$  generates  $|\alpha|(Z \cap M'_\alpha)$ ; by (iii) the element  $a_\alpha$  is well-defined modulo  $Z^0 \cap M'_\alpha$ . Note that if  $\alpha$  is orthogonal to  $\Delta_M$  then  $a_\alpha \in Z_{\Delta_M}^\perp$  (see proof of III.7 Corollary) and  $\tau_{a_\alpha}$  is a unit of  $\mathcal{Z}_M(V)$ . If  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  the element  $\tau_{a_\alpha}$  of  $\mathcal{Z}_Z(V_{U \cap M^0})$  does not depend on the choice of  $a_\alpha$ , so we write it  $\tau_\alpha$ .

**Proof of the proposition** Assume that  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ , and take  $z \in Z \cap M'_\alpha$ ; then, for  $z' \in Z$  (in particular for  $z' \in Z^0$ ),  $zz'z^{-1}z'^{-1}$  belongs to  $Z^0 \cap M'_\alpha$  (because  $Z/Z^0$  is abelian and  $Z \cap M'_\alpha$  is normal in  $Z$ ), so we get  $\psi_V(zz'z^{-1}z'^{-1}) = 1$ . That shows that  $z$  normalizes  $\psi_V$  and belongs to  $Z_{\psi_V}$ , hence (i).

Let us introduce the isotropic part  $\tilde{\mathbf{M}}_\alpha = \mathbf{M}_\alpha^{\text{is}}$  of the simply connected covering of the derived group of  $\mathbf{M}_\alpha$ , its minimal Levi subgroup  $\tilde{\mathbf{Z}}_\alpha$  lifting  $\mathbf{Z}$ , and the maximal split torus  $\tilde{\mathbf{S}}_\alpha$  of  $\tilde{\mathbf{Z}}_\alpha$ . Write  $j$  for the canonical map  $\tilde{\mathbf{M}}_\alpha \rightarrow \mathbf{M}_\alpha$ . We have  $M'_\alpha = j(\tilde{M}_\alpha)$  and  $j^{-1}(Z) = \tilde{Z}_\alpha$ , so  $Z \cap M'_\alpha = j(\tilde{Z}_\alpha)$ .

Let  $v_Z : Z \rightarrow X_*(\mathbf{S}) \otimes \mathbb{Q}$  be the map of [HV1, 3.2]; its kernel is the maximal compact subgroup of  $Z$  and by [loc. cit., 6.2]  $Z^0$ , which is the kernel of the Kottwitz invariant  $w_Z$  of  $Z$ , is equal to  $\text{Ker } v_Z \cap \text{Ker } w_G$  where  $w_G$  is the Kottwitz invariant of  $G$  [Kot, §7]. We

have the analogous map  $v_{\tilde{Z}_\alpha}$  and a commutative diagram

$$\begin{array}{ccc} \tilde{Z}_\alpha & \xrightarrow{v_{\tilde{Z}_\alpha}} & X_*(\tilde{\mathbf{S}}_\alpha) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{v_Z} & X_*(\mathbf{S}) \otimes \mathbb{Q} \end{array}$$

where the vertical maps are induced by  $j$ .

As  $\tilde{\mathbf{M}}_\alpha$  is semisimple and simply connected,  $w_{\tilde{\mathbf{M}}_\alpha}$  is trivial and by functoriality of the Kottwitz invariant  $w_G$  is trivial on  $M'_\alpha = j(\tilde{M}_\alpha)$ ; in particular  $Z^0 \cap M'_\alpha = \text{Ker } v_Z \cap M'_\alpha$ . The vertical map on the right of the above diagram is injective so  $j^{-1}(Z^0 \cap M'_\alpha) = \text{Ker } v_{\tilde{Z}_\alpha}$ . Thus  $(Z \cap M'_\alpha)/(Z^0 \cap M'_\alpha)$  is isomorphic to  $\tilde{Z}_\alpha/\text{Ker } v_{\tilde{Z}_\alpha}$ , i.e. to the image of  $v_{\tilde{Z}_\alpha}$ . Since  $\tilde{\mathbf{S}}_\alpha$  has dimension 1, that image is isomorphic to  $\mathbb{Z}$ . Now for  $z \in \tilde{Z}_\alpha$  we have  $|\alpha|(j(z)) = q^{-\langle \alpha, v_Z(j(z)) \rangle} = q^{-\langle \alpha, v_{\tilde{Z}_\alpha}(z) \rangle}$  and (ii), (iii) follow.  $\square$

**Remark** From the above proof it is clear that  $v_Z(a_\alpha)$  is a (negative) rational multiple of  $\alpha^\vee$ . See also IV.11 Example 3.

**III.17.** Let us derive consequences of III.16.

**Proposition** *Assume Hypothesis (H) (III.15). Let  $\alpha \in \Delta$  be orthogonal to  $\Delta_M$ . Then the following conditions are equivalent:*

- (i)  $\sigma$  is trivial on  $Z \cap M'_\alpha$ ,
- (ii)  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  and  $(\varphi\tau_\alpha)(v) = \varphi(v)$  for  $v \in V^{U_{\text{op}} \cap M^0}$ .

**Proof** Apply first III.12 Lemma to get

$$(*) \quad (\varphi\tau_\alpha)(v) = a_\alpha^{-1} \varphi(v)$$

for  $v \in V^{U_{\text{op}} \cap M^0}$ . Now assume (i). By III.15 Lemma,  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ ; then, since  $\alpha$  is orthogonal to  $\Delta_M$ ,  $a_\alpha$  belongs to  $Z_{\Delta_M}^\perp$  and (\*) implies (ii). Conversely assume (ii). Applying III.16 Proposition and (\*) again we get that  $Z \cap M'_\alpha$  acts trivially on the line  $\varphi(V^{U_{\text{op}} \cap M^0})$ . But as  $\alpha$  is orthogonal to  $\Delta_M$ ,  $M$  normalizes  $M'_\alpha$  and hence also  $Z \cap M'_\alpha$ ; consequently the set of fixed points of  $Z \cap M'_\alpha$  in  $W$  is invariant under  $M$ . As it contains  $\varphi(V^{U_{\text{op}} \cap M^0})$  it contains  $\varphi(V)$  since  $V^{U_{\text{op}} \cap M^0}$  generates  $V$  over  $M^0$ , and by hypothesis (H),  $Z \cap M'_\alpha$  acts trivially on  $W$ .  $\square$

**Corollary** *Assume Hypothesis (H) and that moreover  $\varphi$  is a  $\mathcal{Z}_M(V)$ -eigenvector with supersingular eigenvalue  $\chi$ . Then  $\Delta(\sigma)$  as in (II.7) is the set of  $\alpha \in \Delta$ , orthogonal to  $\Delta_M$ , such that  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  and  $\chi(\tau_\alpha) = 1$ .*

**Proof** Assume  $\alpha \in \Delta(\sigma)$  is not orthogonal to  $\Delta_M$ . By II.7 Corollary 2 and Remark 5, there is a proper parabolic subgroup  $Q = M_Q N_Q$  of  $M$  (containing  $M \cap B$ ) such that  $\sigma$  is trivial on  $N_Q$  and is a subrepresentation of  $\text{Ind}_Q^M(\sigma|_{M_Q})$ . By III.12 Proposition (iii), no eigenvalue of  $\sigma$  can be supersingular. Consequently any  $\alpha$  in  $\Delta(\sigma)$  is orthogonal to  $\Delta_M$  and the result follows from the proposition.  $\square$

In particular, we have determined  $P(\sigma)$  for a supersingular representation  $\sigma$  of  $M$ .

**E) Weights and eigenvalues of  $I(P, \sigma, Q)$**

**III.18.** In this section, for a supersingular triple  $(P, \sigma, Q)$  (I.5), we determine the weights and eigenvalues of  $I(P, \sigma, Q)$ . A slightly more general situation is useful in part G) though.

**Proposition** *Consider a B-triple  $(P, \sigma, Q)$  as in I.5 with  $P = MN$ , and assume that  $\Delta(\sigma)$  is orthogonal to  $\Delta_M$ . Let  $V$  be an irreducible representation of  $K$ , with parameter  $(\psi_V, \Delta(V))$ .*

- 1) *The following conditions are equivalent:*
  - (i)  $V$  is weight of  $I(P, \sigma, Q)$ ,
  - (ii)  $V_{N^0}$  is a weight of  $\sigma$  and  $\Delta(V) \cap \Delta(\sigma) = \Delta_Q \cap \Delta(\sigma)$ .
- 2) *If  $V$  is a weight of  $I(P, \sigma, Q)$ , then the eigenvalues of  $\mathcal{Z}_G(V)$  in  $I(P, \sigma, Q)$  are in bijection with those of  $\mathcal{Z}_M(V_{N^0})$  in  $\sigma$  via  $\mathcal{S}_M^G$ .*

The proof of 1) is in III.19–III.21 below, that of 2) in III.22, which actually gives more precise information.

**Remark 1** Consider the case where  $P = B$  and  $\sigma$  is the trivial representation of  $B$ . Then  $P(\sigma) = G$  and  $I(B, \sigma, Q) = \text{St}_Q^G$ . From [Ly1, §8] we get that  $\text{St}_Q^G$  has a unique weight  $V_Q^G$ , with multiplicity one, and parameter  $(1, \Delta_Q)$ . That weight also occurs with multiplicity one in  $\text{Ind}_Q^G 1$  and the natural map  $\text{Hom}_K(V_Q^G, \text{Ind}_Q^G 1) \rightarrow \text{Hom}_K(V_Q^G, \text{St}_Q^G)$  is an isomorphism; similarly  $V_Q^G$  occurs with multiplicity one in  $\text{Ind}_B^G 1$  and the natural map  $\text{Hom}_K(V_Q^G, \text{Ind}_B^G 1) \rightarrow \text{Hom}_K(V_Q^G, \text{Ind}_Q^G 1)$  is an isomorphism. Those isomorphisms are  $\mathcal{H}_G(V_Q^G)$ -equivariant, and the algebra  $\mathcal{H}_G(V_Q^G)$ , isomorphic to the monoid algebra  $C[Z^+/Z^0]$ , acts via the augmentation character sending  $\tau_z$  to 1 for  $z \in Z^+$ . That special case will be used in the proof of part 2) of the proposition.

The proposition may be applied to a supersingular triple, by III.17 Corollary.

**Corollary** *Assume  $(P, \sigma, Q)$  is a supersingular triple; if  $V$  is a weight of  $I(P, \sigma, Q)$  then for any eigenvalue  $\chi$  of  $\mathcal{Z}_G(V)$  in  $I(P, \sigma, Q)$ , we have  $\Delta_0(\chi) = \Delta_M$ .*

**Proof** By part 2) of the proposition,  $\chi$  extends to a character of  $\mathcal{Z}_M(V_{N^0})$  so  $\Delta_0(\chi) \subset \Delta_M$ . On the other hand the extended character is an eigenvalue of  $\sigma$  which is supersingular so  $\Delta_M \subset \Delta_0(\chi)$ .  $\square$

**Remark 2** In the context of the corollary, if  $P \neq G$ , then no eigenvalue of  $I(P, \sigma, Q)$  is supersingular.

**III.19.** By III.12 Proposition, we immediately reduce the proof of part 1) of the proposition to the case where  $P(\sigma) = G$ . In the course of the proof we shall glean more information on the weights and eigenvalues.

We put  $\Delta_1 = \Delta_M$  and  $\Delta_2 = \Delta(\sigma)$ , so that  $\Delta$  is the union of two orthogonal subsets  $\Delta_1$  and  $\Delta_2$ . As in II.4 we introduce the group  $\tilde{G} = \mathbf{G}^{\text{is}}$ . It appears as the product of two factors  $\tilde{G}_1$  and  $\tilde{G}_2$  attached to  $\Delta_1, \Delta_2$ . Note that  $\tilde{G}$  and  $G$  have the same semisimple building and their actions on it are compatible. Let  $\tilde{K}$  be the parahoric subgroup of  $\tilde{G}$  attached to the point  $\mathbf{x}_0$ . It decomposes as  $\tilde{K}_1 \times \tilde{K}_2$  where for  $i = 1, 2$ ,  $\tilde{K}_i = \tilde{K} \cap \tilde{G}_i$  is a parahoric subgroup of  $\tilde{G}_i$ . Write  $\iota$  for the natural map  $\tilde{G} \rightarrow G$ . For  $i = 1, 2$ , let  $M_i$  be the Levi subgroup  $M_{\Delta_i}$  of  $G$ . Then  $M'_i = \iota(\tilde{G}_i)$  and  $M_i = ZM'_i$ . By II.7 Remark 4,  $M'_1$  and  $M'_2$  commute with each other,  $Z$  normalizes each of them and  $G = ZM'_1M'_2$ .

**Proposition** (i)  $\tilde{K} = \iota^{-1}(K)$ ,  $\tilde{Z}^0 = \iota^{-1}(Z^0)$  and  $\iota(\tilde{K}_i) = K \cap M'_i$  for  $i = 1, 2$ .

(ii) Let  $\alpha \in \Phi$ ; then  $\iota$  induces a group isomorphism of  $\tilde{U}_\alpha^0 = \tilde{U}_\alpha \cap \tilde{K}$  onto  $U_\alpha^0 = U_\alpha \cap K$ .

Here  $\tilde{U}_\alpha$  denotes the root subgroup of  $\tilde{G}$  attached to  $\alpha \in \Phi$ .

**Proof** By functoriality of the Kottwitz invariant, since  $\tilde{G}$  is semisimple simply connected,  $w_G \circ \iota$  is trivial; on the other hand an element  $x \in \tilde{G}$  fixes the point  $\mathbf{x}_0$  if and only if  $\iota(x)$  fixes  $\mathbf{x}_0$ . So we have  $\tilde{K} = \iota^{-1}(K)$  and intersecting with  $\tilde{Z} = \iota^{-1}(Z)$  we get  $\tilde{Z}^0 = \iota^{-1}(Z^0)$ . If  $x \in \tilde{K}_i$  then  $\iota(x) \in K \cap \iota(\tilde{G}_i) = K \cap M'_i$ . Conversely if  $x \in \tilde{G}_i$  and  $\iota(x) \in K$  then  $x \in \tilde{K} \cap \tilde{G}_i = \tilde{K}_i$ . This proves (i).

(ii) Let  $\alpha \in \Phi$ . As  $\iota(\tilde{K}) \subset K$  we have  $\iota(\tilde{U}_\alpha^0) \subset U_\alpha^0$ . Conversely for  $x \in \tilde{U}_\alpha$ ,  $\iota(x) \in U_\alpha^0$  implies  $x \in \tilde{U}_\alpha \cap \iota^{-1}(K) = \tilde{U}_\alpha^0$  by (i).  $\square$

**Corollary** We have  $K = Z^0 \iota(\tilde{K})$ . For  $i = 1, 2$ ,  $M_i^0 = Z^0 \iota(\tilde{K}_i)$ .

**Proof** This comes from (ii) of the proposition, given III.7 Lemma.  $\square$

**Remark** By II.7 Remark 4,  $M'_1 \cap M'_2$  is finite and central in  $G$ . As it is contained in  $\text{Ker } w_G$ , it follows that  $Z^0$  contains  $M'_1 \cap M'_2$ , which is equal to  $\iota(\tilde{K}_1) \cap \iota(\tilde{K}_2)$ .

**III.20.** Let now  $(\rho, V)$  be an irreducible representation of  $K$ . We want to write  $V$  as a tensor product adapted to the orthogonal decomposition  $\Delta = \Delta_1 \sqcup \Delta_2$ .

Write  $(\tilde{\rho}, \tilde{V})$  for the representation of  $\tilde{K}$  obtained from  $\rho$  via  $\iota : \tilde{K} \rightarrow K$ . By III.19 Proposition (ii)  $\iota(\tilde{K})$  contains  $\overline{G'}$ , so by III.11 Corollary  $\tilde{\rho}$  is irreducible. Since  $\tilde{K} = \tilde{K}_1 \times \tilde{K}_2$ ,  $\tilde{V}$  decomposes as a tensor product  $\tilde{V}_1 \otimes \tilde{V}_2$  where for  $i = 1, 2$ ,  $\tilde{V}_i$  is an irreducible representation of  $\tilde{K}_i$  which is trivial on  $\tilde{K}_{3-i}$ .

To decompose  $V$  as a tensor product  $V_1 \otimes V_2$  of irreducible representations of  $K$ , where  $V_1$  restricts to  $\tilde{V}_1$  via  $\iota$ , and  $V_2$  to  $\tilde{V}_2$ , we have to take some care, as  $K$  is not the direct product  $M_1^0 \times M_2^0$ .

**Proposition** (i) For  $i = 1, 2$ , let  $V_i$  be an irreducible representation of  $K$  trivial on  $K \cap M'_{3-i}$ . Then  $V_1 \otimes V_2$  is irreducible with parameter  $(\psi_{V_1} \psi_{V_2}, \Delta(V_1) \cap \Delta(V_2))$ . Moreover,  $\Delta(V_i)$  contains  $\Delta_{3-i}$ .

(ii) Let  $V$  be an irreducible representation of  $K$ . If  $V_2$  is an irreducible representation of  $K$  trivial on  $K \cap M'_1$  with  $\text{Hom}_{K \cap M'_2}(V_2, V) \neq 0$ , then  $V_1 = \text{Hom}_{K \cap M'_2}(V_2, V)$  is an irreducible representation of  $K$  trivial on  $K \cap M'_2$  and  $V \simeq V_1 \otimes V_2$ .

(iii) Let  $V$  be an irreducible representation of  $K$ . Then  $V \simeq V_1 \otimes V_2$  with  $V_i$  as in (i) if and only if  $V$  is trivial on  $M'_1 \cap M'_2$ .

We will not need part (iii), we only included it for completeness.

**Proof** (i) Let  $\tilde{V}_i$  be the pullback of  $V_i$  to  $\tilde{K}$  via  $\iota$ . Then  $\tilde{V}_i$  is trivial on  $\tilde{K}_{3-i}$ , so  $\tilde{V}_1 \otimes \tilde{V}_2$  is an irreducible representation of  $\tilde{K}$ . Hence  $V := V_1 \otimes V_2$  is an irreducible representation of  $K$ . If  $Q = M_Q N_Q$  is a parabolic subgroup containing  $B$ , then

$$V_{N_Q^0} \simeq (V_1)_{N_Q^0} \otimes (V_2)_{N_Q^0}, \text{ as } N_Q^0 = (N_Q^0 \cap M'_1) \times (N_Q^0 \cap M'_2).$$

Hence by III.10,  $\Delta_Q \subset \Delta(V)$  if and only if  $\Delta_Q \subset \Delta(V_i)$  for  $i = 1, 2$ , so  $\Delta(V) = \Delta(V_1) \cap \Delta(V_2)$ . Taking  $Q = B$ , we deduce  $\psi_V = \psi_{V_1} \psi_{V_2}$ . As  $K \cap M'_{3-i}$  is trivial on  $V_i$ , we get  $\Delta_{3-i} \subset \Delta(V_i)$ .

(ii) This follows from Clifford theory [Abe, Lemma 5.3].



(iii) The “if” direction is obvious. Assume that  $V$  is trivial on  $M'_1 \cap M'_2$ . Let  $W$  be an irreducible representation of  $K \cap M'_2$  such that  $\text{Hom}_{K \cap M'_2}(W, V) \neq 0$ . Via  $\iota$ ,  $W$  is an irreducible representation of  $\tilde{K}_2$ , which we consider as a representation  $\tilde{W}$  of  $\tilde{K}$  trivial on  $\tilde{K}_1$ . As  $V$ , hence  $W$ , is trivial  $\iota(\tilde{K}_1) \cap \iota(\tilde{K}_2)$  by assumption, it follows that  $\tilde{W}$  is trivial on  $\text{Ker } \iota$ , so we have extended  $W$  to an irreducible representation of  $K \cap G'$ , which is trivial on  $K \cap M'_1$ . We may view  $W$  as an irreducible representation of  $\overline{G'}$  and we choose an irreducible representation  $V_2$  of  $\overline{G}$  such that  $W$  occurs in  $V_2|_{\overline{G'}}$ . By III.11 Corollary  $W \simeq V_2|_{\overline{G'}}$  and hence  $\text{Hom}_{K \cap M'_2}(V_2, V) \neq 0$ . By part (ii),  $V \simeq V_1 \otimes V_2$  with  $V_i$  as in (i).  $\square$

**III.21.** Let  $(P, \sigma, Q)$  be a  $B$ -triple with  $P(\sigma) = G$ . We are now finally ready to determine the weights of  ${}^e\sigma \otimes \text{St}_Q^G$ . We keep the notation of III.19. Recall that by construction  ${}^e\sigma$  is trivial on  $M'_2$  and  $\text{St}_Q^G$  is trivial on  $M'_1$ .

Let us fix a weight  $V$  of  $I(P, \sigma, Q) = {}^e\sigma \otimes \text{St}_Q^G$ . We decompose the pullback  $\tilde{V}$  of  $V$  to a representation of  $\tilde{K} = \tilde{K}_1 \times \tilde{K}_2$ , via  $\iota$ , as  $\tilde{V} \simeq \tilde{V}_1 \otimes \tilde{V}_2$ . Therefore  $\text{Hom}_K(V, {}^e\sigma \otimes \text{St}_Q^G)$  injects into

$$\text{Hom}_{\tilde{K}}(\tilde{V}, {}^e\sigma \otimes \text{St}_Q^G) \simeq \text{Hom}_{\tilde{K}}(\tilde{V}_1, {}^e\sigma) \otimes \text{Hom}_{\tilde{K}}(\tilde{V}_2, \text{St}_Q^G),$$

where we used that  $\tilde{K}_1$  acts trivially on  $\tilde{V}_2, \text{St}_Q^G$  and  $\tilde{K}_1$  acts trivially on  $\tilde{V}_1, {}^e\sigma$ . As  $\text{St}_Q^G$  has a unique weight (III.18),  $\tilde{V}_2$  is the pullback via  $\iota$  of the unique weight  $V_2$  of  $\text{St}_Q^G$ . By lifting via  $\iota : \tilde{K}_2 \rightarrow \iota(\tilde{K}_2) = K \cap M'_2$ , we deduce  $\text{Hom}_{K \cap M'_2}(V_2, V) = \text{Hom}_{\tilde{K}_2}(\tilde{V}_2, \tilde{V}) \neq 0$ . By III.20 Proposition (ii),  $V \simeq V_1 \otimes V_2$  for some irreducible representation  $V_1$  of  $K$  trivial on  $K \cap M'_2$ . We also see by III.20 Proposition (i) and III.18 Remark 1 that  $\Delta(V) \cap \Delta_2 = \Delta(V_2) \cap \Delta_2 = \Delta_Q \cap \Delta_2$ . The natural injection  $\text{Hom}_K(V_2, \text{St}_Q^G) \hookrightarrow \text{Hom}_{K \cap M'_2}(V_2, \text{St}_Q^G)$  is an isomorphism of 1-dimensional vector spaces, because the right-hand side is isomorphic to  $\text{Hom}_{\tilde{K}}(\tilde{V}_2, \text{St}_Q^G)$  via  $\iota$ . Thus the following lemma, in our situation, implies that  $V_1$  is a weight of  ${}^e\sigma$ , so  $V_{N^0}$  is a weight of  $\sigma$ . This proves that (i) implies (ii) in III.18 Proposition 1).

**Lemma** *Let  $\sigma_1$  be a representation of  $G$  trivial on  $M'_2$ ,  $\sigma_2$  a representation of  $G$  trivial on  $M'_1$ . Let  $V_1$  be an irreducible representation of  $K$  trivial on  $K \cap M'_2$ ,  $V_2$  an irreducible representation of  $K$  trivial on  $K \cap M'_1$ . Assume that the inclusion  $\text{Hom}_K(V_2, \sigma_2) \rightarrow \text{Hom}_{K \cap M'_2}(V_2, \sigma_2)$  is an isomorphism. Then the natural inclusion of  $\text{Hom}_K(V_1, \sigma_1) \otimes \text{Hom}_K(V_2, \sigma_2)$  into  $\text{Hom}_K(V_1 \otimes V_2, \sigma_1 \otimes \sigma_2)$  is an isomorphism.*

**Proof** Look first at points fixed by  $K \cap M'_2$  in  $\text{Hom}(V_1 \otimes V_2, \sigma_1 \otimes \sigma_2)$ . As  $K \cap M'_2$  acts trivially in  $V_1$  and  $\sigma_1$ , it is simply  $\text{Hom}(V_1, \sigma_1) \otimes \text{Hom}_{K \cap M'_2}(V_2, \sigma_2)$ , so by the assumption it is also  $\text{Hom}(V_1, \sigma_1) \otimes \text{Hom}_K(V_2, \sigma_2)$ . Now  $K$  acts trivially on  $\text{Hom}_K(V_2, \sigma_2)$ , so taking fixed points under  $K$  indeed gives  $\text{Hom}_K(V_1, \sigma_1) \otimes \text{Hom}_K(V_2, \sigma_2)$ .  $\square$

We now prove that (ii) implies (i) in III.18 Proposition 1). Let  $V$  be an irreducible representation of  $K$  satisfying (ii). From III.12 Proposition (i),  $V$  is a weight of  $\text{Ind}_P^G \sigma \simeq {}^e\sigma \otimes \text{Ind}_P^G 1$ . Therefore,  $V$  is a weight of  $I(P, \sigma, Q')$  for some parabolic  $Q' \supset P$ . As we have already proved that (i) implies (ii) in III.18 Proposition 1), we deduce that  $\Delta_{Q'} \cap \Delta_2 = \Delta_Q \cap \Delta_2$ , so  $Q' = Q$ .  $\square$

**III.22.** It remains to prove part 2) of III.18 Proposition. We in fact establish something more precise, which gives what we need by III.12 Proposition. Also, by that proposition we may assume  $P(\sigma) = G$ .

**Lemma 1** *Let  $(\rho, V)$  be a weight of  $I(P, \sigma, Q)$  where  $P(\sigma) = G$ .*

(i) *The quotient map  $\text{Ind}_Q^G 1 \rightarrow \text{St}_Q^G$  induces an  $\mathcal{H}_G(V)$ -isomorphism*

$$\text{Hom}_K(V, \text{Ind}_Q^G {}^e\sigma) \longrightarrow \text{Hom}_K(V, I(P, \sigma, Q)).$$

(ii) *The inclusion  $\text{Ind}_Q^G 1 \rightarrow \text{Ind}_P^G 1$  induces an  $\mathcal{H}_G(V)$ -isomorphism*

$$\text{Hom}_K(V, \text{Ind}_Q^G {}^e\sigma) \longrightarrow \text{Hom}_K(V, \text{Ind}_P^G \sigma).$$

**Proof** It is clear that the maps in (i), (ii) are  $\mathcal{H}_G(V)$ -equivariant. As in III.21 write  $V$  as  $V_1 \otimes V_2$  where  $V_2$  is the unique weight of  $\text{St}_Q^G$  (it has parameter  $(1, \Delta_Q)$ ). By III.21 Lemma (the hypothesis is verified by pulling back via  $\iota$ , as in III.21), we get isomorphisms

$$\begin{aligned} \text{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \text{St}_Q^G) &\simeq \text{Hom}_K(V_1, {}^e\sigma) \otimes \text{Hom}_K(V_2, \text{St}_Q^G), \\ \text{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \text{Ind}_Q^G 1) &\simeq \text{Hom}_K(V_1, {}^e\sigma) \otimes \text{Hom}_K(V_2, \text{Ind}_Q^G 1), \\ \text{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \text{Ind}_P^G 1) &\simeq \text{Hom}_K(V_1, {}^e\sigma) \otimes \text{Hom}_K(V_2, \text{Ind}_P^G 1). \end{aligned}$$

The maps  $\text{Ind}_Q^G 1 \rightarrow \text{St}_Q^G$  and  $\text{Ind}_Q^G 1 \rightarrow \text{Ind}_P^G 1$  induce on each side vertical maps which give commutative diagrams. As the vertical maps on the right-hand side are isomorphisms by III.18 Remark 1, so are the vertical maps on the left-hand side, and (i), (ii) are implied by the following well-known lemma.  $\square$

**Lemma 2** *Let  $H'$  be a closed subgroup of a locally profinite group  $H$  and  $\text{ind}_{H'}^H$  the smooth compact induction functor. Let  $V$  be a smooth representation of  $H'$  and  $W$  a smooth representation of  $H$ . Then there is an isomorphism  $\Phi$  of representations of  $H$ ,  $W \otimes \text{ind}_{H'}^H V \xrightarrow{\sim} \text{ind}_{H'}^H(W \otimes V)$ , given by the formula*

$$\Phi(w \otimes f) : h \longmapsto hw \otimes f(h) \quad \text{for } w \in W, f \in \text{ind}_{H'}^H V.$$

## F) Irreducibility of $I(P, \sigma, Q)$

**III.23. Proposition** *Let  $(P, \sigma, Q)$  be a supersingular triple. Then  $I(P, \sigma, Q)$  is irreducible.*

**Proof** It is enough to prove that if  $V$  is an irreducible representation of  $K$  and  $\varphi \in \text{Hom}_K(V, I(P, \sigma, Q))$  is a  $\mathcal{Z}_G(V)$ -eigenvector with eigenvalue  $\chi$ , then the subrepresentation  $X$  of  $I(P, \sigma, Q)$  generated by  $\varphi(V)$  is  $I(P, \sigma, Q)$ . So we fix such a situation and write  $(\psi_V, \Delta(V))$  for the parameter of  $V$ . We prove the result by induction on the cardinality of  $\Delta(V)$ .

By III.14 Corollary 1 we have  $X = I(P, \sigma, Q)$  if  $\Delta(V) \subset \Delta_{P(\sigma)}$ , so we assume that this is not the case. We pick  $\alpha$  in  $\Delta(V)$  but not in  $\Delta_{P(\sigma)}$ , and let  $V'$  be an irreducible representation of  $K$  with parameters  $(\psi_V, \Delta(V) - \{\alpha\})$ . Note that  $V'_{U^0}$  and  $V_{U^0}$  are isomorphic, so that  $\chi$  defines a character of  $\mathcal{Z}_G(V')$  via the Satake isomorphism, which we also denote by  $\chi$ .

Via  $\varphi$ ,  $X$  is a quotient of  $\chi \otimes_{\mathcal{Z}_G(V)} \text{ind}_K^G V$ . By III.18 Corollary  $\Delta_0(\chi) = \Delta_M$ , hence  $\alpha \notin \Delta_0(\chi)$ . By the change of weight theorem (IV.2 Corollary),  $\chi \otimes_{\mathcal{Z}_G(V)} \text{ind}_K^G V$  and  $\chi \otimes_{\mathcal{Z}_G(V')} \text{ind}_K^G V'$  are isomorphic unless  $\alpha$  is orthogonal to  $\Delta_0(\chi)$ ,  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$

and  $\chi(\tau_\alpha) = 1$  (see III.16 for the notation  $\tau_\alpha$ ). By induction then, we are reduced to the case where  $\alpha$  is orthogonal to  $\Delta_0(\chi)$ ,  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  and  $\chi(\tau_\alpha) = 1$ . As  $\Delta_0(\chi) = \Delta_M$ , the conditions imply (III.17 Corollary) that  $\alpha$  belongs to  $\Delta(\sigma) \subset \Delta_{P(\sigma)}$  contrary to assumption.  $\square$

### G) Injectivity of the parametrization

**III.24.** Let  $(P_1, \sigma_1, Q_1)$  and  $(P_2, \sigma_2, Q_2)$  be supersingular triples such that

$$I(P_1, \sigma_1, Q_1) \simeq I(P_2, \sigma_2, Q_2).$$

Let  $V$  be a weight of  $I(P_1, \sigma_1, Q_1)$ , with parameter  $(\psi_V, \Delta(V))$ , and  $\chi$  an eigenvalue of  $\mathcal{Z}_G(V)$  in  $I(P_1, \sigma_1, Q_1)$ . We have seen  $\Delta_0(\chi) = \Delta_{P_1}$  (III.19 Corollary) so we deduce  $\Delta_{P_1} = \Delta_{P_2}$  and  $P_1 = P_2$ . Write  $P_i = M_i N_i$  as usual. By III.18 Proposition,  $V_{N_i^0}$  is a weight of  $\sigma_i$  with supersingular eigenvalue  $\chi$  (via  $\mathcal{S}_{M_i}^G$ ). Then III.17 Corollary implies that  $P(\sigma_1) = P(\sigma_2)$ . Taking the ordinary part functor [Eme, Vig3] with respect to  $P(\sigma_1)$ , we deduce that  ${}^e\sigma_1 \otimes \text{St}_{Q_1}^{P(\sigma_1)}$  and  ${}^e\sigma_2 \otimes \text{St}_{Q_2}^{P(\sigma_2)}$  are isomorphic as representations of  $P(\sigma_1) = P(\sigma_2)$ . From II.8 Remark, we get  $Q_1 = Q_2$  and  $\sigma_1 \simeq \sigma_2$ . This completes the proof of the uniqueness in I.5 Theorem 4.

We insert here a consequence of the irreducibility of  $I(P, \sigma, Q)$  and of the injectivity of the parametrization, which we shall use in part H) and generalize in Chapter VI.

**Proposition** *Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ , and  $\sigma$  a supersingular representation of  $M$ , inflated to  $P$ . Then the irreducible components of  $\text{Ind}_P^G \sigma$  are the  $I(P, \sigma, Q)$ ,  $Q$  a parabolic subgroup of  $G$  with  $P \subset Q \subset P(\sigma)$ ; each occurs with multiplicity 1. In particular  $\text{Ind}_P^G \sigma$  has finite length.*

**Proof** The representation  $\text{Ind}_P^{P(\sigma)} \sigma$  is isomorphic to  ${}^e\sigma \otimes \text{Ind}_P^{P(\sigma)} 1$  (III.22 Lemma 2), which has a filtration with subquotients  ${}^e\sigma \otimes \text{St}_Q^{P(\sigma)}$ , one for each parabolic subgroup  $Q$  with  $P \subset Q \subset P(\sigma)$ . The proposition then follows from III.23 Proposition by parabolic induction from  $P(\sigma)$  to  $G$ .  $\square$

### H) Surjectivity of the parametrization

**III.25.** Let  $(\pi, W)$  be an irreducible admissible representation of  $G$ . To prove that  $\pi$  has the form  $I(P, \sigma, Q)$  for a supersingular triple  $(P, \sigma, Q)$ , we use induction on the semisimple rank of  $G$ .

If  $\Delta_0(\chi) = \Delta$  for all weights  $V$  of  $\pi$  and corresponding eigenvalues  $\chi$ , then  $\pi$  is supersingular and  $\pi \simeq I(G, \pi, G)$ . So we fix a weight  $V$  for  $\pi$  with  $\mathcal{Z}_G(V)$ -eigenvalue  $\chi$  such that  $\Delta_0(\chi) \neq \Delta$ . By construction  $\pi$  is a quotient of  $\chi \otimes_{\mathcal{Z}_G(V)} \text{ind}_K^G V$ .

Let  $P = MN$  be the parabolic subgroup such that  $\Delta_P = \Delta_0(\chi)$ . Consider  $\sigma = \chi \otimes \text{ind}_{M^0}^M V_{N^0}$ . By the filtration theorem (I.6 Theorem 6, proved in Chapter V),  $\chi \otimes \text{ind}_K^G V$  has a filtration with subquotients  $I_e(P, \sigma, Q) = \text{Ind}_{P_e}^G ({}^e\sigma \otimes \text{St}_Q^{P_e})$  where  $P \subset Q \subset P_e$ . So  $\pi$  is a quotient of some  $I_e(P, \sigma, Q)$ . If  $P_e \neq G$ , then by [HV2, Proposition 7.9] (note that  $\sigma$  has a central character by III.12 Lemma) there is an irreducible admissible representation  $\rho$  of the Levi quotient of  $P_e$  such that  $\pi$  is a quotient of  $\text{Ind}_{P_e}^G \rho$ . By the induction hypothesis and III.24 Proposition,  $\rho$  is an irreducible constituent of  $\text{Ind}_{P_1}^{P_e} \rho_1$  where  $P_1$  is a parabolic subgroup of  $P_e$  containing  $B$ , and  $\rho_1$  is a supersingular representation of the Levi quotient

of  $P_1$ . Then  $\pi$  is an irreducible constituent of  $\text{Ind}_{P_1}^G \rho_1$ , so by III.24 Proposition it is isomorphic to  $I(P_1, \rho_1, Q')$  for some  $Q'$ .

If  $P_e = G$ ,  $\pi$  is a quotient of some  ${}^e\sigma \otimes \text{St}_Q^G$ . By II.8 Proposition and Remark,  $\pi$  is isomorphic to  ${}^e\sigma_\pi \otimes \text{St}_Q^G$  for some irreducible admissible representation  $\sigma_\pi$  of  $M$ . The eigenvalues of  $\sigma_\pi$  are those of  $\pi$  by III.18 Proposition, and since  $\Delta_M = \Delta_0(\chi)$ ,  $\sigma_\pi$  has a supersingular eigenvalue. As  $\Delta_M \neq \Delta$ , the induction hypothesis implies that  $\sigma_\pi$  is supersingular, cf. III.18 Remark 2, and  $\pi \simeq I(P, \sigma_\pi, Q)$ .  $\square$

**III.26.** It is worth commenting on the admissibility assumptions in our results. The reader may notice that, since admissibility plays no rôle in Chapters IV and V, our results would still be true if instead of irreducible admissible representations, we considered irreducible representations  $(\sigma, W)$  such that for some weight  $(\rho, V)$  of  $\sigma$ ,  $\text{Hom}_K(V, W)$  contains an eigenvector for  $\mathcal{Z}_G(V)$ . But the classification thus obtained would depend on the choice of  $K, \mathbf{S}, \mathbf{B}$ , whereas we shall see in Chapter VI that with the admissibility assumption it does not depend on those choices. Of course one may hope that the condition above actually implies admissibility or even, as is the case for complex representations, that any irreducible representation of  $G$  is admissible. Note that because of our admissibility condition we do not assert that  $G$  has any supersingular representation. One of us (F.H.) can prove, by a global argument, that when  $G = \text{GL}_n(F)$  and  $F$  has characteristic 0, supersingular representations of  $G$  exist.

#### IV. CHANGE OF WEIGHT

**IV.1.** The main goal of this chapter is to establish our change of weight theorem (IV.2 Corollary). Before commenting on the method of proof, let us state precisely what we prove here. We fix an irreducible representation  $\rho$  of  $K$  on a space  $V$ , with parameter  $(\psi_V, \Delta(V))$ . We consider the “universal” representation  $\text{ind}_K^G V$ , which we see as a subrepresentation of  $\text{Ind}_B^G(\text{ind}_{Z_0}^{Z_0}(V_{U^0}))$  via the injective canonical intertwiner (III.13.1).

We assume that  $\Delta(V)$  is non-empty, and we choose  $\alpha \in \Delta(V)$  and let  $(\rho', V')$  be the irreducible representation of  $K$  with parameter  $(\psi_V, \Delta(V) - \{\alpha\})$ . Similarly we consider the universal representation  $\text{ind}_K^G V'$  as a subrepresentation of  $\text{Ind}_B^G(\text{ind}_{Z_0}^{Z_0} V'_{U^0})$ .

To compare the two universal representations, we fix non-zero vectors  $v$  in  $V$  and  $v'$  in  $V'$  which are invariant under  $U_{\text{op}}^0$ . The image of  $v$  in  $V_{U^0}$  is then a basis of  $V_{U^0}$ , and similarly for  $v'$ . Using those images as basis vectors, we obtain embeddings of  $\text{ind}_K^G V$  and  $\text{ind}_K^G V'$  into the same representation  $\text{Ind}_B^G(\text{ind}_{Z_0}^{Z_0} \psi_V)$ . Moreover the Satake isomorphism induces an algebra homomorphism  $\mathcal{H}_G(V) \rightarrow \mathcal{H}_Z(\psi_V)$ ; the algebra  $\mathcal{H}_Z(\psi_V)$  acts on  $\text{ind}_{Z_0}^{Z_0} \psi_V$ , hence on  $\text{Ind}_B^G(\text{ind}_{Z_0}^{Z_0} \psi_V)$ , and the embedding  $\text{ind}_K^G(V) \rightarrow \text{Ind}_B^G(\text{ind}_{Z_0}^{Z_0} \psi_V)$  is  $\mathcal{H}_G(V)$ -equivariant. We have similar properties for  $V'$ . Note that  $\mathcal{H}_G(V)$  and  $\mathcal{H}_G(V')$  have the same image in  $\mathcal{H}_Z(\psi_V)$ , so we identify them with that common image, which we write  $\mathcal{H}_G$ , and similarly we write  $\mathcal{Z}_G$  for their common centre.

For  $z$  in  $Z$  normalizing  $\psi_V$ , we have the function  $\tau_z$  in  $\mathcal{H}_Z(\psi_V)$  with support  $Z^0 z$  and value  $1_C$  at  $z$ . Recall from III.16 the notation  $a_\alpha$  and  $\tau_\alpha = \tau_{a_\alpha}$ , when  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ .

**Theorem** *Let  $z \in Z^+$ . Assume that  $z$  normalizes  $\psi_V$  and that  $|\alpha|(z) < 1$ . We have:*

- (i)  $\tau_z(\text{ind}_K^G V) \subset \text{ind}_K^G V'$ .
- (ii) *If  $\psi_V$  is not trivial on  $Z^0 \cap M'_\alpha$ , then  $\tau_z(\text{ind}_K^G V') \subset \text{ind}_K^G V$ .*

(iii) If  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ , then  $\tau_z(1 - \tau_\alpha)(\text{ind}_K^G V') \subset \text{ind}_K^G V$ .

**Remark** In (iii)  $\tau_z(1 - \tau_\alpha) = \tau_z - \tau_{za_\alpha}$  belongs to  $\mathcal{Z}_G(V)$  if  $z \in Z_{\psi_V}$  and  $za_\alpha$  belongs to  $Z^+$ ; moreover, if  $|\alpha|(z)$  is small enough,  $za_\alpha$  belongs to  $Z^+$ .

**IV.2.** We obtain our change of weight theorem:

**Corollary** Let  $\chi$  be a character of  $\mathcal{Z}_G$  and assume that  $\alpha \notin \Delta_0(\chi)$ . Then  $\chi \otimes_{\mathcal{Z}_G} \text{ind}_K^G V$  and  $\chi \otimes_{\mathcal{Z}_G} \text{ind}_K^G V'$  are isomorphic unless  $\alpha$  is orthogonal to  $\Delta_0(\chi)$ ,  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$  and  $\chi(\tau_\alpha) = 1$ .

We remark that  $\chi(\tau_\alpha)$  is well defined if  $\alpha$  is orthogonal to  $\Delta_0(\chi)$  (III.4, III.16 Notation).

**Proof** Choose  $z$  as in the theorem, with  $\chi(\tau_z) \neq 0$ . For example, we can take for  $z$  the element  $z_\alpha$  of III.4, since  $\alpha \notin \Delta_0(\chi)$ : then  $\chi(\tau_{z_\alpha}) \neq 0$ . Multiplying by  $\tau_z$  in  $\text{Ind}_B^G(\text{ind}_{Z^0}^Z \psi_V)$  is  $\mathcal{Z}_G$ -linear, so, when  $\psi_V$  is not trivial on  $Z^0 \cap M'_\alpha$ , by (i) and (ii) of the theorem,  $\tau_z$  induces  $G$ -equivariant maps from  $\text{ind}_K^G V$  to  $\text{ind}_K^G V'$  and back. The composites in both directions are given by the action of  $\tau_z^2$ . Tensoring with  $\chi$ , we see that the representations  $\chi \otimes_{\mathcal{Z}_G} \text{ind}_K^G V$  and  $\chi \otimes_{\mathcal{Z}_G} \text{ind}_K^G V'$  are isomorphic, because  $\chi(\tau_z^2) \neq 0$ . That gives the desired result when  $\psi_V$  is non-trivial on  $Z^0 \cap M'_\alpha$ .

Assume then that  $\psi_V$  is trivial on  $Z^0 \cap M'_\alpha$ . Replacing  $z$  by a positive power, we may assume  $za_\alpha \in Z^+$ . If  $\alpha$  is not orthogonal to  $\Delta_0(\chi)$  then there is  $\beta$  in  $\Delta_0(\chi)$  with  $|\beta|(za_\alpha) < 1$  and then  $\chi(\tau_{za_\alpha}) = 0$ , so the same reasoning applies, using (iii) instead of (ii). It similarly applies if  $\alpha$  is orthogonal to  $\Delta_0(\chi)$  and  $\chi(\tau_\alpha) \neq 1$ .  $\square$

**IV.3.** Let us now comment on the proof of IV.1 Theorem. We abbreviate  $\psi = \psi_V$ ,  $J = \Delta(V)$ ,  $J' = J - \{\alpha\}$ , and  $\mathcal{X} = \text{Ind}_B^G(\text{ind}_{Z^0}^Z \psi)$ .

We first remark that  $\text{ind}_K^G V$  is generated, as a representation of  $G$ , by a single element, the function with support  $K$  and value  $v$  at  $1_G$ . We write  $f$  for its image in  $\mathcal{X}$ : it is described explicitly in IV.4 below. Similarly we have a function  $f'$  in  $\mathcal{X}$ , corresponding to  $v'$ , which generates the subrepresentation  $\text{ind}_K^G V'$ .

Let  $I$  be the pro- $p$  Iwahori subgroup of  $G$  which is the inverse image in  $K$  of  $U_k^{\text{op}14}$ . We use work of the fourth-named author [Vig4] which determines the structure of the Hecke algebra  $\mathcal{H} = \mathcal{H}(G, I)$ , the intertwining algebra in  $G$  of the trivial character of  $I$ . The space  $\mathcal{X}^I$  is a right module over  $\mathcal{H}$ , and for  $x \in \mathcal{X}^I$  and  $T$  in  $\mathcal{H}$ ,  $xT$  belongs to the  $G$ -subspace generated by  $x$ . By construction  $f$  and  $f'$  belong to  $\mathcal{X}^I$  and to prove the theorem we show that: for (i)  $\tau_z f \in f' \mathcal{H}$ ; for (ii)  $\tau_z f' \in \mathcal{Z}_G f + f \mathcal{H}$ ; for (iii)  $\tau_z(1 - \tau_\alpha) f' \in \mathcal{Z}_G f + f \mathcal{H}$ . That is not an easy matter and takes up the rest of this chapter.

**IV.4.** Let us first identify  $f$  as an element of  $\mathcal{X}^I$ ; the obvious analogue will hold for  $f'$ .

As  $G = BK$  it is enough to specify  $f$  at  $g \in K$ . Going through the construction of the embedding  $\text{ind}_K^G V \rightarrow \text{Ind}_B^G(\text{ind}_{Z^0}^Z \psi)$  we get that for  $g$  in  $K$ ,  $f(g)$  is the function in  $\text{ind}_{Z^0}^Z \psi$  with support  $Z^0$  and value  $\varepsilon(g)$  at 1, where  $\overline{g}v = \varepsilon(g)\overline{v}$  in  $V_{U^0}$ , bars indicating the images under  $V \rightarrow V_{U^0}$ .

The value  $\varepsilon(g)$  depends only on the image  $\overline{g}$  of  $g$  in  $K/K(1)$ , we write accordingly  $\varepsilon(\overline{g})$ . By [HV2, Corollary 3.19] we have  $\varepsilon(\overline{g}) \neq 0$  if and only if  $\overline{g}$  belongs to  $B_k P_{J,k} B_k^{\text{op}}$  (recall from III.9 Definition that  $P_{J,k}$  is the stabilizer in  $G_k$  of the kernel of the quotient map  $V \rightarrow V_{U^0}$ ); that last set is also  $P_{J,k} U_k^{\text{op}}$ . We can be more precise; we obviously have

<sup>14</sup>Beware of the notation: here, for convenience, we write  $I$  for a pro- $p$  “lower” Iwahori subgroup.

$\varepsilon(\bar{g}x) = \varepsilon(\bar{g})$  for  $x \in U_k^{\text{op}}$ , so it is enough to describe  $\varepsilon|_{P_{J,k}}$ . Since  $P_{J,k}$  is the stabilizer in  $G_k$  of the kernel of  $V \rightarrow V_{U^0}$ , the restriction  $\varepsilon|_{P_{J,k}}$  is a character  $P_{J,k} \rightarrow C^\times$ ; as such it is trivial on unipotent elements. On  $Z_k$  it is given by the action of  $Z_k$  on  $V_{U_k}^{\text{op}}$  or  $V_{U_k}$ , so it is equal to  $\psi$  there. In other words, on  $P_{J,k}$  the character  $\varepsilon$  is simply the (unique) extension of  $\psi$  to  $P_{J,k}$ .

**IV.5.** To relate  $f$  and  $f'$  we shall express both of them in terms of Hecke operators in the subalgebra  $\mathcal{H}(K, I)$  of  $\mathcal{H}(G, I)$  acting on a single function  $f_0$  in  $\mathcal{X}^I$ .

We first describe the double coset spaces  $I \backslash G / I$  and  $B \backslash G / I$ . Recall that the Weyl group  $W_0$  of  $G$  can be seen as  $\mathcal{N}^0 / Z^0$  or  $\mathcal{N}_k / Z_k$ . As  $G = BK$  the inclusion of  $K$  in  $G$  induces a bijection  $B^0 \backslash K / I \simeq B \backslash G / I$ ; as moreover  $I$  contains the normal subgroup  $K(1)$  of  $K$ , reduction mod  $K(1)$  induces a bijection  $B^0 \backslash K / I \simeq B_k \backslash G_k / U_k^{\text{op}}$  and the Bruhat decomposition in  $G_k$  gives a bijection  $\mathcal{N}_k / Z_k \simeq B_k \backslash G_k / U_k^{\text{op}}$ . All in all, we see that the map  $\mathcal{N}^0 \rightarrow B \backslash G / I$   $g \mapsto BgI$  induces a bijection  $W_0 = \mathcal{N}^0 / Z^0 \simeq B \backslash G / I$ .

On the other hand, the map  $\mathcal{N} \rightarrow I \backslash G / I$  induces a bijection  $\mathcal{N} / (Z \cap K(1)) \simeq I \backslash G / I$  and, by restriction, a bijection  $\mathcal{N}^0 / (Z \cap K(1)) \simeq I \backslash K / I$ . Under reduction modulo  $K(1)$  we get the bijection  $\mathcal{N}_k \simeq U_k^{\text{op}} \backslash G_k / U_k^{\text{op}}$  given by the Bruhat decomposition.

**Notation** We put  $Z(1) = Z \cap K(1)$ ; it is a normal subgroup of  $Z$ , and the maximal pro- $p$  subgroup of  $Z^0$ . We write  ${}_1W$  for the group  $\mathcal{N} / Z(1)$  and  ${}_1W_0$  for the group  $\mathcal{N}^0 / Z(1)$  (naturally isomorphic to  $\mathcal{N}_k$ ),  $W$  for the group  $\mathcal{N} / Z^0$ . We have obvious exact sequences of groups

$$\begin{aligned} 1 &\longrightarrow Z_k \longrightarrow {}_1W_0 \longrightarrow W_0 \longrightarrow 1, \\ 1 &\longrightarrow Z_k \longrightarrow {}_1W \longrightarrow W \longrightarrow 1. \end{aligned}$$

Moreover  $W$  is the semi-direct product of  $\Lambda = Z / Z^0$  with  $W_0$  viewed as  $\mathcal{N}^0 / Z^0$ . We also put  ${}_1\Lambda = Z / Z(1)$  and  ${}_1\Lambda^+ = Z^+ / Z(1)$ .

For  $g$  in  $G$  we write  $T(g)$  for the double coset  $IgI$  viewed as an element of  $\mathcal{H}(G, I)$ . On an element  $\varphi$  in  $\mathcal{X}^I$  it acts via

$$(IV.5.1) \quad (\varphi T(g))(h) = \sum_{x \in I / (I \cap g^{-1}Ig)} \varphi(hxg^{-1}) \text{ for } h \in G.$$

When  $g \in \mathcal{N}$ ,  $T(g)$  depends only on the class  $w$  of  $g$  modulo  $Z(1)$ , and we write  $T(w)$  for  $T(g)$ . In a similar manner, reduction modulo  $K(1)$  gives an isomorphism of  $\mathcal{H}(K, I)$  onto  $\mathcal{H}(G_k, U_k^{\text{op}})$ ; accordingly for  $g \in K$ ,  $T(g)$  depends only on the reduction  $\bar{g}$  of  $g$  in  $G_k$  and we write also  $T(\bar{g})$ . In fact we shall also have use of the Hecke algebras with integer coefficients  $\mathcal{H}_{\mathbb{Z}}(G, I)$  and  $\mathcal{H}_{\mathbb{Z}}(K, I)$  (isomorphic to  $\mathcal{H}_{\mathbb{Z}}(G_k, U_k^{\text{op}})$ ) and we use the same notations  $T(g)$ ,  $T(w)$ ,  $T(\bar{g})$ .

**IV.6.** Basic generators and relations for  $\mathcal{H}_{\mathbb{Z}}(G, I)$  and  $\mathcal{H}_{\mathbb{Z}}(K, I)$  are given in [Vig4]. By tensoring with  $C$  they give generators and relations for  $\mathcal{H}(G, I)$  and  $\mathcal{H}(K, I)$ . We now state the results we use, referring to loc. cit. for details. We need a bit more notation, though.

For  $\beta \in \Delta$ , we let  $s_\beta$  be the corresponding reflection in  $W_0$ . We put  $\Sigma_0 = \{s_\beta \mid \beta \in \Delta\}$ . The pro- $p$  Iwahori subgroup  $I$  is attached to an alcove  $\mathfrak{a}$  in the (semisimple) Bruhat-Tits building of  $G$ , with vertex the special point  $\mathbf{x}_0$ , and we let  $\Sigma$  be the set of reflections across the walls of  $\mathfrak{a}$ , so that  $\Sigma_0$  appears as the subset of reflections across walls passing through  $\mathbf{x}_0$ . Then  $\Sigma$  generates an affine Weyl group  $W^a$  canonically identified with the subgroup

$(\mathcal{N} \cap \text{Ker } w_G)/Z^0$  of  $W$ ; also  $W$  is the semi-direct product of its normal subgroup  $W^a$  and the subgroup  $\Omega$  stabilizing the alcove  $\mathbf{a}$ . We let  $\ell$  be the length function of the Coxeter system  $(W^a, \Sigma)$  and we extend it to  $W$ , trivially on  $\Omega$ , i.e. so that  $\ell(w\omega) = \ell(w)$  for  $w \in W^a$ ,  $\omega \in \Omega$ ; on  $W_0$  it restricts to the length function of the Coxeter system  $(W_0, \Sigma_0)$ . Inflating through  ${}_1W \rightarrow W$  we get a length function on  ${}_1W$  and  ${}_1W_0$ , still written  $\ell$ . The operators  $T(w)$  in  $\mathcal{H}_{\mathbb{Z}}(G, I)$  for  $w \in {}_1W$  satisfy the ‘‘braid relations’’

$$(IV.6.1) \quad T(w)T(w') = T(ww') \text{ when } \ell(ww') = \ell(w) + \ell(w').$$

There are other relations, the ‘‘quadratic relations’’ [Vig4, Proposition 4.3]. Essentially there is one such relation for each  $s \in \Sigma$ . It comes directly from the finite field case, treated in [CE, 6.8]. For  $s \in \Sigma_0$ ,  $s = s_\beta$  for some  $\beta \in \Delta$ , we may describe the relation as follows: let  $n_s$  be a lift of  $s_\beta$  in  $\mathcal{N}_k \cap M'_{\beta, k}$  (so that  $n_s^2$  belongs to  $Z_k \cap M'_{\beta, k}$ ); then the quadratic relation for  $T(n_s)$  is:

$$(IV.6.2) \quad T(n_s)(T(n_s) - c_{n_s}) = q_s T(n_s^2),$$

where  $q_s$  is a power of  $p$ ,  $q_s > 1$  and  $c_{n_s} = \sum_{t \in Z_{k,s}} c_{n_s}(t) T(t)$  for positive integers  $c_{n_s}(t)$  constant on each coset of  $\{xs(x)^{-1} \mid x \in Z_k\}$ , of sum  $q_s - 1$ , and  $c_{n_s} \equiv c_s \pmod{p}$  where

$$(IV.6.3) \quad c_s := (q_s - 1) |Z_{k,s}|^{-1} \left( \sum_{t \in Z_{k,s}} T(t) \right),$$

and  $Z_{k,s}$  is the group  $Z_k \cap M'_{\beta, k}$ . We have  $T(n_s)c_{n_s} = c_{n_s}T(n_s)$ .

**Remark** In the  $C$ -algebra  $\mathcal{H}(G, I)$ ,  $q_s$  equals 0 and  $c_{n_s}$  equals  $-|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} T(t)$ , so the relations simplify somewhat. We always embed the group algebra of  $Z_k$  over  $C$  into  $\mathcal{H}(G, I)$  by sending  $t$  to  $T(t)$ ; for  $s = s_\beta$  as above we have  $\psi(c_{n_s}) = -1$  if  $\psi$  is trivial on  $Z_{k,s}$  (i.e.  $\beta$  belongs to the set  $\Delta(\psi)$  of III.8, which contains  $J$ ), and  $\psi(c_{n_s}) = 0$  otherwise.

**Proposition** *There is a unique extension of the map  $s \mapsto n_s$  from  $\Sigma_0$  to  $\mathcal{N}_k$  to a map  $w \mapsto n_w$  from  $W_0$  to  $\mathcal{N}_k$  such that  $n_{ww'} = n_w n_{w'}$  for  $w, w'$  in  $W_0$  such that  $\ell(ww') = \ell(w) + \ell(w')$ .*

**Proof** (Another proof is in [Vig4, Proposition 3.4].) Uniqueness is obvious, as we must have  $n_w = n_{s_1} \cdots n_{s_r}$  for each reduced decomposition  $w = s_s \cdots s_r$  of  $w$  in  $W_0$  with the  $s_i$  in  $\Sigma_0$ . Existence will be consequence of [Bk, §1, n<sup>o</sup> 5, Proposition 5] once we prove:

(\*) For  $s, s'$  distinct in  $\Sigma_0$ , and  $m$  the order of  $ss'$ , then  $(n_s n_{s'})^\ell = (n_{s'} n_s)^\ell$  if  $m = 2\ell$  and  $(n_s n_{s'})^\ell n_s = (n_{s'} n_s)^\ell n_{s'}$  if  $m = 2\ell + 1$ .

To prove (\*) we may assume that  $\mathbf{G}_k$  is semisimple simply connected of relative rank 2, with  $W_0$  generated by  $s$  and  $s'$ , corresponding to the two simple roots  $\beta$  and  $\beta'$ . But then the result follows from [BT1, 6.1.8] applied to the valued root datum associated to  $(\mathbf{G}_k, \mathbf{S}_k, \mathbf{B}_k)$ : indeed we can always put reduced roots of  $\Phi$  in a ‘‘circular order’’ as required by loc. cit., with  $\beta$  first and  $\beta'$  in the  $m$ -th position, in which case loc. cit. formula (9) gives exactly the required equality (\*) above.  $\square$

Henceforward we use the extension  $w \mapsto n_w$ , and we put  $\nu_w = n_w^{-1}$  for  $w \in W_0$ ; in particular if  $w, w'$  in  $W_0$  satisfy  $\ell(ww') = \ell(w) + \ell(w')$ , then  $\nu_{ww'} = \nu_w \nu_{w'}$ .

**IV.7.** We are now ready to define  $f_0$  (as promised in IV.5) and study the action of  $\mathcal{H}(K, I)$  on it. We let  $w_0$  be the longest element in  $W_0$ .

**Definition** *The function  $f_0$  in  $\mathcal{X}^I$  has support  $B\nu_{w_0}I$  and its value at  $\nu_{w_0}$  is the function  $e_\psi$  in  $\text{ind}_{Z^0}^Z \psi$  with support  $Z^0$  and equal to  $\psi$  on  $Z^0$ .*

Note the abuse of notation: we should choose a representative  $\tilde{\nu}_{w_0}$  of  $\nu_{w_0}$  in  $\mathcal{N}^0$  but neither the coset  $B\tilde{\nu}_{w_0}I$  nor the value at  $\tilde{\nu}_{w_0}$  depend on that choice. We shall allow similar abuse of notation below. Note also that  $f_0$  depends on the choice of  $\nu_{w_0}$ .

**Notation** For  $z \in Z_k$  and  $w \in W_0$  we put  $w \cdot z = n_w z n_w^{-1}$  (it is simply the natural action of  $w \in W_0 = \mathcal{N}_k/Z_k$  on  $Z_k$ ); more generally we shall use a dot to denote a conjugation action, which will be clear from the context.

**Lemma** *For  $z \in Z_k$  we have  $z^{-1}f_0 = \psi(w_0 \cdot z^{-1})f_0 = f_0T(z) = \tau_{w_0 \cdot z}f_0$ .*

The last equality in the lemma will be generalized below (IV.10).

**Proof** Since  $Z^0$  normalizes  $I$ , the first equality in the lemma comes from an immediate computation, whereas the equality  $f_0T(z) = z^{-1}f_0$  comes from (IV.5.1). The equality  $\tau_z f_0 = \psi(z^{-1})f_0$  is equally easy.  $\square$

**Proposition** *Let  $w \in W_0$ . Then  $f_0T(n_w)$  has support  $B\nu_{w_0w}I$  and value  $e_\psi$  at  $\nu_{w_0w}$ .*

**Proof** As  $f_0T(n_w)$  is  $I$ -invariant, it is enough to compute its value at  $\nu_{w'}$  for  $w'$  in  $W_0$ . By definition  $(f_0T(n_w))(g) = \sum f_0(ghn_w^{-1})$  for  $g \in G$ , where the sum runs over  $h$  in  $I/(n_w^{-1}In_w \cap I)$ . Assume that for such an  $h$ ,  $f_0$  is not 0 at  $\nu_{w'}hn_w^{-1}$ . Then looking modulo  $K(1)$ , we get that  $\nu_{w'}U_k^{\text{op}}n_w^{-1} \cap B_k\nu_{w_0}U_k^{\text{op}}$  is non-empty, and, multiplying on the right by  $\nu_{w_0}^{-1}$ , that  $\nu_{w'}U_k^{\text{op}}n_w^{-1}\nu_{w_0}^{-1} \cap B_k \neq \emptyset$  and hence  $B_k\nu_{w'}U_k^{\text{op}} \cap B_k\nu_{w_0}n_wU_k^{\text{op}} \neq \emptyset$ ; by the Bruhat decomposition in  $G_k$ , that implies  $w' = w_0w$ . Assume that  $w' = w_0w$ ; then  $h$  belongs to  $\nu_{w'}^{-1}B^0\nu_{w_0}In_w$ . However note that  $\ell(w_0w) + \ell(w^{-1}) = \ell(w_0)$  (because  $w_0$  is the longest element in  $W_0$ ), so that  $\nu_{w'}\nu_{w^{-1}} = \nu_{w_0}$ ; we deduce that the image of  $h$  in  $G_k$  belongs to  $n_w^{-1}B_k^{\text{op}}n_w \cap U_k^{\text{op}} = n_w^{-1}U_k^{\text{op}}n_w \cap U_k^{\text{op}}$ . But that shows that  $h$  belongs to  $n_w^{-1}In_w \cap I$  and consequently  $(f_0T(n_w))(\nu_{w_0w}) = f_0(\nu_{w_0w}n_w^{-1}) = f_0(\nu_{w_0}) = e_\psi$ .  $\square$

**Corollary**  $f = \sum_{w \in w_0W_J} f_0T(n_w)$ .

**Proof** By the description in IV.4, for  $w$  in  $W_0$ ,  $f(\nu_w)$  is equal to  $e_\psi$  if  $w$  belongs to  $W_J$  and is 0 otherwise: we only have to remark that  $P_{J,k}U_k^{\text{op}} = B_kW_JU_k^{\text{op}}$ , and since  $\psi(Z_k \cap M'_{\beta,k}) = 1$  for  $\beta \in J$ , the character  $\varepsilon$  of IV.4 is trivial on  $\nu_w$  for  $w \in W_J$ .  $\square$

**IV.8.** We need to determine the action of  $c_{n_s}$  on  $f_0T(n_w)$  for  $s = s_\beta$ ,  $\beta \in J$ . We recall that  $J \subset \Delta(\psi)$ .

**Proposition** *Let  $\beta \in \Delta(\psi)$ ,  $s = s_\beta$  and  $z \in Z_k \cap M'_{\Delta(\psi),k}$ . For  $w \in w_0W_{\Delta(\psi)}$ , we have*

$$\begin{aligned} f_0T(n_w)T(z) &= f_0T(n_w) \text{ and} \\ f_0T(n_w)c_{n_s} &= -f_0T(n_w). \end{aligned}$$

*In particular  $fc_{n_s} = -f$ .*

**Proof** By III.10 Example 2,  $\psi$  is trivial on  $Z_k \cap M'_{\Delta(\psi),k}$ . By IV.7 Lemma then, we get  $f_0T(z) = f_0$  for  $z \in Z_k \cap w_0M'_{\Delta(\psi),k}w_0^{-1}$ . The braid relation gives  $T(n_w)T(t) = T(w \cdot t)T(n_w)$  for  $t \in Z_k$ ,  $w \in W_0$ . For  $z \in Z_k \cap M'_{\Delta(\psi),k}$  we have  $w \cdot z \in Z_k \cap M'_{\Delta(\psi),k}$  for



$w \in W_{\Delta(\psi)}$ , hence  $(w_0w) \cdot z \in Z_k \cap w_0M'_{\Delta(\psi),k}w_0^{-1}$ , and consequently  $f_0T(n_{w_0w})T(z) = f_0T(n_{w_0w})$ . That gives the first assertion.

The second one comes from the expression of  $c_{n_s}$  in (IV.6.3), noting that  $q_s$  gives 0 in  $C$ ; the last assertion follows from IV.7 Corollary.  $\square$

**IV.9. Notation** Let  $w_J$  be the longest element in  $W_J \subset W_0$  and put  $w^J = w_0w_J$  (note that  $w_J$  and  $w_0$  have order 2). We put  $f_J = f_0T(n_{w^J})$ .

**Lemma 1** For  $w \in W_J$  we have (i)  $\ell(w^Jw) = \ell(w^J) + \ell(w)$ , (ii)  $T(n_{w^Jw}) = T(n_{w^J})T(n_w)$ , and (iii)  $f_0T(n_{w^Jw}) = f_JT(n_w)$ .

**Proof** We have  $\ell(w^Jw) = \ell(w_0w_Jw) = \ell(w_0) - \ell(w_Jw)$ ; if  $w \in W_J$  we also have  $\ell(w_Jw) = \ell(w_J) - \ell(w)$  so we get  $\ell(w^Jw) = \ell(w^J) + \ell(w)$ ; by the braid relation  $T(n_{w^Jw}) = T(n_{w^J})T(n_w)$ , and the last assertion follows.  $\square$

By Lemma 1, and IV.7 Corollary, IV.8 Proposition, we have  $f = \sum_{w \in W_J} f_JT(n_w)$  and for  $w \in W_J$

$$(IV.9.1) \quad f_JT(n_w)c_{n_s} = -f_JT(n_w).$$

For  $s \in \Sigma_0$  we put  $T^*(n_s) = T(n_s) - c_{n_s}$ , so that in  $\mathcal{H}_{\mathbb{Z}}(K, I)$  we get

$$T(n_s)T^*(n_s) = T^*(n_s)T(n_s) = q_sT(n_s^2) \quad (= 0 \text{ in } \mathcal{H}(K, I)).$$

That definition can be extended to defining  $T^*(n_w)$  for  $w \in W_0$ , so that  $T^*(n_{ww'}) = T^*(n_w)T^*(n_{w'})$  if  $\ell(ww') = \ell(w) + \ell(w')$  [Vig4, Proposition 4.13]. We now use the Bruhat order  $\leq$  on the Coxeter group  $W_J$ .

**Proposition** For  $w \in W_J$  we have

$$\begin{aligned} f_J\left(\sum_{v \leq w} T(n_v)\right) &= f_JT^*(n_w) \text{ and in particular} \\ f &= f_JT^*(n_{w^J}) = f_0T(n_{w^J})T^*(n_{w^J}). \end{aligned}$$

A similar proposition can be found in [Oll2, Lemma 5.1].

**Proof** We use induction on  $\ell(w)$ . The result is true for  $w = 1$ . If  $\ell(w) = \ell \geq 1$ , we write  $w = w's$  with  $\ell(w') = \ell - 1$ ,  $\ell(s) = 1$ . As  $\ell(w) = \ell(w') + \ell(s)$  we have  $T^*(n_w) = T^*(n_{w'})T^*(n_s) = T^*(n_{w'})(T(n_s) - c_{n_s})$ . By induction  $f_JT^*(n_{w'}) = \sum_{v \leq w'} f_JT(n_v)$ . Remembering that for  $v$  in  $W_J$  we have  $T(n_{w^J})T(n_v) = T(n_{w^Jv})$  and by (IV.9.1)  $f_JT(n_v)c_{n_s} = -f_JT(n_v)$ . So finally we obtain

$$f_JT^*(n_w) = f_JT^*(n_{w'})(T(n_s) + 1).$$

By induction  $f_JT^*(n_{w'}) = \sum_{v \leq w'} f_JT(n_v)$ , so we want to compute  $A = \sum_{v \leq w'} f_JT(n_v)T(n_s)$ .

Divide the set of  $v \leq w'$  in the disjoint union  $X \sqcup Y \sqcup Ys$  where

$$\begin{aligned} Y &= \{v \in W_J, v < vs \leq w'\}, \\ Ys &= \{v \in W_J, vs < v \leq w'\}, \\ X &= \{v \in W_J, v \leq w' \text{ and } vs \not\leq w'\}. \end{aligned}$$

In  $A$ , the subsum over  $Y \sqcup Ys$  is

$$\sum_{v \in Y} f_J(T(n_{vs}) + T(n_v))T(n_s).$$

But for  $v \in Y$ , we have  $v < vs$  so  $T(n_{vs}) = T(n_v)T(n_s)$  and  $f_J(T(n_{vs}) + T(n_v))T(n_s) = f_J T(n_v)(T(n_s) + 1)T(n_s)$ . By (IV.9.1) that equals  $f_J T(n_v)T^*(n_s)T(n_s)$  which is 0 because  $T^*(n_s)T(n_s) = 0$  in  $\mathcal{H}$ . So  $A = \sum_{v \in X} f_J T(n_v)T(n_s)$ . Since for  $v \in X$ , we have  $v < vs$  we

get  $A = \sum_{v \in X} f_J T(n_{vs})$ .

The proof will be complete once we get:

**Lemma 2**  $Xs = \{v \in W_J, v \leq w \text{ and } v \not\leq w'\}$ .

**Proof** We use properties of the Bruhat order [Deo, Theorem 1.1 (II) (ii)]. Let  $a, b$  in  $W_J$  with  $a \leq b$ . Then:

(1) If  $a < as$  then  $a \leq bs$ ; (2) if  $b > bs$  then  $as \leq b$ .

Let  $v \in X$ , i.e.  $v \leq w'$ ,  $vs \not\leq w'$ . Then by (2) applied to  $a = v, b = w$ , we get  $vs \leq w$ . Conversely let  $v \in W_J$  verify  $v \leq w$  and  $v \not\leq w'$ ; if  $v < vs$  then  $v \leq w'$  by (1) applied to  $a = v, b = w$ , which is a contradiction; so  $vs < v \leq w$ , which gives  $vs \leq w'$  by (1) applied to  $a = vs$  and  $b = w$ . That proves the lemma.  $\square$

**IV.10.** We now turn to the promised generalization of IV.7 Lemma which will be used in IV.15.

**Proposition** Let  $z \in Z$  with  $z^{-1} \in Z^+$ . Assume that  $\nu_{w_0} \cdot z$  normalizes  $\psi$ . Then  $f_0 T(z) = \tau_{\nu_{w_0} \cdot z} f_0$ .

**Remark** If  $z^{-1}$  belongs to  $Z^+$ ,  $\nu_{w_0} \cdot z$  also belongs to  $Z^+$ , and conversely.

**Proof** As both terms are  $I$ -invariant, we only need to check that they are equal at  $\nu_w$  for  $w \in W_0$ . Now  $(f_0 T(z))(g) = \sum f_0(ghz^{-1})$  for  $g \in G$ , where the sum runs over  $h \in I/(z^{-1}Iz \cap I)$ . But  $I$  has an Iwahori decomposition and the assumption that  $z^{-1}$  belongs to  $Z^+$  gives  $z^{-1}(I \cap U)z \subset I \cap U$ ,  $z^{-1}(I \cap U_{\text{op}})z \supset I \cap U_{\text{op}}$ , thus the inclusion of  $I \cap U$  into  $I$  induces a bijection of  $(I \cap U)/(z^{-1}Iz \cap U)$  onto  $I/(z^{-1}Iz \cap I)$ , and it is enough to let  $h$  run through  $(I \cap U)/(z^{-1}Iz \cap U)$ . For such an  $h$ ,  $\nu_w h z^{-1}$  belongs to  $B\nu_{w_0}I$  only if  $w = w_0$ : indeed  $\nu_w h z^{-1} \in Bn_w U$  and  $B\nu_{w_0}I \subset Bn_{w_0}U$ , so the Bruhat decomposition in  $G$  implies  $w = w_0$ . Consequently both terms of the desired equality vanish at  $\nu_w$  for  $w \neq w_0$ .

Consider now  $(f_0 T(z))(\nu_{w_0})$ . Let  $h \in I \cap U$  with  $\nu_{w_0} h z^{-1} = b\nu_{w_0}j$  for some  $b$  in  $B$ ,  $j$  in  $I$ ; again by the Iwahori decomposition of  $I$ , we may assume that  $j$  belongs to  $I \cap U$  and then the equality  $h = (\nu_{w_0}^{-1}b\nu_{w_0})z(z^{-1}jz)$ , where  $\nu_{w_0}^{-1}b\nu_{w_0}z \in B^{\text{op}}$  and  $z^{-1}jz \in U$ , shows that  $h$  is equal to  $z^{-1}jz$  and belongs to  $z^{-1}Iz \cap U$ ; consequently  $(f_0 T(z))(\nu_{w_0}) = f_0(\nu_{w_0} z^{-1}) = f_0((\nu_{w_0} \cdot z^{-1})\nu_{w_0}) = (\nu_{w_0} \cdot z^{-1})f_0(\nu_{w_0})$ . That is equal to  $(\nu_{w_0} \cdot z^{-1})e_\psi$ , which sends  $z'$  to  $e_\psi(z'(\nu_{w_0} \cdot z^{-1}))$ . On the other hand if  $\nu_{w_0} \cdot z$  normalizes  $\psi$ , we have  $(\tau_{\nu_{w_0} \cdot z} f)(\nu_{w_0}) = \tau_{\nu_{w_0} \cdot z} e_\psi$ , sending  $z'$  to  $e_\psi((\nu_{w_0} \cdot z^{-1})z')$ . That gives the result since  $\nu_{w_0} \cdot z$  normalizes  $\psi$ .  $\square$

**IV.11.** To go further, we need other generators, indeed other bases, of  $\mathcal{H}_{\mathbb{Z}}(G, I)$ . They are constructed in [Vig4, Ch. 5] using (spherical) orientations. We need not know what such an object is, only that it is determined by a Weyl chamber in the vector space  $V_{\text{ad}} = X_*(\mathbf{S}_{\text{ad}}) \otimes \mathbb{R}$ , where  $\mathbf{S}_{\text{ad}}$  is the torus image of  $\mathbf{S}$  in the adjoint group  $\mathbf{G}_{\text{ad}}$  of  $\mathbf{G}$ . We have the dominant Weyl chamber  $\mathcal{D}^+ = \{v \in V_{\text{ad}}, \beta(v) > 0 \text{ for } \beta \in \Delta\}$ , and the antidominant Weyl chamber  $\mathcal{D}^- = -\mathcal{D}^+ = w_0\mathcal{D}^+$ . Of course  $W_0$  acts, on the left, on the set of Weyl chambers, hence on orientations; but as in loc. cit., we let  $W_0$  (and hence  ${}_1W$  via  ${}_1W \rightarrow W_0$ ) act **on the right** on orientations by  $(o, w) \mapsto o \cdot w$ , so that if an orientation  $o$  corresponds to the Weyl chamber  $\mathcal{D}$ ,  $o \cdot w$  corresponds to  $w^{-1}(\mathcal{D})$ .

We recall the natural map  $\nu : Z \rightarrow V_{\text{ad}}$  used in [Vig4, 3.3]: the action of  $z \in Z$  on  $V_{\text{ad}}$  is via translation by  $\nu(z)$ . We remark that  $\nu$  is the composite of  $-v_Z : Z \rightarrow X_*(\mathbf{S}) \otimes \mathbb{R}$  with  $X_*(\mathbf{S}) \otimes \mathbb{R} \rightarrow V_{\text{ad}}$ . By loc. cit.  $Z^+$  is the set of  $z \in Z$  such that  $\nu(z)$  belongs to the closure of  $\mathcal{D}^-$  (i.e.  $\beta \circ \nu(z) \leq 0$  for  $\beta \in \Delta$ ). The map  $\nu$  factors through  ${}_1\Lambda$  and  $\Lambda$ , and we still write  $\nu$  for the corresponding maps.

Note however that in citing loc. cit. Ch. 5, some care is needed.

The first thing is that the roots in loc. cit. are in the reduced root system  $\Phi_a$  on  $V_{\text{ad}}$  attached to the collection of affine root hyperplanes in  $V_{\text{ad}}$  (it is denoted  $\Sigma$  in loc. cit.). It is **not** in general the root system  $\Phi$  attached to  $(\mathbf{G}_{\text{ad}}, \mathbf{S}_{\text{ad}})$ . Let us describe what is happening. The space  $V_{\text{ad}} = X_*(\mathbf{S}_{\text{ad}}) \otimes \mathbb{R}$  is naturally a quotient of  $X_*(\mathbf{S}) \otimes \mathbb{R}$ , and its dual  $X^*(\mathbf{S}_{\text{ad}}) \otimes \mathbb{R}$  appears as the subspace of  $X^*(\mathbf{S}) \otimes \mathbb{R}$  generated by the roots in  $\Phi$ , which are then the same for  $(\mathbf{G}, \mathbf{S})$  and  $(\mathbf{G}_{\text{ad}}, \mathbf{S}_{\text{ad}})$ . The coroot in  $V_{\text{ad}}$  attached to a given root  $\beta$  in  $\Phi$  is the image of  $\beta^\vee \in X_*(\mathbf{S})$ , we also write it  $\beta^\vee$ . The root system  $\Phi_a$  on  $V_{\text{ad}}$  can be described from  $\Phi$  as follows. For each  $\beta \in \Phi$ , there is a positive integer  $e_\beta$  such that  $\Phi_a$  is the set of  $\beta_a := e_\beta\beta$  for  $\beta \in \Phi$  – in particular  $e_{2\beta} = e_\beta/2$  if  $2\beta \in \Phi$ . The root systems  $\Phi_a$  and  $\Phi$  share the same Weyl group  $W_0$ , and consequently the same Weyl chambers. The choice of Weyl chamber defining  $\Phi^+$  also defines  $\Phi_a^+$  and  $\beta \mapsto \beta_a$  gives a bijection of  $\Delta$  onto the set  $\Delta_a$  of simple roots in  $\Phi_a$ . Note also that  $(\beta_a \circ \nu)(\Lambda) \subset \mathbb{Z}$  and that the coroot in  $V_{\text{ad}}$  associated to  $\beta_a \in \Phi_a$  is  $\beta_a^\vee = e_\beta^{-1}\beta^\vee$ .

**Examples** 1) If  $\mathbf{G}$  is split, then  $\Phi_a = \Phi$ ,  $e_\beta = 1$  for  $\beta \in \Phi$ .

2) For  $G = \text{GL}_r(D)$ , where  $D$  is a central division algebra over  $F$ , of finite degree  $d^2$ , then  $e_\beta = d$  for all  $\beta \in \Phi$ .

3) Assume that  $\mathbf{G}$  is semisimple simply connected of relative rank 1. Then there is only one positive root  $\beta$  and  $\beta_a \circ \nu(\Lambda) = 2\mathbb{Z}$  (loc. cit. 5.14). Going back to the situation of III.16 with no condition on the reductive group  $\mathbf{G}$  we deduce that  $\nu(a_\beta) = \beta_a^\vee$ , since  $\langle \beta_a, \beta_a^\vee \rangle = 2$ . In particular  $v_Z(a_\beta) = -e_\beta^{-1}\beta^\vee$ .

The second thing to be careful about is that the choice of Iwahori subgroup corresponds to a choice of alcove with vertex  $\mathbf{x}_0$ , and positivity conditions are with respect to that choice. As we work with the “lower” pro- $p$  Iwahori subgroup  $I$ , the alcove with vertex  $\mathbf{x}_0$  which corresponds to  $I$  is the one contained in  $\mathcal{D}^-$ , so positive roots in loc. cit. Ch. 5 correspond to negative roots here. In citing loc. cit. Ch. 5 therefore we either have to exchange positive and negative roots, or to replace  $\nu$  with  $-\nu$ ; we choose the first solution. For example  $\Sigma^+, \mathcal{D}^+$  in loc. cit. correspond to  $\Phi_a^-, \mathcal{D}^-$  here.

**IV.12.** Let  $o$  be an orientation. By [Vig4, Corollary 5.26] it gives a basis  $(E_o(w))_{w \in {}_1W}$  for  $\mathcal{H}_{\mathbb{Z}}(G, I)$ . In  $\mathcal{H}_{\mathbb{Z}}(G, I)$  some computations are easier because it is a “characteristic

zero" algebra. The above basis of  $\mathcal{H}_{\mathbb{Z}}(G, I)$  specializes to a basis  $(E_o(w))_{w \in {}_1W}$  of  $\mathcal{H}$  over  $C$ : we use the same notation, making the context precise when necessary.

To  $w \in {}_1W$  is attached an element  $q_w$  in  $\mathbb{Z}$ , such that  $q_{n_s} = q_s$  for  $s \in \Sigma_0$  and  $q_w = 1$  if  $\ell(w) = 0$ . The main relations in  $\mathcal{H}_{\mathbb{Z}}(G, I)$  satisfied by the  $E_o(w)$  are the following relations: for  $w, w'$  in  ${}_1W$ ,

$$(IV.12.1) \quad E_o(w)E_{o \cdot w}(w') = q_{w, w'} E_o(ww') \text{ with } q_{w, w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2}.$$

**Beware** that in general  $o \cdot w \neq o$ , although it is the case when  $w \in {}_1\Lambda$ . Note that  $q_{w, w'} = 1$  if and only if  $\ell(ww') = \ell(w) + \ell(w')$ , and  $q_{w, w'}$  gives 0 in  $C$  otherwise (loc. cit., Remark 4.18 and Lemma 4.19).

In particular, if  $\mathcal{A}_o$  is the subspace of  $\mathcal{H}$  with basis  $(E_o(\lambda))$  for  $\lambda \in {}_1\Lambda$ , the multiplication in  $\mathcal{A}_o$  is straightforward:

$$(IV.12.2) \quad E_o(\lambda)E_o(\lambda') = \begin{cases} E_o(\lambda\lambda') & \text{if } \ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda'), \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $\mathcal{A}_o$  is a subalgebra of  $\mathcal{H}$ . In fact the condition  $\ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda')$  means that  $\nu(\lambda)$  and  $\nu(\lambda')$  belong to the same closed Weyl chamber in  $V_{\text{ad}}$ , cf. loc. cit. 5.12.

If  $o$  is an orientation, we let  $\Lambda_o$  be the set of  $\lambda \in \Lambda$  such that  $\nu(\lambda)$  belongs to the closure of the corresponding Weyl chamber; we similarly define  ${}_1\Lambda_o$ . For  $\lambda$  in  ${}_1\Lambda_o$ , we have  $E_o(\lambda) = T(\lambda)$ , loc. cit. Example 5.30.

We shall need the orientation  $o_I$  attached to a subset  $I$  of  $\Delta$ : by definition it is the orientation corresponding to the Weyl chamber  $w_I(\mathcal{D}^-)$ . Hence  $o_{\Delta}$  corresponds to  $\mathcal{D}^+$ ,  $o_{\emptyset}$  corresponds to  $\mathcal{D}^-$ ,  $o_I = o_{\Delta} \cdot w^I$ ,  $\Lambda_{o_I} = w_I \cdot \Lambda^+$  (hence  ${}_1\Lambda_{o_I} = \nu_{w_I} \cdot {}_1\Lambda^+$ ). For  $w \in W_I$  we then have  $E_{o_I}(n_w) = T(n_w)$ , loc. cit. Example 5.32. (Note that  $w_I(\mathcal{D}^-)$  here equals  $w_I(\mathcal{D}^+)$  in loc. cit. which corresponds to  $o_{w_I(\Delta)}$  in loc. cit.)

**IV.13.** We need some length formulas (loc. cit. Corollary 5.10 and 5.11). We have to be careful to remember that  $\Sigma^+$  in loc. cit. corresponds to  $\Phi_a^-$ . For  $\lambda \in \Lambda$ ,  $w \in W_0$ , we have

$$(IV.13.1) \quad \ell(w \cdot \lambda) = \sum_{\beta \in \Phi_a^+} |\beta \circ \nu(\lambda)| = \ell(\lambda),$$

$$(IV.13.2) \quad \ell(w\lambda) = \sum_{\beta \in \Phi_a^+ \cap w^{-1}(\Phi_a^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Phi_a^+ \cap w^{-1}(\Phi_a^-)} |\beta \circ \nu(\lambda) - 1|,$$

$$(IV.13.3) \quad \ell(\lambda w) = \sum_{\beta \in \Phi_a^+ \cap w(\Phi_a^+)} |\beta \circ \nu(\lambda)| + \sum_{\beta \in \Phi_a^+ \cap w(\Phi_a^-)} |\beta \circ \nu(\lambda) + 1|.$$

Note that for  $\beta \in \Delta$  and  $w = s_{\beta} = w^{-1}$ ,  $s_{\beta}$  permutes  $\Phi_a^+ - \{\beta_a\}$  and sends  $\beta_a$  to  $-\beta_a$  so  $\Phi_a^+ \cap s_{\beta}(\Phi_a^-) = \{\beta_a\}$ .

**Lemma** *Let  $I \subset \Delta$ . Then, for  $\lambda \in \Lambda_{o_I}$ ,  $\ell(w^I \lambda) = \ell(w^I) + \ell(\lambda)$ .*

**Proof** By (IV.13.2) we need to check that  $\beta \circ \nu(\lambda) \leq 0$  for  $\beta \in \Phi^+ \cap (w^I)^{-1}(\Phi^-)$ ; but  $\lambda \in \Lambda_{o_I}$  means that  $\beta \circ \nu(\lambda) \geq 0$  for  $\beta \in w_I(\Phi^-) = (w^I)^{-1}(\Phi^+)$ .  $\square$

**IV.14.** An important result in this chapter is the following (see [Oll, Section 5] for  $\mathrm{GL}_n$ ).

**Theorem** *Let  $w \in W_J$ . Then for  $\lambda \in {}_1\Lambda$ ,*

$$f_J T^*(n_w) E_{o_J}(\lambda) = \begin{cases} \tau((\nu_{w_J} n_w) \cdot \lambda) f_J T^*(n_w) & \text{if } (\nu_{w_J} n_w) \cdot \lambda \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } (\nu_{w_J} n_w) \cdot \lambda \notin {}_1\Lambda^+. \end{cases}$$

The proof of the theorem is in IV.15–IV.18. Taking  $w = w_J$  we get by IV.9 Proposition:

**Corollary** *For  $\lambda \in {}_1\Lambda$ ,*

$$f E_{o_J}(\lambda) = \begin{cases} \tau(\lambda) f & \text{if } \lambda \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } \lambda \notin {}_1\Lambda^+. \end{cases}$$

### Remarks

- 1) We have used the notation  $\tau(\mu)$  for  $\mu \in {}_1\Lambda^+$  to mean  $\tau_z$  for  $z \in Z^+$  with image  $\mu \in {}_1\Lambda^+$ . The shift of indices is only for typographical convenience.
- 2) As  $\psi$  extends to a character of  $M_{J,k}$  by IV.4, each  $n_w$  for  $w \in W_J$  normalizes  $\psi$ , and it follows that  $\lambda$  normalizes  $\psi$  if and only if so does  $(\nu_{w_J} n_w) \cdot \lambda$ .
- 3) The subspace of  $\mathcal{A}_{o_J}$  generated by the  $E_{o_J}(\lambda)$  for  $\lambda$  in  ${}_1\Lambda$  normalizing  $\psi$  is a subalgebra  $\mathcal{A}_{o_J}(\psi)$  of  $\mathcal{A}_{o_J}$ . The map  $\mathcal{A}_{o_J}(\psi) \rightarrow \mathcal{H}_Z(\psi)$  sending  $E_{o_J}(\lambda)$  to  $\tau((\nu_{w_J} n_w) \cdot \lambda)$  if  $(\nu_{w_J} n_w) \cdot \lambda \in {}_1\Lambda^+$  and to 0 otherwise is an algebra homomorphism  $\theta_{\nu_{w_J} n_w}$ , and for  $T \in \mathcal{A}_{o_J}(\psi)$  we have

$$f_J T^*(n_w) T = \theta_{\nu_{w_J} n_w}(T) f_J T^*(n_w).$$

- 4) The theorem says nothing when  $(\nu_{w_J} n_w) \cdot \lambda \in {}_1\Lambda^+$  and does not normalize  $\psi$ . We do not use this case.

**IV.15.** We prove the theorem by induction on  $\ell(w)$ . We treat first the case where  $w = 1$ . Recalling that  $f_J = f_0 T(n_{w^J})$ , we want to compute  $f_0 T(n_{w^J}) E_{o_J}(\lambda)$ . By IV.12, we have  $T(n_{w^J}) = E_{o_\Delta}(n_{w^J})$ , so we look at  $E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda)$ .

Assume first that  $\nu_{w_J} \cdot \lambda$  belongs to  ${}_1\Lambda^+$ , i.e. that  $\lambda$  belongs to  ${}_1\Lambda_{o_J}$ , and that  $\nu_{w_J} \cdot \lambda$  normalizes  $\psi$ . Then IV.13 Lemma gives  $\ell(n_{w^J}) + \ell(\lambda) = \ell(n_{w^J} \lambda)$ , hence  $E_{o_\Delta}(n_{w^J} \lambda) = E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda)$ . Since  $\ell(n_{w^J} \cdot \lambda) = \ell(\lambda)$  by (IV.13.1), we also obtain  $\ell(n_{w^J} \cdot \lambda) + \ell(n_{w^J}) = \ell(n_{w^J} \lambda)$  hence  $E_{o_\Delta}(n_{w^J} \lambda) = E_{o_\Delta}(n_{w^J} \cdot \lambda) E_{o_\Delta}(n_{w^J})$ , and finally

$$E_{o_\Delta}(n_{w^J} \cdot \lambda) E_{o_\Delta}(n_{w^J}) = E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda) = T(n_{w^J}) E_{o_J}(\lambda).$$

We can apply IV.10 Proposition to  $n_{w^J} \cdot \lambda$ . Indeed  $\nu_{w_0} \cdot (n_{w^J} \cdot \lambda) = (\nu_{w_0} n_{w^J}) \cdot \lambda$  and  $\nu_{w_0} n_{w^J} = \nu_{w_J}$ . Since by IV.12  $E_{o_\Delta}(n_{w^J} \cdot \lambda) = T(n_{w^J} \cdot \lambda)$ , that gives  $f_0 E_{o_\Delta}(n_{w^J} \cdot \lambda) = \tau(\nu_{w_J} \cdot \lambda) f_0$ , so  $\tau(\nu_{w_J} \cdot \lambda) f_J = f_0 E_{o_\Delta}(n_{w^J} \cdot \lambda) T(n_{w^J}) = f_J E_{o_J}(\lambda)$ , which is the desired formula when  $\nu_{w_J} \cdot \lambda$  belongs to  ${}_1\Lambda^+$ .

Fix a regular such  $\lambda$  and let  $\lambda' \in {}_1\Lambda - {}_1\Lambda_{o_J}$ . Then  $E_{o_J}(\lambda) E_{o_J}(\lambda') = 0$  by (IV.12.2), and  $f_J E_{o_J}(\lambda) E_{o_J}(\lambda') = 0$ , implying  $\tau(\nu_{w_J} \cdot \lambda) f_J E_{o_J}(\lambda') = 0$ . Since  $\tau(\nu_{w_J} \cdot \lambda)$  is invertible in  $\mathcal{H}_Z(\psi)$ , we get  $f_J E_{o_J}(\lambda') = 0$ , which is the formula we want for  $\lambda'$ . The theorem is proved for  $w = 1$ .

**IV.16.** Now let  $\ell(w) = \ell \geq 1$ , and write  $w = w's$  with  $\ell(w') = \ell - 1$  and  $s = s_\beta$  for some  $\beta \in J$  – note that  $w'(\beta) \in \Phi^+$  since  $\ell(w's) = \ell(w') + 1$ . In particular  $n_w = n_{w'}n_s$  and  $T^*(n_w) = T^*(n_{w'})T^*(n_s)$ .

We need to investigate  $T^*(n_s)E_{o_J}(\lambda)$  for  $\lambda \in {}_1\Lambda$ . Suppose we can prove

$$(*) \quad f_J T^*(n_{w'}) T^*(n_s) E_{o_J}(\lambda) = f_J T^*(n_{w'}) E_{o_J}(n_s \cdot \lambda) T^*(n_s);$$

then the desired formula follows from the induction hypothesis. So we need to compare  $E_{o_J}(n_s \cdot \lambda) T^*(n_s)$  and  $T^*(n_s) E_{o_J}(\lambda)$ . By loc. cit. Corollary 5.53 we have, for any orientation  $o$  such that  $\text{Ker } \beta$  is a wall of the Weyl chamber corresponding to  $o$ :

$$(IV.16.1) \quad \begin{aligned} \text{If } \beta \circ \nu(\lambda) = 0, & \quad E_o(n_s \cdot \lambda) E_o(n_s) = E_o(n_s) E_o(\lambda); \\ \text{if } \beta \circ \nu(\lambda) > 0, & \quad E_o(n_s \cdot \lambda) E_o(n_s) = E_{o \cdot s}(n_s) E_o(\lambda); \\ \text{if } \beta \circ \nu(\lambda) < 0, & \quad E_o(n_s \cdot \lambda) E_{o \cdot s}(n_s) = E_o(n_s) E_o(\lambda). \end{aligned}$$

We now apply the results in loc. cit. §5.4 to our case, where  $o = o_J$ . (We need to point out that since  $\beta \in J$ ,  $\text{Ker}(\beta)$  is a wall of the Weyl chamber corresponding to  $o_J$ ; also loc. cit. uses the notation  $s$  for an element of  ${}_1W_0$ , where we use  $n_s$ , but we do have  $n_s^2 \in Z_k$  as required by loc. cit. 5.35 and 5.36.) Since  $\beta \in J$ , we have  $E_{o_J}(n_s) = T(n_s)$  (IV.12) and  $E_{o_J \cdot s}(n_s) = T^*(n_s)$  by loc. cit. Example 5.32. So we get:

$$(IV.16.2) \quad \begin{aligned} \text{If } \beta \circ \nu(\lambda) = 0, & \quad E_{o_J}(n_s \cdot \lambda) T(n_s) = T(n_s) E_{o_J}(\lambda); \\ \text{if } \beta \circ \nu(\lambda) > 0, & \quad E_{o_J}(n_s \cdot \lambda) T(n_s) = T^*(n_s) E_{o_J}(\lambda); \\ \text{if } \beta \circ \nu(\lambda) < 0, & \quad E_{o_J}(n_s \cdot \lambda) T^*(n_s) = T(n_s) E_{o_J}(\lambda). \end{aligned}$$

**IV.17.** Accordingly we distinguish the three cases.

Assume first  $\beta \circ \nu(\lambda) = 0$ ; then formula (\*) of IV.16 follows from (IV.16.2) and the following lemma.

**Lemma** *Assume  $\beta \circ \nu(\lambda) = 0$ . Then  $E_{o_J}(n_s \cdot \lambda) c_{n_s} = c_{n_s} E_{o_J}(\lambda)$ .*

**Proof** We work within the Levi subgroup  $M_\beta$  of  $G$ . As  $\beta \circ \nu(\lambda) = 0$ ,  $\lambda$  normalizes  $K \cap M_\beta$  (III.7 Corollary). (Note that  $K \cap M_\beta$  is the parahoric subgroup of  $M_\beta$  attached to our special point  $\mathbf{x}_0$ ;  $\lambda$  also normalizes the pro- $p$  radical  $K(1) \cap M_\beta$  of  $K \cap M_\beta$ .) Consequently  $\lambda$  acts via conjugation on  $M_{\beta,k}$ ; that action stabilizes  $U_{\beta,k}$  and  $U_{\beta,k}^{\text{op}}$ , so it also stabilizes the subgroup  $M'_{\beta,k}$  they generate. Consequently  $\lambda$  acts via conjugation on  $Z_{k,s} = Z_k \cap M'_{\beta,k}$ . On the other hand, an element  $t$  in  $Z_{k,s}$  has length 0, implying  $E_{o_J}(n_s \cdot \lambda) T(t) = E_{o_J}((n_s \cdot \lambda)t)$  and  $T(t) E_{o_J}(\lambda) = E_{o_J}(t\lambda)$ . Now, computing in  ${}_1W$ ,  $(n_s \cdot \lambda)t\lambda^{-1} = (n_s \lambda n_s^{-1} \lambda^{-1})(\lambda t \lambda^{-1})$ . As  $t$  runs through  $Z_{k,s}$ , so does  $\lambda t \lambda^{-1}$ ; on the other hand, by construction  $n_s$  belongs to  $M'_{\beta,k}$  so  $n_s \lambda n_s^{-1} \lambda^{-1}$  belongs to  $Z_{k,s}$ . The result follows.  $\square$

**IV.18.** Assume now that  $\beta \circ \nu(\lambda) < 0$ . Since  $w'(\beta)$  is positive,  $(w_J w' s)(\beta) = -w_J w'(\beta)$  is positive too. But  $((w_J w' s)(\beta)) \circ \nu$ , evaluated on  $\nu_{w_J n_{w'}}(\lambda)$  gives  $(s(\beta) \circ \nu)(\lambda) = -\beta \circ \nu(\lambda) > 0$  so  $\nu_{w_J n_{w'}}(\lambda)$  is not in  ${}_1\Lambda^+$ , and consequently  $f_J T^*(n_{w'}) E_{o_J}(\lambda) = 0$  by the induction hypothesis. But by (IV.16.2)

$$f_J T^*(n_{w'}) [T^*(n_s) E_{o_J}(\lambda) - E_{o_J}(n_s \cdot \lambda) T^*(n_s)] = -f_J T^*(n_{w'}) c_{n_s} E_{o_J}(\lambda).$$

Since  $f_J T^*(n_{w'}) c_{n_s} = -f_J T^*(n_{w'})$  by IV.8 Proposition,  $-f_J T^*(n_{w'}) c_{n_s} E_{o_J}(\lambda)$  is equal to  $f_J T^*(n_{w'}) E_{o_J}(\lambda)$ , which is 0 by the above, and (\*) is true in that case too.

The case where  $\beta \circ \nu(\lambda) > 0$  is dealt with similarly: in that case we find

$$f_J T^*(n_{w'}) [T^*(n_s) E_{o_J}(\lambda) - E_{o_J}(n_s \cdot \lambda) T^*(n_s)] = f_J T^*(n_{w'}) E_{o_J}(n_s \cdot \lambda) c_{n_s}$$

by (IV.16.2) and that is 0 by induction because  $\nu_{w_J} n_w(\lambda)$  is not in  ${}_1\Lambda^+$  (as  $(w_J w(\beta)) \circ \nu$  is positive on it). This completes the proof of IV.14 Theorem.  $\square$

**IV.19.** We now reach the easier part of our change of weight, which is a consequence of the following theorem.

**Theorem** *Assume that  $\lambda \in {}_1\Lambda$  normalizes  $\psi$ . Then*

$$f' E_{o_{J'}}(\lambda n_{w_J w_{J'}}^{-1}) T^*(n_{w_J w_{J'}}) = \begin{cases} \tau(\lambda) f & \lambda \in {}_1\Lambda^+ \text{ and } \alpha \circ \nu(\lambda) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Taking  $z \in Z^+$ , normalizing  $\psi$ , and with  $|\alpha|(z) < 1$ , we get  $f' T = \tau_z f$  for some  $T$  in  $\mathcal{H}$ , which gives IV.1 Theorem (i). To prove the theorem, we first prove:

**Lemma**  $f' = f_J T^*(n_{w_J w_{J'} w_J}) T(n_{w_J w_{J'}})$ .

**Proof** By IV.7 Corollary,  $f' = f_0 \sum_{w \in w_0 W_{J'}} T(n_w)$ , which can also be written as  $f' = f_0 \sum_{v \in W_{J'}} T(n_{w_0 v w_{J'}})$ . For  $v$  in  $W_{J'}$ , write  $w_0 v w_{J'} = w^J (w_J v w_J) (w_J w_{J'})$ . We have

$$\ell(w_0 v w_{J'}) = \ell(w_0) - \ell(v w_{J'}) = \ell(w_0) - \ell(w_{J'}) + \ell(v) \text{ (since } v \in W_{J'}),$$

and

$$\ell(w^J) = \ell(w_0) - \ell(w_J), \ell(w_J v w_J) = \ell(v), \ell(w_J w_{J'}) = \ell(w_J) - \ell(w_{J'}),$$

so  $\ell(w_0 v w_{J'}) = \ell(w^J) + \ell(w_J v w_J) + \ell(w_J w_{J'})$ . Consequently

$$\sum_{v \in W_{J'}} T(n_{w_0 v w_{J'}}) = T(n_{w^J}) \left( \sum_{v \in W_{J'}} T(n_{w_J v w_J}) \right) T(n_{w_J w_{J'}})$$

and  $f' = f_J \left( \sum_{v \in W_{J'}} T(n_{w_J v w_J}) \right) T(n_{w_J w_{J'}})$ .

Now  $J'' = -w_J(J')$  is a subset of  $J$  and  $w_J W_{J'} w_J = W_{J''}$ ; the element  $w_J w_{J'} w_J$  is the longest element of that group, hence

$$\sum_{v \in W_{J'}} T(n_{w_J v w_J}) = \sum_{v \leq w_J w_{J'} w_J} T(n_v).$$

By IV.9 Proposition

$$f_J \left( \sum_{v \leq w_J w_{J'} w_J} T(n_v) \right) = f_J T^*(n_{w_J w_{J'} w_J})$$

so  $f' = f_J T^*(n_{w_J w_{J'} w_J}) T(n_{w_J w_{J'}})$ .  $\square$

**Proof of the theorem** Put  $v = w_J w_{J'}$ . Note that since  $v \in W_J$ ,  $n_v \cdot \lambda$  normalizes  $\psi$ , see IV.14 Remark 2).

By the relations (IV.12.1) we get

$$q_{n_v, \lambda n_v^{-1}} E_{o_J}(n_v \cdot \lambda) = E_{o_J}(n_v) E_{o_{J \cdot v}}(\lambda n_v^{-1}).$$

On the other hand  $E_{o_J}(n_w) = T(n_w)$  for  $w \in W_J$ , so we get

$$T(n_v) E_{o_{J \cdot v}}(\lambda n_v^{-1}) = q_{n_v, \lambda n_v^{-1}} E_{o_J}(n_v \cdot \lambda).$$

We now compute

$$\begin{aligned} f' E_{o_J \cdot v}(\lambda n_v^{-1}) &= f_J T^*(n_{w_J w_{J'} w_J}) T(n_v) E_{o_J \cdot v}(\lambda n_v^{-1}) \\ &= q_{n_v, \lambda n_v^{-1}} f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v \cdot \lambda). \end{aligned}$$

By IV.14 Theorem we see that  $f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v \cdot \lambda)$  is 0 if  $\lambda \notin {}_1\Lambda^+$ . If  $\alpha \circ \nu(\lambda) = 0$ , since  $v(\alpha) \in \Phi^-$ ,  $\ell(\lambda v^{-1}) > \ell(\lambda) - \ell(v^{-1}) = \ell(v \cdot \lambda) - \ell(v)$  by IV.13, so  $q_{n_v, \lambda n_v^{-1}} = 0$ .

Assume  $\alpha \circ \nu(\lambda) < 0$  and  $\lambda \in {}_1\Lambda^+$ . Let  $\beta \in \Phi^+$  with  $v(\beta) \in \Phi^-$ . Since  $v \in W_J$ ,  $\beta$  is a linear combination of roots in  $J$  (with non-negative integer coefficients). Moreover  $w_{J'}(\beta) \in \Phi^+$ , so the coefficient of  $\alpha$  in  $\beta$  is positive. Then for  $\beta \in \Phi^+ \cap v^{-1}(\Phi^-)$  we have  $\beta \circ \nu(\lambda) \leq \alpha \circ \nu(\lambda)$  by the above, so  $\beta \circ \nu(\lambda) < 0$ , which implies by IV.13 that  $\ell(\lambda v^{-1}) = \ell(v \cdot \lambda) - \ell(v^{-1})$  and  $f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v \cdot \lambda) = \tau(\lambda) f_J T^*(n_{w_J w_{J'} w_J})$  by IV.14 Theorem (indeed,  $\ell(w_J w_{J'} w_J) + \ell(v) = \ell(w_J)$  implies  $n_{w_J w_{J'} w_J} n_v = n_{w_J}$ , so  $\nu_{w_J} n_{w_J w_{J'} w_J} n_v = 1$ ). The theorem follows on multiplying by  $T^*(n_v)$ , noting  $T^*(n_{w_J w_{J'} w_J}) T^*(n_v) = T^*(n_{w_J})$  and  $o_{J'} = o_J \cdot v$ .  $\square$

**IV.20.** We now turn to the other part of the change of weight, which is harder. From now on, we put  $s = s_\alpha$ .

**Lemma**  $f' = f - f_J T^*(n_{w_{J_s}}) = f_J T^*(n_{w_{J_s}}) T(n_s)$ .

**Proof** By IV.9 we have  $f = f_J \left( \sum_{w \leq w_J} T(n_w) \right)$  and  $f_J T^*(n_{w_{J_s}}) = f_J \left( \sum_{w \leq w_{J_s}} T_w \right)$  so  $f - f_J T^*(n_{w_{J_s}}) = f_J \left( \sum_{w \leq w_J} T(n_w) \right)$  where the sum runs over  $w$  in  $W_J$  with  $w \not\leq w_{J_s}$ ; but for  $w \in W_J$ ,  $w \leq w_{J_s}$  is equivalent to  $s \leq w_J w$ , so  $w \not\leq w_{J_s}$  means that  $w_J w$  belongs to  $W_{J'}$ . Consequently  $f - f_J T^*(n_{w_{J_s}}) = f_J \left( \sum_{w \in W_{J'}} T(n_{w_J w_{J'} w}) \right) = f_0 T(n_{w^J}) \left( \sum_{w \in W_{J'}} T(n_{w_J w_{J'} w}) \right)$ .

For  $w$  in  $W_{J'}$ ,  $\ell(w_J w_{J'} w) = \ell(w_J) - \ell(w_{J'}) = \ell(w_J) - \ell(w_{J'}) + \ell(w) = \ell(w_J w_{J'}) + \ell(w)$  so  $T(n_{w_J w_{J'} w}) = T(n_{w_J w_{J'}}) T(n_w)$ . On the other hand,  $\ell(w^J) + \ell(w_J w_{J'}) = \ell(w_0) - \ell(w_J) + \ell(w_J) - \ell(w_{J'}) = \ell(w^{J'})$  so  $T(n_{w^J}) = T(n_{w^{J'}}) T(n_{w_J w_{J'}})$ . It follows that  $f - f_J T^*(n_{w_{J_s}}) = f_0 T(n_{w^{J'}}) \left( \sum_{w \in W_{J'}} T(n_w) \right) = f'$  by IV.7 Corollary applied to  $J'$ . Moreover,

as  $\ell(w_{J_s}) + \ell(s) = \ell(w_J)$  we have  $T^*(n_{w_{J_s}}) = T^*(n_{w_{J_s}}) T^*(n_s)$  and  $f = f_J T^*(n_{w_{J_s}}) = f_J T^*(n_{w_{J_s}}) (T(n_s) + 1)$ , as seen in IV.9 above, so  $f' = f_J T^*(n_{w_{J_s}}) T(n_s)$ .  $\square$

**IV.21.** Let now  $\lambda \in {}_1\Lambda^+$  and put  $\lambda' = n_s \cdot \lambda$ . It is the element  $f E_{o_J \cdot s}(n_s \lambda')$  that we want to relate to  $f'$ . To get an expression for it, we again need to distinguish cases, according to the integer  $r = -\alpha_a \circ \nu(\lambda) \geq 0$  (recall that  $\alpha_a$  is the simple root in  $\Phi_a$  corresponding to  $\alpha$ ). We first deal with the ‘‘easy’’ relations in  $\mathcal{H}$ .

**Lemma** (i)  $\lambda'(n_s \cdot \lambda') = n_s \cdot (\lambda \lambda') \in {}_1\Lambda^+$ .

(ii) If  $r > 0$ ,  $\ell(n_s \lambda') = \ell(\lambda') - 1$  and  $T(n_s) E_{o_J \cdot s}(n_s \lambda') = T(n_s^2) E_{o_J}(\lambda')$ .

(iii) If  $r \geq 2$ , then  $E_{o_J}(\lambda') E_{o_J}(n_s \lambda') = 0$ .

(iv) If  $r = 1$ , then  $E_{o_J \cdot s}(n_s \lambda') = E_{o_J}(n_s \lambda')$  and

$$E_{o_J}(\lambda') E_{o_J}(n_s \lambda') = E_{o_J}(\lambda'(n_s \cdot \lambda')) T(n_s).$$

**Proof** (i) The first equality is clear. Let us prove that  $\lambda'(n_s \cdot \lambda')$  is in  ${}_1\Lambda^+$ . We have  $\alpha_a \circ \nu(\lambda'(n_s \cdot \lambda')) = 0$ . For  $\beta \in \Delta$ ,  $\beta \neq \alpha$  we compute

$$\begin{aligned} \beta_a \circ \nu(\lambda'(n_s \cdot \lambda')) &= \beta_a \circ \nu((n_s \cdot \lambda)(n_s^2 \cdot \lambda)) \\ &= (\beta_a + s(\beta_a))(\nu(\lambda)). \end{aligned}$$



It is  $\leq 0$  since  $\beta_a, s(\beta_a) > 0$  and  $\lambda$  is in  ${}_1\Lambda^+$ . So we get (i).

(ii) Assume  $r > 0$ . We need to work in  $\mathcal{H}_{\mathbb{Z}}$ , and then specialize to  $\mathcal{H}$ . By (IV.13.2), we get  $\ell(n_s\lambda') = \ell(\lambda') - 1$  because  $\alpha \circ \nu(\lambda') > 0$ . So the relation (IV.12.1) gives  $E_{o_J \cdot s}(n_s)E_{o_J}(\lambda') = q_s E_{o_J \cdot s}(n_s\lambda')$ . We also have  $E_{o_J}(n_s)E_{o_J \cdot s}(n_s) = q_s E_{o_J}(n_s^2)$ , which gives

$$q_s E_{o_J}(n_s)E_{o_J \cdot s}(n_s\lambda') = q_s E_{o_J}(n_s^2)E_{o_J}(\lambda').$$

Cancelling  $q_s$ , using  $E_{o_J}(n_s^2) = T(n_s^2)$ , and specializing to  $\mathcal{H}$  we get (ii).

(iii) We proved  $\ell(n_s\lambda') = \ell(\lambda') - 1$  in (ii), so  $\ell(\lambda') + \ell(n_s\lambda') = 2\ell(\lambda') - 1$ . On the other hand  $\lambda'n_s\lambda' = \lambda'(n_s \cdot \lambda')n_s$  and  $\alpha_a \circ \nu(\lambda'(n_s \cdot \lambda')) = 0$  so  $\ell(\lambda'n_s\lambda') = \ell(\lambda'(n_s \cdot \lambda')) + 1$  by (IV.13.3). But  $\ell(\lambda'(n_s \cdot \lambda')) = \ell(\lambda') + \ell(n_s \cdot \lambda') - 2r$  by (IV.13.1) so we get  $\ell(\lambda') + \ell(n_s\lambda') - \ell(\lambda'n_s\lambda') = 2r - 2$ . This is  $> 0$  if  $r \geq 2$ , so in that case  $E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = 0$  by the relations (IV.12.1).

(iv) Assume now that  $r = 1$ . The first formula is given by loc. cit. Lemma 5.34. In the proof of (ii) we have seen that  $\ell(\lambda') + \ell(n_s\lambda') = \ell(\lambda'n_s\lambda')$  so we get  $E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = E_{o_J}(\lambda'n_s\lambda')$ . On the other hand  $\lambda'n_s\lambda' = \lambda'(n_s \cdot \lambda')n_s$  and we have seen  $\ell(\lambda'n_s\lambda') = \ell(\lambda'(n_s \cdot \lambda')) + 1$ , so  $E_{o_J}(\lambda'n_s\lambda') = E_{o_J}(\lambda'(n_s \cdot \lambda'))E_{o_J}(n_s) = E_{o_J}(\lambda'(n_s \cdot \lambda'))T(n_s)$ .  $\square$

**IV.22.** In the sequel it is convenient to put  $\varphi = f_J T^*(n_{w_J s})$  so that  $f' = \varphi T(n_s)$ ,  $f = \varphi + f'$ . From IV.14 Theorem, we get the following: for  $\mu \in {}_1\Lambda$ ,

$$(IV.22.1) \quad \varphi E_{o_J}(n_s \cdot \mu) = \begin{cases} \tau(\mu)\varphi & \text{if } \mu \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } \mu \notin {}_1\Lambda^+. \end{cases}$$

Put  $E = E_{o_J \cdot s}(n_s\lambda')$  with  $\lambda' = n_s \cdot \lambda$  as in IV.21 – note that  $\lambda'$  also normalizes  $\psi$ .

By (ii) of IV.21 Lemma,  $T(n_s)E = T(n_s^2)E_{o_J}(\lambda')$ , so  $\varphi T(n_s)E = \tau(n_s^2)\varphi E_{o_J}(\lambda')$  by (IV.22.1). But  $\tau(n_s^2)\varphi = \varphi$  because  $n_s^2$ , which belongs to  $Z_k \cap M'_{\alpha, k}$ , acts trivially on  $\varphi$  by IV.7 Lemma. We deduce  $\varphi T(n_s)E = \varphi E_{o_J}(\lambda') = \tau(\lambda)\varphi$ , again by (IV.22.1).

We are now ready to prove a change of weight formula, in the special case where  $\lambda \in {}_1\Lambda^+$  normalizes  $\psi$  and  $\alpha_a \circ \nu(\lambda) = -1$ . Indeed by (IV.22.1) and (iv) of IV.21 Lemma we get  $\tau(\lambda)\varphi E = \varphi E_{o_J}(\lambda')E = \varphi E_{o_J}(\lambda'(n_s \cdot \lambda'))T(n_s)$ , hence  $\tau(\lambda)\varphi E = \tau(\lambda\lambda')\varphi T(n_s)$ , using again (IV.22.1). We deduce that  $\varphi E = \tau(\lambda')\varphi T(n_s)$ , as  $\tau(\lambda)$  is invertible in  $\mathcal{H}_{\mathbb{Z}}(\psi)$ .

Consequently  $fE = \varphi E + \varphi T(n_s)E = \tau(\lambda)\varphi + \tau(\lambda')\varphi T(n_s) = \tau(\lambda)(f - f') + \tau(\lambda')f'$ . We have proved:

**Proposition** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $\alpha_a \circ \nu(\lambda) = -1$ . Then*

$$\tau(\lambda)f - fE = (\tau(\lambda) - \tau(\lambda'))f'.$$

**Remark** Note that  $\tau(\lambda)f$  belongs to  $\text{ind}_K^G V$  because  $\lambda \in {}_1\Lambda^+$ , so we see that  $\text{ind}_K^G V$  contains  $(\tau(\lambda) - \tau(\lambda'))(\text{ind}_K^G V')$ . Note also that  $\tau(\lambda)f'$  belongs to  $\text{ind}_K^G V'$  for the same reason; but  $\tau(\lambda')f'$  does not necessarily belong to  $\text{ind}_K^G V'$  because  $\lambda'$  is not in  ${}_1\Lambda^+$ .

**IV.23.** We now seek a similar formula in the case where  $\lambda \in {}_1\Lambda^+$  normalizes  $\psi$ ,  $r = -\alpha_a \circ \nu(\lambda) \geq 2$ , still with  $\lambda' = n_s \cdot \lambda$  and  $E = E_{o_J \cdot s}(n_s\lambda')$ . By loc. cit., Proposition 5.48 we have, in  $\mathcal{H}_{\mathbb{Z}}$ , an identity

$$(*) \quad E_{o_J \cdot s}(n_s\lambda') - E_{o_J}(n_s\lambda') = \sum_{k=1}^{r-1} q(k, \lambda') q_s^{-1} c(k, \lambda') E_{o_J}(\mu(k, \lambda'))$$

and by loc. cit. Proposition 5.49, in  $\mathcal{H}$  only the terms for  $k = 1$  and  $k = r - 1$  may be non-zero, so we get, in  $\mathcal{H}$ ,

$$E = E_{o_J}(n_s \lambda') + c_1 E_{o_J}(\mu_1 \lambda') + c_{r-1} E_{o_J}(\mu_{r-1} \lambda')$$

where the last term disappears if  $r = 2$ . For the moment we need not know what  $c_1, c_{r-1}$  are in  $C[Z_k]$ , nor what  $\mu_1$  and  $\mu_{r-1}$  are in  ${}_1\Lambda$  except that they do not depend on  $\lambda$  and (loc. cit., formula (90))  $\nu(\mu_k) = -k\alpha_a^\vee$ , so  $\nu(n_s^{-1} \cdot \mu_k) = k\alpha_a^\vee$ . From that it follows that  $(n_s^{-1} \cdot \mu_1)\lambda$  is in  ${}_1\Lambda^+$ , but not  $(n_s^{-1} \cdot \mu_{r-1})\lambda$  if  $r > 2$ . Also by loc. cit. 5.49, the  $q$ -terms in the identity (\*) above give 1 in  $C$  for  $k = 1$  or  $k = r - 1$ . Indeed we have to show that  $\ell(\lambda') - \ell(\mu_{-\alpha_a}^{-1} \lambda') = 2$ : remarking that  $\nu(\mu_{-\alpha_a}^{-1} \lambda') = \nu(\lambda') - \alpha_a^\vee$ , that comes from the length formula in IV.13. As in IV.22 we have  $\varphi T(n_s)E = \tau(\lambda)\varphi$ . On the other hand

$$\varphi E = \varphi E_{o_J}(n_s \lambda') + \varphi c_1 E_{o_J}(\mu_1 \lambda') + \varphi c_{r-1} E_{o_J}(\mu_{r-1} \lambda')$$

where the last term disappears if  $r = 2$ .

But  $\tau(\lambda)\varphi = \varphi E_{o_J}(\lambda')$  by (IV.22.1), so  $\tau(\lambda)\varphi E_{o_J}(n_s \lambda') = \varphi E_{o_J}(\lambda')E_{o_J}(n_s \lambda')$  which is 0 by IV.21 Lemma (iii), and hence  $\varphi E_{o_J}(n_s \lambda') = 0$ . For  $z \in Z_k$  we have  $\varphi E_{o_J}(n_s \cdot z) = \tau_z \varphi = \psi(z^{-1})\varphi$  so we get  $\varphi c_1 E_{o_J}(\mu_1 \lambda') = \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda')$ , with the obvious notation for the conjugation action on  $C[Z_k]$ , and the obvious extension of  $\psi^{-1}$  from  $Z_k$  to  $C[Z_k]$ . Similarly, if  $r \geq 3$ ,  $\varphi c_{r-1} E_{o_J}(\mu_{r-1} \lambda') = \psi^{-1}(n_s^{-1} \cdot c_{r-1})\varphi E_{o_J}(\mu_{r-1} \lambda')$ , which is 0 by (IV.22.1) because  $(n_s^{-1} \cdot \mu_{r-1})\lambda$  is not in  ${}_1\Lambda^+$ . Thus for  $r \geq 2$ ,

$$\varphi E = \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda').$$

As  $\varphi T(n_s)E = \tau(\lambda)\varphi$  we obtain:

**Proposition** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $-\alpha_a \circ \nu(\lambda) \geq 2$ . Then*

$$fE = \tau(\lambda)\varphi + \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda').$$

**IV.24.** We now apply the formulas given by IV.22 Proposition and IV.23 Proposition to the case where  $\lambda \in {}_1\Lambda^+$  normalizes  $\psi$ , and deduce IV.1 Theorem (ii) and (iii). We first assume  $\alpha_a \circ \nu(\lambda) = -1$ . As we have seen in IV.22 Remark,  $\lambda'$  normalizes  $\psi$  and  $(\tau(\lambda) - \tau(\lambda'))(\text{ind}_K^G V') \subset \text{ind}_K^G V$ .

**Proposition** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $\alpha_a \circ \nu(\lambda) = -1$ . Then  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$  and  $\tau(\lambda') = \tau(\lambda)\tau_\alpha$ .*

**Proof** We work within  $M_\alpha$ . The semisimple Bruhat-Tits building of  $M_\alpha$  is a tree, the apartment corresponding to  $\mathbf{S}$  is the line in  $V_{\text{ad}}$  generated by  $\alpha_a^\vee$ ; the group  $Z$  acts on that line via its quotient  $\Lambda$ , and  $\lambda \in \Lambda$  acts via translation by  $v$  with  $\alpha_a \circ \nu(\lambda) = \alpha_a(v)$  and as  $\alpha_a \circ \nu(\lambda) = -1$ ,  $\lambda$  sends the (special) vertex  $\mathbf{x}_0$  to the adjacent (special) vertex  $\mathbf{x}_1 = \mathbf{x}_0 - \frac{1}{2}\alpha_a^\vee$  in the apartment. We shall later prove the following claim.

For the claim the situation is the following:

**Assumption** Assume that  $\mathbf{G}$  has relative semisimple rank 1, and let  $\mathbf{x}_1$  be a vertex in  $V_{\text{ad}}$  (a line) adjacent to  $\mathbf{x}_0$ , and  $K_1$  the corresponding (special) parahoric subgroup of  $G$ . Let  $\mathbf{G}_{1,k}$  be the group over  $k$  attached to the parahoric subgroup  $K_1$ . (Note that both  $K = K_0$  and  $K_1$  contain  $Z^0$  and  $G_k, G_{1,k}$  contain  $Z_k$ .)

**Claim** *The subgroup of  $Z_k$  generated by  $Z_k \cap G'_k$  and  $Z_k \cap G'_{1,k}$  is the image of  $Z^0 \cap G'$  in  $Z_k$ .*

We apply the claim to  $\mathbf{M}_\alpha$ . Since  $\lambda$  sends  $\mathbf{x}_0$  to  $\mathbf{x}_1$ , it conjugates  $K_0$  to  $K_1$ , and conjugation by  $\lambda$  induces an isomorphism of  $M_{\alpha,k}$  onto  $M_{\alpha,1,k}$  and of  $M'_{\alpha,k}$  onto  $M'_{\alpha,1,k}$ . As  $\psi$  is trivial on  $Z_k \cap M'_{\alpha,k}$  by hypothesis, and  $\lambda$  stabilizes  $\psi$ ,  $\psi$  is also trivial on  $Z_k \cap M'_{\alpha,1,k}$  and by the claim  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ . By the second line after formula (90) in loc. cit., from  $\alpha_a \circ \nu(\lambda) = -1$  we get  $\nu(\lambda^{-1}\lambda') = \alpha_a^\vee$ ; but  $\lambda' = n_s \cdot \lambda$  by definition, so  $\lambda^{-1}\lambda' = \lambda^{-1}n_s\lambda n_s^{-1}$ . Take  $z \in Z$  with image  $\lambda$  in  ${}_1\Lambda$  and  $\tilde{n}_s$  in  $K \cap M'_\alpha \cap \mathcal{N}$  with image  $n_s$  in  $M_{\alpha,k}$  (the existence follows from III.7 Lemma, for instance). Since  $M'_\alpha$  is normal in  $M_\alpha$ ,  $z^{-1}\tilde{n}_s z$  is in  $M'_\alpha$  so  $\lambda^{-1}n_s\lambda n_s^{-1}$  is the image in  ${}_1\Lambda$  of an element of  $Z \cap M'_\alpha$ . It follows that we can take  $\lambda^{-1}\lambda'$  as the image in  ${}_1\Lambda$  of  $a_\alpha$  of III.16 Notation (which verifies  $\nu(a_\alpha) = \alpha_a^\vee$ , cf. IV.11 Example 3), and then  $\tau(\lambda') = \tau(\lambda)\tau_\alpha$ .  $\square$

From the above proposition and IV.22 Proposition, we get case (iii) of IV.1 Theorem when  $\alpha_a \circ \nu(\lambda) = -1$ .

**Corollary** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $\alpha_a \circ \nu(\lambda) = -1$ . Then  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$  and  $\tau(\lambda)(1 - \tau_\alpha) \text{ind}_K^G V' \subset \text{ind}_K^G V$ .*

We note that  $\lambda a_\alpha \notin Z^+$  so in particular  $\tau(\lambda)(1 - \tau_\alpha) \notin \mathcal{Z}_G$ .

**IV.25.** We investigate the term  $\psi^{-1}(n_s^{-1} \cdot c_1) \varphi E_{O_J}(\mu_1 \lambda')$  in IV.23 Proposition.

**Proposition** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $-\alpha_a \circ \nu(\lambda) \geq 2$ .*

- (i) *The element  $n_s^{-1} \cdot \mu_1 \in {}_1\Lambda$  is in the image of  $Z \cap M'_\alpha$ .*
- (ii) *If  $\psi$  is not trivial on  $Z^0 \cap M'_\alpha$ , then  $\psi^{-1}(n_s^{-1} \cdot c_1) = 0$ .*
- (iii) *If  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ , then  $\psi^{-1}(n_s^{-1} \cdot c_1) = -1$  and  $\tau((n_s^{-1} \cdot \mu_1)\lambda) = \tau(\lambda)\tau_\alpha$ .*

Note that from (i) and III.16 Proposition (i),  $n_s^{-1} \cdot \mu_1$  normalizes  $\psi$  if  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ . In particular, in (iii) the element  $\tau((n_s^{-1} \cdot \mu_1)\lambda)$  is defined. Using IV.23 Proposition and (IV.22.1) we get

$$fE = \begin{cases} \tau(\lambda)(f - f') & \text{if } \psi \text{ is not trivial on } Z^0 \cap M'_\alpha, \\ \tau(\lambda)(1 - \tau_\alpha)(f - f') & \text{if } \psi \text{ is trivial on } Z^0 \cap M'_\alpha. \end{cases}$$

This formula immediately yields IV.1 Theorem (ii), (iii) when  $-\alpha_a \circ \nu(\lambda) \geq 2$  (note that this implies  $\lambda a_\alpha \in {}_1\Lambda^+$ ):

**Corollary** *Let  $\lambda \in {}_1\Lambda^+$  normalize  $\psi$ , and assume  $-\alpha_a \circ \nu(\lambda) \geq 2$ .*

- (i) *If  $\psi$  is not trivial on  $Z^0 \cap M'_\alpha$  then  $\tau(\lambda) \text{ind}_K^G V' \subset \text{ind}_K^G V$ .*
- (ii) *If  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$  then*

$$\tau(\lambda)(1 - \tau_\alpha) \text{ind}_K^G V' \subset \text{ind}_K^G V.$$

To prove the proposition we need to know precisely what  $c_1$  and  $\mu_1$  are. We have to distinguish cases:  $\alpha_a \circ \nu(\Lambda) = \delta\mathbb{Z}$  for  $\delta = 1$  or  $2$  (cf. loc. cit. Remark 5.3). The generic case is  $\delta = 1$ , which we tackle first. In that case choose  $\lambda_s \in \Lambda$  with  $\alpha_a \circ \nu(\lambda_s) = 1$ ; then  $\mu_1 = (n_s \cdot \lambda_s)\lambda_s^{-1}$  and  $c_1 = (n_s \cdot \lambda_s) \cdot c_{n_s}$ . Recall that  $c_{n_s} = \frac{-1}{|Z_{k,s}|} \sum_{z \in Z_{k,s}} z$  in  $C[Z_k]$ . In particular,  $\psi^{-1}(n_s^{-1} \cdot c_1) = \frac{-1}{|Z_{k,s}|} \sum_{z \in Z_{k,s}} \psi^{-1}(\lambda_s \cdot z)$ . So we see that  $\psi^{-1}(n_s^{-1} \cdot c_1)$  is non-zero if and only if  $\psi$  is trivial on  $\lambda_s Z_{k,s} \lambda_s^{-1}$ , in which case it is equal to  $-1$ . Reasoning as in IV.24 with  $\lambda_s$  instead of  $\lambda$  we see that  $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$  if and only if  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$  and the other assertions of the proposition are obtained as in IV.24 as well (when  $\delta = 1$ ), noting that  $\tau_\alpha$  is in the centre of  $\mathcal{H}_Z(\psi)$ .

**IV.26.** We continue the proof of IV.25 Proposition. Now assume that  $\delta = 2$ . One situation where this may happen is when  $\mathbf{G}$  has relative semisimple rank 1, or more generally when the connected component of the relative Dynkin diagram of  $\mathbf{G}$  containing  $\alpha$  has rank 1. In that case, let  $\tilde{s}$  be the reflection in the affine Weyl group of  $\mathbf{M}_\alpha$  corresponding to the affine root  $\alpha_a + 1$ ; it corresponds to a vertex  $\mathbf{x}_1$  in the semisimple Bruhat-Tits building of  $\mathbf{M}_\alpha$  (a tree) adjacent to the vertex  $\mathbf{x}_0$ . As in IV.24 we let  $K_1$  be the parahoric subgroup of  $M_\alpha$  corresponding to the vertex  $\mathbf{x}_1$  (which is special), and  $K_1(1)$  its pro- $p$  radical. Then  $Z \cap K_1 = Z^0$ ,  $Z \cap K_1(1) = Z(1)$ . The image of  $\mathcal{N} \cap K_1$  in  $K_1/K_1(1) = M_{\alpha,1,k}$  is the group  $\mathcal{N}_{1,k}$  of  $k$ -points of the normalizer of  $\mathbf{Z}_k$  in  $\mathbf{M}_{\alpha,1,k}$  and we can choose in  $\mathcal{N}_{1,k}$  a lift  $n_{\tilde{s}}$  of  $\tilde{s}$  which actually belongs to  $M'_{\alpha,1,k}$  – note that  $\tilde{s}$  generates  $(\mathcal{N} \cap K_1)/Z^0$  which we identify, via reduction with  $\mathcal{N}_{1,k}/Z_k$ . Then, inside  ${}_1W = \mathcal{N}/Z(1)$ , we can take (loc. cit. Notation 5.37)  $\lambda_s = n_s n_{\tilde{s}}$ ,  $\mu_1 = \lambda_s^{-1}$ ,  $c_1 = c_{\tilde{s}} n_s^2$ , where  $c_{\tilde{s}} = \frac{-1}{|Z_{k,\tilde{s}}|} \sum_{z \in Z_{k,\tilde{s}}} z$ , with  $Z_{k,\tilde{s}} = Z_k \cap M'_{\alpha,1,k}$ .<sup>15</sup>

We see that  $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$  if and only if  $\psi$  is trivial on  $Z_{k,\tilde{s}}$ . As  $\psi$  is already trivial on  $Z_{k,s}$ , we get by IV.24 Claim that  $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$  if and only if  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ , in which case  $\psi^{-1}(n_s^{-1} \cdot c_1) = -1$ . On the other hand,  $n_s^{-1} \cdot \mu_1$  is in the image of  $Z \cap M'_\alpha$  (by lifting  $n_s$  and  $n_{\tilde{s}}$  to  $\mathcal{N} \cap M'_\alpha$  as in IV.24). Moreover, by construction  $\nu(\mu_1) = -\alpha_a^\vee$  and as in IV.24 we deduce that we can take the image of  $a_\alpha$  in  ${}_1\Lambda$  to be  $n_s^{-1} \cdot \mu_1$  and that  $\tau((n_s^{-1} \cdot \mu_1)\lambda) = \tau(\lambda)\tau_\alpha$  if  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ .

**IV.27.** The only other case when  $\delta = 2$  may happen is when the connected component of the Dynkin diagram of  $\Phi_a$  containing  $\alpha$  has type  $C_n$ ,  $n \geq 2$ , and  $\alpha$  is a long root (loc. cit. Proposition 5.14). Let then  $\tilde{\alpha}_a$  be the highest root in  $\Phi_a^+$  lying in the same component as  $\alpha$ , and  $\tilde{s}$  be the reflection associated with  $\tilde{\alpha}_a + 1$ . Then (loc. cit. Lemma 5.15 and Notation 5.37)  $\mu_{-\alpha_a} = s w \tilde{s} w^{-1}$  for some  $w \in W^a$  such that  $\ell(\mu_{-\alpha_a}) = 2\ell(w) + 2$  and  $w \tilde{s} w^{-1}$  is the reflection  $s'$  associated with the affine root  $\alpha_a + 1$  (whereas  $s$  is associated with  $\alpha_a$ ). Moreover  $\mu_{-\alpha_a} = s s'$  satisfies  $\nu(\mu_{-\alpha_a}) = \alpha_a^\vee$ . In that case (loc. cit.)  $c_1 = (w \cdot c_{\tilde{s}}) n_s^2$  and  $\lambda_s = n_s (w \cdot n_{\tilde{s}})$ ,  $\mu_1 = \lambda_s^{-1}$  with  $n_{\tilde{s}}$ ,  $c_{\tilde{s}}$  defined similarly as before (loc. cit., §4); but conjugating by  $w$  yields  $w \cdot c_{\tilde{s}} = c_{s'}$  and  $w \cdot n_{\tilde{s}} = n_{s'}$  where now  $c_{s'}$ ,  $n_{s'}$  have a similar meaning, but in the relative semisimple rank 1 group  $M_\alpha$ . The same reasoning as in IV.26 then gives the desired result.

**IV.28.** To finish the proof of IV.25 Proposition we need only prove IV.24 Claim. It is convenient to deal first with the case where  $\mathbf{G} = \mathbf{G}^{\text{is}}$ . Then  $W = W^a$  is generated by the involutions  $s_0$  (generating  $\mathcal{N}^0/Z^0$ ) and  $s_1$  (generating  $(\mathcal{N} \cap K_1)/Z^0$ ). As  $s_0 s_1$  acts as a non-trivial translation on the apartment,  $s_0 s_1$  has infinite order.

Identify  $\mathcal{N}^0/Z(1)$  with  $\mathcal{N}_k$  and similarly  $(\mathcal{N} \cap K_1)/Z(1)$  with the group  $\mathcal{N}_{1,k}$  of  $k$ -points of the normalizer of  $\mathbf{Z}_k$  in  $\mathbf{G}_{1,k}$ . Choose a lifting  $n_0$  of  $s_0$  in  $\mathcal{N}_k \cap G'_k \subset {}_1W$  and a lifting  $n_1$  of  $s_1$  in  $\mathcal{N}_{1,k} \cap G'_{1,k} \subset {}_1W$ . An element  $w$  of  $W$  has a unique reduced expression  $w = \sigma_1 \cdots \sigma_h$  with  $\sigma_i = s_0$  or  $s_1$  and we put  $n_w = x_1 \cdots x_h$  with  $x_i = n_0$  if  $\sigma_i = s_0$ ,  $x_i = n_1$  if  $\sigma_i = s_1$ . We let  $X$  be the subgroup of  $Z_k$  generated by  $Z_k \cap G'_k$  and  $Z_k \cap G'_{1,k}$ , and put  $Y = \{n_w x \mid w \in W, x \in X\}$ .

**Lemma 1** *X and Y are normal subgroups of  ${}_1W$ .*

<sup>15</sup>In principle those elements are defined in loc. cit. with respect to  $\mathbf{G}$ , not  $\mathbf{M}_\alpha$ , but the above choices in  $M_\alpha$  also work in  $G$ . The same remark applies in IV.27.

**Proof** Let  $x \in Z_k$ ; then  $n_0^{-1}xn_0x^{-1}$  belongs to  $Z_k$ ; but  $Z_k$  normalizes  $G'_k$  so  $n_0^{-1}xn_0x^{-1}$  belongs to  $Z_k \cap G'_k$ . Similarly  $n_1^{-1}xn_1x^{-1}$  belongs to  $Z_k \cap G'_{1,k}$ . In particular,  $n_0$  and  $n_1$  normalize  $X$ . Since  $Z_k$  also normalizes  $X$ , so  ${}_1W$  itself normalizes  $X$ . As  $n_0^2$  and  $n_1^2$  belong to  $X$ , we deduce that for  $w, w' \in W$   $n_w n_{w'} \in n_{ww'}X$  and  $n_w^{-1} \in n_{w^{-1}}X$ , so  $Y$  is indeed a normal subgroup of  ${}_1W$ .  $\square$

Now let  $H = IYI$  with the usual abuse of notation.

**Lemma 2**  $H$  is a normal subgroup of  $G$  and  $(H \cap Z^0)/Z(1) = X$

**Proof** We first prove that  $H$  is a subgroup of  $G$ . By Lemma 1,  $H$  is closed under inverses. Working in  $\mathcal{H}_{\mathbb{Z}}$ , it is enough to show that for  $y, y'$  in  $Y$ , the product  $T(y)T(y')$  in  $\mathcal{H}_{\mathbb{Z}}$  is a linear combination of  $T(y'')$  for  $y''$  in  $Y$ . But that is given by the relations in  $\mathcal{H}_{\mathbb{Z}}$ : the braid relations and the two quadratic relations  $T(n_i)^2 = q_i T(n_i^2) + c_i T(n_i)$  where  $q_i \in \mathbb{Z}$  and  $c_i \in \mathbb{Z}[Z_k \cap G'_{i,k}]$  for  $i = 0, 1$ .

As  $Z^0$  normalizes  $I$ , and  $Z_k$  normalizes  $Y$ ,  $Z^0$  normalizes  $H$ . The normalizer of  $H$  contains  $n_0, n_1$  (which belong to  $H$ ),  $Z^0$  and  $I$ , so it is  $G$  itself. If an element  $x$  of  $H$  in a class  $IyI$ ,  $y \in Y$ , is in  $Z^0$  then  $y$  has to belong to  $Z_k$  so by the very definition of  $Y$ ,  $y$  belongs to  $X$  and  $x$  itself has image  $y$  in  $Z^0/Z(1) = Z_k$ . That gives the last assertion of the lemma.  $\square$

Clearly  $H$  is not central in  $G$ , so  $H = G$  because the only non-central normal subgroup of  $G$  is  $G$  itself (II.3 Proposition). But then  $H \cap Z^0 = G \cap Z^0 = Z^0$  so  $X = Z_k$ , which gives the claim for  $\mathbf{G} = \mathbf{G}^{\text{is}}$ .

Let us now prove IV.24 Claim in the general case. We show first that the claim is equivalent to

$$(*) \quad Z(1)(Z^0 \cap G') = Z(1)\langle Z^0 \cap \langle U^0, U_{\text{op}}^0 \rangle, Z^0 \cap \langle U \cap K_1, U_{\text{op}} \cap K_1 \rangle \rangle.$$

It suffices to show that the image of  $Z^0 \cap \langle U^0, U_{\text{op}}^0 \rangle$  in  $Z_k$  equals  $Z_k \cap G'_k$  (and similarly for the other term). It is clear that an arbitrary element of  $Z_k \cap G'_k$  lifts to an element of  $\langle U^0, U_{\text{op}}^0 \rangle \cap Z^0 K(1)$ . Using the Iwahori decomposition of  $K(1)$  (III.7) we can modify the lift so that it is contained in  $Z^0 \cap \langle U^0, U_{\text{op}}^0 \rangle$ .

The only non-trivial part of the equality (\*) is the inclusion  $\subset$ . The inclusion is true for  $G^{\text{is}}$ , and we deduce it for  $G$  by applying the natural homomorphism  $\iota : G^{\text{is}} \rightarrow G$ , using that  $(Z^{\text{is}})^0 = \iota^{-1}(Z^0)$  (III.19 Proposition) and that  $Z(1)$  is the pro- $p$  Sylow of  $Z^0$ . This completes the proof of IV.24 Claim and hence of IV.1 Theorem.  $\square$

## V. UNIVERSAL MODULES

**V.1.** In this chapter our goal is, for an irreducible representation  $V$  of  $K$ , to study the “universal” representation  $\text{ind}_K^G V$  as a module over the centre  $\mathcal{Z}_G(V)$  of the Hecke algebra  $\mathcal{H}_G(V)$ . In fact that structure is difficult to elucidate, so we consider various algebra homomorphisms  $\chi : \mathcal{Z}_G(V) \rightarrow A$  and the corresponding  $A$ -module  $A \otimes_{\chi} \text{ind}_K^G V$ . As an application, for a character  $\chi : \mathcal{Z}_G(V) \rightarrow C$ , we prove Theorem 6 of the introduction – used in Chapter III at the end of our classification – which gives a nice filtration of  $C \otimes_{\chi} \text{ind}_K^G V$  as a representation of  $G$ . In this chapter we fix an irreducible representation  $V$  of  $K$  and let  $(\psi, \Delta(V))$  be its parameter.

### A) Freeness of the supersingular quotient of $\text{ind}_K^G V$

**V.2.** Until V.11 we fix a parabolic subgroup  $P = MN$  of  $G$  containing  $B$ . Recall from III.12 the subgroup  $Z_{\Delta_M}^\perp$  of  $Z$  consisting of those  $z \in Z$  with  $|\beta|(z) = 1$  for all  $\beta \in \Delta_M$ . We write  $Z^{+M}$  for the set of  $z \in Z$  with  $|\beta|(z) \leq 1$  for  $\beta \in \Delta_M$ . Recall from III.4 that  $\mathcal{Z}_Z(V_{U^0})$  is spanned by the  $\tau_z$  for  $z \in Z_\psi$ , and that the natural image of  $\mathcal{Z}_M(V_{N^0})$  in  $\mathcal{Z}_Z(V_{U^0})$  (via  $\mathcal{S}_Z^G$ ) is spanned by the  $\tau_z$  for  $z \in Z^{+M} \cap Z_\psi$  – we identify  $\mathcal{Z}_M(V_{N^0})$  with that image.

**Notation** We let  $R_M$  be the quotient of  $\mathcal{Z}_M(V_{N^0})$  by the ideal of elements supported on  $(Z^{+M} \cap Z_\psi) - Z_{\Delta_M}^\perp$ .

As  $\mathcal{Z}_M(V_{N^0})$  is viewed as a subset of  $\mathcal{Z}_Z(V_{U^0})$ , we emphasize that the supports above are subsets of  $Z$ . Note that the elements of  $\mathcal{Z}_M(V_{N^0})$  supported on  $Z_{\Delta_M}^\perp$  form a subalgebra which maps isomorphically onto  $R_M$ .

Our first main result in this chapter is:

**Theorem** *Let  $P = MN$  be a parabolic subgroup of  $G$  containing  $B$ . Then  $R_M \otimes \text{ind}_K^G V$  is free over  $R_M$ , where the tensor product is via the composite map  $\mathcal{Z}_G(V) \rightarrow \mathcal{Z}_M(V_{N^0}) \rightarrow R_M$ .*

The proof of that theorem is rather long (V.3 to V.11). We first treat the case where  $P = G$  (V.3 Proposition). The proof then proceeds by comparing with situations with a more regular weight (i.e. smaller  $\Delta(V)$ ). Using the change of weight results of Chapter IV, we reduce the proof in general to a special case where, in particular,  $\Delta_M$  is orthogonal to  $\Delta - \Delta_M$  (V.7). Finally we use a filtration argument (V.8 to V.11).

**V.3. Proposition**  $R_G \otimes \text{ind}_K^G V$  is free over  $R_G$ .

The proof in V.4 requires several lemmas. We use again the Kottwitz invariant map  $w_G$  and the map  $v_G$  (III.16).

**Lemma 1** *Let  $z, z_1, z_2$  in  $Z$ . If  $zz_1z_2 \in Kz_1Kz_2K$ , then  $w_G(z) = 0$ .*

**Proof** The Kottwitz invariant  $w_G$  is a homomorphism of  $G$  into a commutative group; the result follows from  $w_G(K) = 0$ .  $\square$

**Lemma 2** *Let  $z_1 \in Z^+$  normalizing  $\psi$ , and  $f \in \mathcal{H}_G(V)$  with support in  $Kz_1K$ . Then  $\mathcal{S}_Z^G(f) \in \mathcal{H}_Z(V_{U^0})$  has support in  $(Z \cap \text{Ker } w_G)z_1$ .*

**Proof** That is immediate from (III.3.2), once we note that  $w_G$  is trivial on  $U$ .  $\square$

**Lemma 3** *Let  $z_1 \in Z^+$  normalizing  $\psi$ , and  $z_2 \in Z$ . If  $f \in \text{ind}_K^G V$  has support in  $Kz_2K$ , then  $\tau_{z_1} * f$  has support in  $K(Z \cap \text{Ker } w_G)z_1z_2K$ .*

**Proof** By definition  $\tau_{z_1}$ , as an element of  $\mathcal{H}_Z(V_{U^0})$ , has support  $Z^0z_1$ . From Lemma 2,  $\tau_{z_1}$ , as an element of  $\mathcal{H}_G(V)$ , has support in  $K(Z \cap \text{Ker } w_G)z_1K$ . The result then follows from the convolution formula in  $\mathcal{H}_G(V)$  and Lemma 1.  $\square$

**Lemma 4**  $Z_{\Delta}^\perp \cap \text{Ker } w_G = Z^0$ .

**Proof** Let  $z \in \text{Ker } w_G$ . Then  $v_G(z) = 0$ . If moreover  $z \in Z_{\Delta}^\perp$ , then  $v_Z(z) = 0$  for the analogous map  $v_Z$ , cf. [HV1, 6.3 Remark 1]; from loc. cit. 6.2 Lemma, (ii) and (iii), it follows that  $z \in Z^0$ . Conversely  $Z^0 \subset Z_{\Delta}^\perp \cap \text{Ker } w_G$  is clear.  $\square$

**V.4.** We prove V.3 Proposition. We decompose  $\text{ind}_K^G V$  as  $\bigoplus I(x)$ ,  $x \in Z/(Z \cap \text{Ker } w_G)$ , where  $I(x)$  consists of the functions in  $\text{ind}_K^G V$  with support in  $Kx(Z \cap \text{Ker } w_G)K$ . For  $z$  in  $Z^+$  normalizing  $\psi$ , we have  $\tau_z * I(x) \subset I(zx)$  by V.3 Lemma 3, with equality if  $z \in Z_\Delta^\perp$  since then  $\tau_z$  has inverse  $\tau_{z^{-1}}$ . For  $x \in Z/(Z \cap \text{Ker } w_G)$ , let  $I^+(x)$  be the sum of the subspaces  $\tau_z * I(y)$  of  $I(x)$ , where  $z \in Z^+ \cap Z_\psi$ ,  $z \notin Z_\Delta^\perp$ ,  $y \in Z/(Z \cap \text{Ker } w_G)$  and  $zy = x$  in  $Z/(Z \cap \text{Ker } w_G)$ . By definition  $R_G \otimes \text{ind}_K^G V$  is the quotient of  $\text{ind}_K^G V$  obtained by killing all the subspaces  $I^+(x)$ ; thus it appears as  $\bigoplus_{x \in Z/(Z \cap \text{Ker } w_G)} (I(x)/I^+(x))$ . Let  $z \in Z_\Delta^\perp \cap Z_\psi$ ; then  $\tau_z * I(x) = I(zx)$ ,  $\tau_z * I^+(x) = I^+(zx)$  for  $x \in Z/(Z \cap \text{Ker } w_G)$ , hence the corresponding element in  $R_G$ , still written  $\tau_z$ , sends  $I(x)/I^+(x)$  isomorphically onto  $I(zx)/I^+(zx)$ . As  $Z_\Delta^\perp \cap \text{Ker } w_G = Z^0$  by V.3 Lemma 4, the image of  $Z_\Delta^\perp \cap Z_\psi$  in  $Z/(Z \cap \text{Ker } w_G)$  acts by multiplication without fixed points on  $Z/(Z \cap \text{Ker } w_G)$ ; choosing a set of representatives  $\Omega$  for the orbits, we deduce that  $R_G \otimes \text{ind}_K^G V$  is isomorphic to the free  $R_G$ -module  $R_G \otimes_C (\bigoplus_{x \in \Omega} I(x)/I^+(x))$ .  $\square$

For further use, we state a result proved in a similar manner.

**Lemma** *Let  $z \in Z^+ \cap Z_\psi$ .*

(i) *If  $v_G(z) \neq 0$ ,  $\tau_z - 1$  acts injectively on  $\text{ind}_K^G V$ ; if moreover  $z \in Z_\Delta^\perp$  then  $\tau_z - 1$  is not a divisor of 0 in  $R_G$ .*

(ii) *Let  $T \in \mathcal{Z}_G(V)$ ; if  $v_G(z)$  is linearly independent from  $v_G(\text{Supp}(T))$ , then  $(\tau_z - 1)\text{ind}_K^G V \cap T\text{ind}_K^G V = (\tau_z - 1)T\text{ind}_K^G V$ .*

**Remark** The condition  $v_G(z) = 0$  is equivalent to  $v_Z(z) \in \mathbb{R}\Delta^\vee \subset X_*(\mathbf{S}) \otimes \mathbb{R}$ .

**Proof** (i) Let  $f \in \text{ind}_K^G V$ , and write as above  $f = \sum f_x$ ,  $x \in Z/(Z \cap \text{Ker } w_G)$ ,  $f_x \in I(x)$ . Then for  $z \in Z^+ \cap Z_\psi$ ,  $\tau_z * f = \sum_x \tau_z * f_x$  with  $\tau_z * f_x \in I(zx)$ . The equality  $\tau_z * f = f$  amounts to  $\tau_z * f_x = f_{zx}$  for all  $x \in Z/(Z \cap \text{Ker } w_G)$ . If  $v_G(z) \neq 0$  then the image of  $z$  in  $Z/(Z \cap \text{Ker } w_G)$  has infinite order; since  $f_x = 0$  for all but a finite number of  $x$ 's,  $\tau_z * f = f$  implies  $f = 0$ , and  $\tau_z - 1$  acts injectively on  $\text{ind}_K^G V$ ; in particular, as  $\mathcal{Z}_G(V)$  acts faithfully on  $\text{ind}_K^G V$ ,  $\tau_z - 1$  is not a divisor of 0 in  $\mathcal{Z}_G(V)$ . If moreover  $z \in Z_\Delta^\perp$  then  $\tau_z - 1$  is not a divisor of 0 in the subalgebra of  $\mathcal{Z}_G(V)$  which maps isomorphically onto  $R_G$ .

(ii) Let  $\Gamma$  be the subgroup of  $Z$  generated by the elements  $\xi$  with  $v_G(\xi) \in v_G(\text{Supp } T)$ . For  $y \in Z/\Gamma$ , let  $J(y)$  be the space of functions in  $\text{ind}_K^G V$  with support in  $Ky\Gamma K$ ; then  $TJ(y) \subset J(y)$  and for  $z \in Z^+ \cap Z_\psi$ ,  $\tau_z * J(y) \subset J(zy)$ . Let  $f, f'$  in  $\text{ind}_K^G V$  with  $(\tau_z - 1)f = f'$ . We have  $\text{ind}_K^G V = \bigoplus_{y \in Z/\Gamma} J(y)$  and decomposing accordingly  $f = \sum f_y$  and  $f' = \sum f'_y$  we get  $\tau_z * f_y = f_{zy} + f'_{zy}$  for  $y \in Z/\Gamma$ . Let  $f' \in T\text{ind}_K^G V$ ; then  $f'_y \in T\text{ind}_K^G V$  for all  $y \in Z/\Gamma$  so if  $f_y$  belongs to  $T\text{ind}_K^G V$ , then so do  $\tau_z * f_y$  and  $f_{zy}$ . The hypothesis on  $z$  in (ii) implies that its image in  $Z/\Gamma$  has infinite order, so  $f_{z^{-r}y}$  is 0 for large  $r$ . So we get, using descending induction on  $r$ , that  $f_y$  does indeed belong to  $T\text{ind}_K^G V$ .  $\square$

**V.5.** We now turn to the general case of V.2 Theorem. For each parabolic subgroup  $P_1 = M_1 N_1$  of  $G$  containing  $P$ , we let  $V_{P_1}$  be the irreducible representation of  $K$  with parameter  $(\psi, \Delta_{P_1} \cap \Delta(V))$  – for  $P_1 = G$  we have  $V_G = V$ ; we choose a basis vector for  $(V_{P_1})_{U^0}$ .

For such a  $P_1$  consider the sequence of canonical (injective) intertwiners:

$$(V.5.1) \quad \text{ind}_K^G V_{P_1} \rightarrow \text{Ind}_{P_1}^G \text{ind}_{M_1^0}^{M_1} (V_{P_1})_{N_1^0} \rightarrow \text{Ind}_P^G \text{ind}_{M^0}^M (V_{P_1})_{N^0} \rightarrow \text{Ind}_B^G \text{ind}_{Z^0}^Z (V_{P_1})_{U^0}.$$

As  $(V_{P_1})_{N_1^0}$  has the same parameter as  $V_{N_1^0}$ , there is a unique isomorphism between them that is compatible with the choice of basis vectors in  $(V_{P_1})_{U^0}$  and  $V_{U^0}$ ; it induces an isomorphism of  $(V_{P_1})_{N^0}$  onto  $V_{N^0}$ . Using those isomorphisms we identify the sequence (V.5.1) with

$$(V.5.2) \quad \text{ind}_K^G V_{P_1} \rightarrow \text{Ind}_{P_1}^G \text{ind}_{M_1^0}^{M_1} V_{N_1^0} \rightarrow \text{Ind}_P^G \text{ind}_{M^0}^M V_{N^0} \rightarrow \text{Ind}_B^G \text{ind}_{Z^0}^Z \psi.$$

The sequence (V.5.1) is equivariant for the sequence of Hecke algebras

$$(V.5.3) \quad \mathcal{H}_G(V_{P_1}) \rightarrow \mathcal{H}_{M_1}((V_{P_1})_{N_1^0}) \rightarrow \mathcal{H}_M((V_{P_1})_{N^0}) \rightarrow \mathcal{H}_Z((V_{P_1})_{U^0})$$

given by the (injective) Satake homomorphisms. The choice of basis vectors gives an isomorphism  $\mathcal{H}_Z((V_{P_1})_{U^0}) \simeq \mathcal{H}_Z(\psi)$ , actually independent of that choice, and inside  $\mathcal{H}_Z(\psi)$  the Hecke algebras in (V.5.3) do not depend on  $P_1$ ; accordingly we write  $\mathcal{H}_G, \mathcal{H}_{M_1}, \mathcal{H}_M, \mathcal{H}_Z$ , and similarly for the centres. The sequence (V.5.2) is then equivariant for the sequence of algebras  $\mathcal{H}_G \rightarrow \mathcal{H}_{M_1} \rightarrow \mathcal{H}_M \rightarrow \mathcal{H}_Z$ .

As in Chapter IV we identify the spaces in (V.5.2) with their images in  $\text{Ind}_B^G \text{ind}_{Z^0}^Z \psi$ , and similarly  $\mathcal{H}_G, \mathcal{H}_{M_1}, \mathcal{H}_M$  with their images in  $\mathcal{H}_Z$ .

**Notation** For  $P_1$  as above containing  $P$ , we let  $\pi_{P_1}$  be the  $\mathcal{Z}_M[G]$ -submodule  $\mathcal{Z}_M \otimes_{\mathcal{Z}_G} \text{ind}_K^G V_{P_1}$  of  $\text{Ind}_P^G \text{ind}_{M^0}^M V_{N^0}$  (which is then  $\pi_P$ ).

**Remark** By III.14 Theorem,  $\pi_{P_1}$  is also  $\mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \text{Ind}_{P_1}^G \text{ind}_{M_1^0}^{M_1} V_{N_1^0}$ , which we also see as  $\text{Ind}_{P_1}^G (\mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \text{ind}_{M_1^0}^{M_1} V_{N_1^0})$ , cf. [HV2] Corollary 1.3.

For further use, let us recall the considerations around loc. cit. Let  $X$  be a locally profinite space with a countable basis. Then the functor  $X \mapsto C_c^\infty(X, A)$  is exact on  $\mathbb{Z}$ -modules  $A$ ,  $C_c^\infty(X, \mathbb{Z})$  is free and  $C_c^\infty(X, \mathbb{Z}) \otimes A \rightarrow C_c^\infty(X, A)$  is an isomorphism; if  $A$  is a free module over some ring  $R$ , then so is  $C_c^\infty(X, A)$  and if  $R \rightarrow R'$  is a ring homomorphism, then  $R' \otimes_R C_c^\infty(X, A) \rightarrow C_c^\infty(X, R' \otimes_R A)$  is an isomorphism of  $R'$ -modules. If  $Y$  is an open subset of  $X$ , we have an exact sequence  $0 \rightarrow C_c^\infty(Y, \mathbb{Z}) \rightarrow C_c^\infty(X, \mathbb{Z}) \rightarrow C_c^\infty(X - Y, \mathbb{Z}) \rightarrow 0$  of free  $\mathbb{Z}$ -modules. We are particularly interested in the case  $X = J \backslash H$  where  $H$  is a locally profinite second countable group, and  $J$  a closed subgroup of  $H$ . If  $A$  is a smooth  $R[J]$ -module for some ring  $R$ , choosing a continuous section of  $H \rightarrow J \backslash H$  gives an isomorphism of  $R$ -modules  $C_c^\infty(J \backslash H, A) \simeq \text{ind}_J^H A$ , so we deduce that  $\text{ind}_J^H$  is an exact functor on smooth  $R[J]$ -modules, that  $\text{ind}_J^H A$  is free over  $R$  if  $A$  is, and that  $R' \otimes_R \text{ind}_J^H A \rightarrow \text{ind}_J^H (R' \otimes_R A)$  is an isomorphism of  $R'[H]$ -modules for any ring homomorphism  $R \rightarrow R'$ .

**V.6.** We gather consequences of the change of weight results of Chapter IV.

**Proposition** *Let  $P_1, P_2$  be parabolic subgroups of  $G$  containing  $P$ , with  $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$ .*

- (i)  $\pi_{P_2} \subset \pi_{P_1}$  with equality if  $\alpha \notin \Delta(V)$  or if  $\psi$  is not trivial on  $Z^0 \cap M'_\alpha$ .
- (ii) If  $\alpha \in \Delta(V)$  and  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ , then  $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2}$  (with  $\tau_\alpha$  as in III.16 Notation). If moreover  $\alpha$  is not orthogonal to  $\Delta_M$ , the inclusion  $\pi_{P_2} \subset \pi_{P_1}$  induces an isomorphism  $R_M \otimes_{\mathcal{Z}_M} \pi_{P_2} \xrightarrow{\sim} R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$ .

**Proof** First remark that if  $\alpha \notin \Delta(V)$  then  $V_{P_1}$  and  $V_{P_2}$  are isomorphic, so  $\pi_{P_2} = \pi_{P_1}$  is immediate. Assume  $\alpha \in \Delta(V)$ . We apply IV.1 Theorem to  $V_{P_2}$  (in lieu of  $V$ ) and  $V_{P_1}$  (in lieu of  $V'$ ). Choose  $z \in Z_\psi$  with  $|\alpha|(z) < 1$  and  $|\beta|(z) = 0$  for  $\beta \in \Delta, \beta \neq \alpha$ ; thus  $\tau_z$  is an invertible element of  $\mathcal{Z}_M$ . By loc. cit. (i), we have the inclusion  $\tau_z \text{ind}_K^G V_{P_2} \subset \text{ind}_K^G V_{P_1}$



of subspaces of  $\text{Ind}_B^G(\text{ind}_{Z^0}^Z \psi)$ . As  $\tau_z$  is invertible in  $\mathcal{Z}_M$ , we get  $\pi_{P_2} \subset \pi_{P_1}$ . If  $\psi$  is not trivial on  $Z^0 \cap M'_\alpha$  then loc. cit. (ii) gives  $\tau_z \text{ind}_K^G V_{P_1} \subset \text{ind}_K^G V_{P_2}$  hence  $\pi_{P_2} = \pi_{P_1}$ . If  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$ , loc. cit. (ii) gives  $\tau_z(1 - \tau_\alpha) \text{ind}_K^G V_{P_1} \subset \text{ind}_K^G V_{P_2}$  so  $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2}$ . Now  $\tau_\alpha = \tau_{a_\alpha}$  for  $a_\alpha \in Z_\psi$  with  $\nu(a_\alpha) = r\alpha^\vee$  with some positive rational number  $r$  (III.16 Proposition (i), IV.12 Example). If  $\alpha$  is not orthogonal to  $\Delta_M$ , we have  $|\beta|(a_\alpha) < 1$  for some  $\beta \in \Delta_M$ ; but  $\tau_\alpha$  is sent to 0 in  $R_M$ . This implies the last assertion.  $\square$

**V.7.** We deduce a reduction for the proof of V.2 Theorem. Let  $\Delta_1 = \Delta_M \cup \{\alpha \in \Delta(V), \alpha \perp \Delta_M, \psi(Z^0 \cap M'_\alpha) = 1\}$  and let  $P_1 = M_1 N_1$  be the corresponding parabolic subgroup of  $G$ . By V.6 Proposition, the inclusion  $\pi_G \subset \pi_{P_1}$  induces an isomorphism  $R_M \otimes \pi_G \simeq R_M \otimes \pi_{P_1}$ . But  $R_M \otimes \pi_{P_1}$  is the same as  $\text{Ind}_{P_1}^G(R_M \otimes \text{ind}_{M_1^0}^{M_1} V_{N_1^0})$  (V.5 Remark); if the  $R_M$ -module inside the induction is free, then so is  $R_M \otimes \pi_{P_1}$  (loc. cit.). As a consequence, it is enough to prove V.2 when  $\Delta_1 = \Delta$ .

**Assumption** (until V.11):  $\Delta = \Delta_M \cup \Delta(V)$ ,  $(\Delta - \Delta_M) \perp \Delta_M$  and  $\psi(Z^0 \cap M'_\alpha) = 1$  for  $\alpha \in \Delta - \Delta_M$ .

**Notation** We put  $\sigma = \text{ind}_{M^0}^M V_{N^0}$ , so  $\pi_P = \text{Ind}_P^G \sigma$ . We also put  $W(M) = \{w \in W_0, w^{-1}(\Delta_M) \subset \Phi^+\}$ .

By V.3 Proposition, we know that  $R_M \otimes \sigma$  is free over  $R_M$ , and so is  $R_M \otimes \pi_P$  (V.5 Remark). We want to deduce the same for  $R_M \otimes \pi_G$ . For that we filter  $\pi_P$  according to the double cosets  $PwB$  for  $w \in W(M)$  (recall that  $G$  is the disjoint union of the double cosets  $PwB$ ,  $w \in W(M)$ ).

We consider upper sets in  $W(M)$ , i.e. subsets  $A$  such that  $v \in A$ ,  $v' \in W(M)$  and  $v' \geq v$  (in the Bruhat order) imply  $v' \in A$ . For an upper set  $A$ ,  $PAB = \bigcup_{v \in A} PvB$  is open in  $G$  and we let  $\pi_{P,A}$  be the subspace of functions in  $\pi_P = \text{Ind}_P^G \sigma$  with support in  $PAB$ ; it is a  $\mathcal{Z}_M$ -submodule of  $\pi_P$ .

Let  $A$  be non-empty upper set in  $W(M)$  and choose a minimal element  $w$  in  $A$ . Put  $A' = A - \{w\}$ ; then  $A'$  is an upper set in  $W(M)$  and we have the submodule  $\pi_{P,A'}$  of  $\pi_{P,A}$ .

Let  $\bar{A}, \bar{A}'$  be the (open) images of  $PAB, PA'B$  in  $P \backslash G$ . We have an exact sequence of free  $\mathbb{Z}$ -modules

$$0 \longrightarrow C_c^\infty(\bar{A}', \mathbb{Z}) \longrightarrow C_c^\infty(\bar{A}, \mathbb{Z}) \longrightarrow C_c^\infty(\bar{A} - \bar{A}', \mathbb{Z}) \longrightarrow 0. \quad (\text{V.5 Remark})$$

Choosing a continuous section of  $G \rightarrow P \backslash G$ , and noting that  $\bar{A} - \bar{A}'$  is the image of  $PwB$  in  $P \backslash G$ , we get from loc. cit. that evaluating functions on  $PwB$  gives an isomorphism of  $\pi_{P,A}/\pi_{P,A'}$  with the  $\mathcal{Z}_M$ -module of locally constant functions  $f : PwB \rightarrow \sigma$  with  $f(pg) = pf(g)$  for  $p \in P, g \in PwB$ , and with compact support in  $P \backslash PwB$ ; equivalently evaluating on  $wU$  gives an isomorphism with the compactly induced representation  $\text{ind}_{w^{-1}Pw \cap U}^U w\sigma$ .

**Lemma** *The inclusion  $\pi_{P,A} \rightarrow \pi_P$  induces an isomorphism of  $R_M \otimes \pi_{P,A}$  onto the subspace of  $R_M \otimes \pi_P = \text{Ind}_P^G(R_M \otimes \sigma)$  consisting of functions with support in  $PAB$ . The sequence*

$$0 \longrightarrow R_M \otimes \pi_{P,A'} \longrightarrow R_M \otimes \pi_{P,A} \longrightarrow R_M \otimes (\pi_{P,A}/\pi_{P,A'}) \longrightarrow 0$$

*is exact, and all three terms are free over  $R_M$ .*

**Proof** Choosing a continuous section of  $G \rightarrow P \backslash G$ ,  $\pi_{P,A}$  appears as  $C_c^\infty(\bar{A}, \mathbb{Z}) \otimes \sigma$ ,  $R_M \otimes \pi_{P,A}$  as  $C_c^\infty(\bar{A}, \mathbb{Z}) \otimes (R_M \otimes \sigma)$ , so the result follows from V.5 Remark via the exact sequence  $0 \rightarrow C_c^\infty(\bar{A}', \mathbb{Z}) \rightarrow C_c^\infty(\bar{A}, \mathbb{Z}) \rightarrow C_c^\infty(\bar{A} - \bar{A}', \mathbb{Z}) \rightarrow 0$ .  $\square$

**V.8.** Let  $A, w, A'$  be as in V.7, and let  $Q$  be a parabolic subgroup of  $G$  containing  $P$ . Then  $\pi_Q \subset \pi_P$  and we let  $\pi_{Q,A} = \pi_{P,A} \cap \pi_Q$ , similarly for  $A'$ , so we get an inclusion of  $\mathcal{Z}_M$ -modules

$$\pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

**Notation** Set  $c_{Q,w} = \prod_{\alpha \in \Delta_Q, w^{-1}(\alpha) < 0} (\tau_\alpha - 1) \in \mathcal{Z}_M$ .

**Remarks** 1) For  $\alpha \in \Delta$ ,  $w^{-1}(\alpha) < 0$  is equivalent to  $s_\alpha w < w$  and it implies  $\alpha \notin \Delta_M$  since  $w \in W(M)$ . In particular for such an  $\alpha$  we have  $v_M(a_\alpha) \neq 0$  by V.4 Remark.

2) By V.4 Lemma (i) (applied to  $M$ )  $c_{Q,w}$  acts injectively on  $\sigma$  hence on  $\pi_{P,A}/\pi_{P,A'}$ ; moreover,  $c_{Q,w}$  does not divide 0 in  $R_M$ .

**Proposition**  $\pi_{Q,A}/\pi_{Q,A'} = c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$  inside  $\pi_{P,A}/\pi_{P,A'}$ .

Before we give the proof, we derive consequences, in particular V.2 Theorem.

**Corollary 1**  $R_M \otimes (\pi_{Q,A}/\pi_{Q,A'}) \rightarrow R_M \otimes (\pi_{P,A}/\pi_{P,A'})$  is injective, and  $R_M \otimes (\pi_{Q,A}/\pi_{Q,A'})$  is free over  $R_M$ .

**Proof** By the proposition, multiplication by  $c_{Q,w}$  induces maps

$$\pi_{P,A}/\pi_{P,A'} \twoheadrightarrow \pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

Tensoring with  $R_M$  over  $\mathcal{Z}_M$  gives

$$R_M \otimes (\pi_{P,A}/\pi_{P,A'}) \twoheadrightarrow R_M \otimes (\pi_{Q,A}/\pi_{Q,A'}) \longrightarrow R_M \otimes (\pi_{P,A}/\pi_{P,A'})$$

whose composite is multiplication by  $c_{Q,w}$ . By the above remark 2)  $c_{Q,w}$  does not divide 0 in  $R_M$ ; since  $R_M \otimes (\pi_{P,A}/\pi_{P,A'})$  is free over  $R_M$  by V.7 Lemma, multiplication by  $c_{Q,w}$  is injective on it so we get an isomorphism  $R_M \otimes (\pi_{P,A}/\pi_{P,A'}) \simeq R_M \otimes (\pi_{Q,A}/\pi_{Q,A'})$ , thus proving Corollary 1.  $\square$

**Corollary 2**  $R_M \otimes \pi_{Q,A} \rightarrow R_M \otimes \pi_{P,A}$  is injective (in particular, for  $A = W(M)$ ,  $R_M \otimes \pi_Q \rightarrow R_M \otimes \pi_P$  is injective).

**Proof** By induction on  $\#A$ ,  $R_M \otimes \pi_{Q,A'} \rightarrow R_M \otimes \pi_{P,A'}$  is injective. By V.7 Lemma,  $R_M \otimes \pi_{P,A'} \rightarrow R_M \otimes \pi_{P,A}$  is injective and by Corollary 1,  $R_M \otimes (\pi_{Q,A}/\pi_{Q,A'}) \rightarrow R_M \otimes (\pi_{P,A}/\pi_{P,A'})$  is injective too. The result follows from the snake lemma applied to the commutative diagram (with exact rows)

$$\begin{array}{ccccccc} R_M \otimes \pi_{Q,A'} & \rightarrow & R_M \otimes \pi_{Q,A} & \rightarrow & R_M \otimes (\pi_{Q,A}/\pi_{Q,A'}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R_M \otimes \pi_{P,A'} & \rightarrow & R_M \otimes \pi_{P,A} & \rightarrow & R_M \otimes (\pi_{P,A}/\pi_{P,A'}) \rightarrow 0 \quad \square \end{array}$$

**Corollary 3**  $R_M \otimes \pi_{Q,A'} \rightarrow R_M \otimes \pi_{Q,A}$  and  $R_M \otimes \pi_{Q,A} \rightarrow R_M \otimes \pi_Q$  are injective, and  $(R_M \otimes \pi_{Q,A})/(R_M \otimes \pi_{Q,A'}) \rightarrow R_M \otimes (\pi_{Q,A}/\pi_{Q,A'})$  is an isomorphism.

**Proof** In the left hand square of the previous diagram, the two vertical maps and the bottom horizontal one are injective, hence so is the top horizontal one, giving the first assertion, which immediately implies the last one. The second one follows from the first by descending induction on  $\#A$ .  $\square$

Now V.2 Theorem follows from the corollaries. Indeed, by Corollary 1 and Corollary 3,  $R_M \otimes \pi_Q$  is a successive extension of free modules. Therefore  $R_M \otimes \pi_Q$  is free.

**V.9.** The proof of V.8 Proposition will involve an induction argument on  $\dim G$ . For this, a further corollary is necessary.

**Corollary 4** *Let  $z \in Z^{+M} \cap Z_\psi$ , and assume  $v_M(z) \neq 0$ . Then  $\pi_Q \cap (\tau_z - 1)\pi_P = (\tau_z - 1)\pi_Q$ .*

The proof is given after a lemma. Let  $A, w, A'$  be as in V.7, and use the notation  $\pi_{P,A}, \pi_{Q,A}$  of V.7, V.8.

**Lemma**  $(\tau_z - 1)\pi_{P,A} = (\tau_z - 1)\pi_P \cap \pi_{P,A}$ .

**Proof** By descending induction on  $\#A$ , the case  $A = W(M)$  being trivial. By V.4 Lemma (i),  $\tau_z - 1$  acts injectively on  $\sigma$ , hence also on  $\pi_{P,A}/\pi_{P,A'}$  which is a direct sum of copies of  $\sigma$  (V.5 Remark). By the snake lemma  $\pi_{P,A'}/(\tau_z - 1)\pi_{P,A'}$  injects into  $\pi_{P,A}/(\tau_z - 1)\pi_{P,A}$  i.e.  $(\tau_z - 1)\pi_{P,A} \cap \pi_{P,A'} = (\tau_z - 1)\pi_{P,A'}$ . The assertion  $(\tau_z - 1)\pi_{P,A} = (\tau_z - 1)\pi_P \cap \pi_{P,A}$  then implies the similar assertion for  $A'$ .  $\square$

**Proof of Corollary 4** Applying V.4 Lemma (ii) to  $T = c_{Q,w} \in \mathcal{Z}_M$  whose support is in  $\text{Ker } v_M$  we get

$$(\tau_z - 1)\sigma \cap c_{Q,w}\sigma = (\tau_z - 1)c_{Q,w}\sigma,$$

hence

$$(\tau_z - 1)(\pi_{P,A}/\pi_{P,A'}) \cap c_{Q,w}(\pi_{P,A}/\pi_{P,A'}) = (\tau_z - 1)c_{Q,w}(\pi_{P,A}/\pi_{P,A'}).$$

But by V.8 Proposition  $c_{Q,w}(\pi_{P,A}/\pi_{P,A'}) = \pi_{Q,A}/\pi_{Q,A'}$ , so we obtain  $(\tau_z - 1)\pi_{P,A} \cap \pi_{Q,A} \subset (\tau_z - 1)\pi_{Q,A} + \pi_{P,A'}$ . By the lemma we get  $(\tau_z - 1)\pi_P \cap \pi_{Q,A} \subset [(\tau_z - 1)\pi_Q \cap \pi_{P,A}] + \pi_{P,A'}$ . As  $(\tau_z - 1)\pi_P \cap \pi_Q$  contains  $(\tau_z - 1)\pi_Q$ , both give the same contribution to  $\pi_{P,A}/\pi_{P,A'}$ . Their equality now follows by induction on  $\#A$ .  $\square$

**V.10.** We now proceed to the proof of V.8 Proposition, keeping its notation. We first deal with the (easier) statement that  $\pi_{Q,A}/\pi_{Q,A'}$  contains  $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ .

**Notation** Let  $\Delta_w = \{\alpha \in \Delta, w^{-1}(\alpha) > 0\}$  and let  $P_w = M_w N_w$  be the corresponding parabolic subgroup of  $G$ ; it contains  $P$ , and  $w$  is in  $W(M_w)$ .

**Lemma** *Let  $A, w, A'$  be as in V.7. Then  $\pi_{P_w,A} \rightarrow \pi_{P,A}/\pi_{P,A'}$  is surjective.*

Assume that lemma for a moment. Since  $\pi_Q \cap P_w$  contains  $\pi_{P_w,A}$ , the map  $\pi_Q \cap P_w \rightarrow \pi_{P,A}/\pi_{P,A'}$  is surjective as well. But by V.6 Proposition  $\pi_Q$  contains  $c_{Q,w} \pi_Q \cap P_w$ , so the image of  $\pi_Q$  in  $\pi_{P,A}/\pi_{P,A'}$  contains  $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ , i.e. the quotient  $\pi_{Q,A}/\pi_{Q,A'}$  contains  $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ .

**Proof** Let  $A_{\geq w} = \{v \in W(M), v \geq w\}$  and  $A_{>w} = \{v \in W(M), v > w\}$ . We use the abbreviations  $\pi_{P,\geq w} = \pi_{P,A_{\geq w}}$ ,  $\pi_{P,>w} = \pi_{P,A_{>w}}$ . Then  $\pi_{P,A} \supset \pi_{P,\geq w}$  and  $\pi_{P,A'} \supset \pi_{P,>w}$ ; moreover  $\pi_{P,A'} \cap \pi_{P,\geq w} = \pi_{P,>w}$ , so  $\pi_{P,\geq w}/\pi_{P,>w}$  injects into  $\pi_{P,A}/\pi_{P,A'}$ . But evaluation on  $P_w B$  identifies both quotients with the same space of functions, so the injection is an isomorphism. Hence it is enough to prove the lemma for  $A = A_{\geq w}$ .

**Sublemma** (i)  $w^{-1}P_w \cap U = w^{-1}Uw \cap U = w^{-1}P_w w \cap U$ .

(ii)  $PA_{\geq w}B = \sqcup_{v \in W(M_w), v \geq w} P_w v B$ .

**Proof** (i) The first equality comes from  $w \in W(M)$ , the second one from  $w \in W(M_w)$ .

(ii) By [Abe, Lemma 4.20],  $w \in W(M)$  implies  $W_M A_{\geq w} = \{v \in W_0, v \geq w\}$  and similarly  $w \in W(M_w)$  implies  $W_{M_w} \{v \in W(M_w), v \geq w\} = \{v \in W_0, v \geq w\}$ . The result follows on taking  $B$ -double cosets.  $\square$

To prove the lemma (for  $A = A_{\geq w}$ ) we need to consider closely the inclusion  $\pi_{P_w} \hookrightarrow \pi_P$ . Both are parabolically induced from  $P_w$ , and the inclusion comes from the injective map  $\Phi : \mathcal{Z}_M \otimes_{\mathcal{Z}_{M_w}} \text{ind}_{M_w^0}^{M_w} V_{N_w^0} \rightarrow \text{Ind}_{P \cap M_w}^{M_w} \sigma$  obtained from the canonical intertwiner (III.13.1), so  $\pi_{P_w}$  is simply the subspace  $\text{Ind}_{P_w}^G(\text{Im } \Phi)$  of  $\pi_P = \text{Ind}_{P_w}^G(\text{Ind}_{P \cap M_w}^{M_w} \sigma)$ . Seeing  $\pi_P$  as induced from  $P_w$ , we let  $\pi'_{P, \geq w}$  be the subspace of functions with support in  $\bigcup_{v \in W(M_w), v \geq w} P_w v B$ , and similarly  $\pi'_{P, > w}$ . An element  $f$  of  $\pi_P = \text{Ind}_P^G \sigma$  is seen as the function  $f'$  in  $\text{Ind}_{P_w}^G(\text{Ind}_{P \cap M_w}^{M_w} \sigma)$  given by  $f'(g) : m \mapsto f(mg)$  for  $g \in G$ ,  $m \in M_w$ . Hence by (ii) of the sublemma  $\pi_{P, \geq w} = \pi'_{P, \geq w}$  and  $\pi_{P, > w} \supset \pi'_{P, > w}$ . By (i) of the sublemma (and V.5 Remark), choosing a continuous section of  $U \rightarrow w^{-1}Uw \cap U \setminus U$  gives a  $\mathcal{Z}_M$ -linear isomorphism  $\iota$  of  $\pi'_{P, \geq w} / \pi'_{P, > w}$  with  $C_c^\infty(w^{-1}Uw \cap U \setminus U, \mathbb{Z}) \otimes \text{Ind}_{P \cap M_w}^{M_w} \sigma$ , a similar isomorphism of  $\pi_{P, \geq w} / \pi_{P, > w}$  with  $C_c^\infty(w^{-1}Uw \cap U \setminus U, \mathbb{Z}) \otimes \sigma$ , and the quotient map  $\pi'_{P, \geq w} / \pi'_{P, > w} \rightarrow \pi_{P, \geq w} / \pi_{P, > w}$  corresponds to evaluation at 1 :  $\text{Ind}_{P \cap M_w}^{M_w} \sigma \rightarrow \sigma$ . But  $(\pi_{P_w} \cap \pi'_{P, \geq w}) / (\pi_{P_w} \cap \pi'_{P, > w})$  is sent by  $\iota$  to  $C_c^\infty(w^{-1}Uw \cap U \setminus U, \mathbb{Z}) \otimes \text{Im } \Phi$  so to get the surjectivity of  $\pi_{P_w} \cap \pi'_{P, \geq w} \rightarrow \pi_{P, \geq w} / \pi_{P, > w}$  it suffices to see that evaluation at 1 :  $\text{Im } \Phi \rightarrow \sigma$  is surjective. But for  $x \in V_{N_w^0}$  the function in  $\text{ind}_{M_w^0}^{M_w} V_{N_w^0}$  with support  $M_w^0$  and value  $x$  at 1, is sent in  $\text{Ind}_{P \cap M_w}^{M_w} \sigma$  to a function with value at 1 the function in  $\sigma$  with support  $M^0$  and value at 1 the projection of  $x$  in  $V_{N^0}$ ; as those last functions, for varying  $x$ , generate  $\sigma$  as a representation of  $M$ , and  $\text{Im } \Phi \rightarrow \sigma$  is  $M$ -equivariant, it is surjective.  $\square$

**V.11.** We turn to the inclusion  $\pi_{Q, A} / \pi_{Q, A'} \subset c_{Q, w}(\pi_{P, A} / \pi_{P, A'})$  in V.8 Proposition. We need auxiliary lemmas, where  $\alpha \in \Delta - \Delta_M$  is fixed; we let  $P^\alpha = M^\alpha N^\alpha$  be the parabolic subgroup corresponding to  $\Delta_M \cup \{\alpha\}$  and we put  $\bar{\sigma} = \sigma / (\tau_\alpha - 1)\sigma$ . Note that Hypothesis (H) of III.15 holds with the map  $\varphi : V_{N^0} \rightarrow \sigma \rightarrow \bar{\sigma}$ . We also note that  $\varphi \tau_\alpha = \tau_\alpha \varphi = \varphi$ .

**Lemma 1**  $\bar{\sigma}$  extends to  $P^\alpha$ , trivially on  $N$ .

**Proof** By II.7 it suffices to prove that  $\bar{\sigma}$  is trivial on  $Z \cap M'_\alpha$ . Since  $\alpha$  is orthogonal to  $\Delta_M$  and  $\psi$  is trivial on  $Z^0 \cap M'_\alpha$  by assumption, that comes from the fact that  $\tau_\alpha$  acts trivially on  $\bar{\sigma}$  (III.17).  $\square$

We write  ${}^e \bar{\sigma}$  for the extension of  $\bar{\sigma}$  to  $P^\alpha$ . Inside of  $\pi_P / (\tau_\alpha - 1)\pi_P \simeq \text{Ind}_P^G \bar{\sigma}$  we have the subspace  $\text{Ind}_{P^\alpha}^G {}^e \bar{\sigma}$ , cf. III.22 Lemma 2.

**Lemma 2** The image of  $\pi_{P^\alpha} \rightarrow \pi_P \rightarrow \pi_P / (\tau_\alpha - 1)\pi_P$  is contained in  $\text{Ind}_{P^\alpha}^G {}^e \bar{\sigma}$ .

**Proof** Since  $\pi_{P^\alpha} \rightarrow \text{Ind}_P^G \bar{\sigma}$  is  $\mathcal{Z}_M[G]$ -equivariant and  $\pi_{P^\alpha}$  is generated as a  $\mathcal{Z}_M[G]$ -module by  $V_{P^\alpha}$  it is enough to prove that the inclusion of  $\text{Hom}_K(V_{P^\alpha}, \text{Ind}_{P^\alpha}^G {}^e \bar{\sigma})$  into  $\text{Hom}_K(V_{P^\alpha}, \text{Ind}_P^G \bar{\sigma})$  is an isomorphism. By Frobenius reciprocity, this means that

$$\text{Hom}_{M^{\alpha 0}}((V_{P^\alpha})_{N^{\alpha 0}}, {}^e \bar{\sigma}) \hookrightarrow \text{Hom}_{M^{\alpha 0}}((V_{P^\alpha})_{N^{\alpha 0}}, \text{Ind}_{P \cap M^\alpha}^{M^\alpha} \bar{\sigma})$$

is an isomorphism. The quotient of  $\text{Ind}_{P \cap M^\alpha}^{M^\alpha} \bar{\sigma}$  by  ${}^e \bar{\sigma}$  is the representation  ${}^e \bar{\sigma} \otimes \text{St}_{P \cap M^\alpha}^{M^\alpha}$  and it is enough to show that  $(V_{P^\alpha})_{N^{\alpha 0}}$  is not a weight of that representation. But the parameter for  $(V_{P^\alpha})_{N^{\alpha 0}}$  is  $(\psi, (\Delta_M \cup \{\alpha\}) \cap \Delta(V))$  and  $\alpha \in \Delta(V)$  whereas by III.18 the weights of  ${}^e \bar{\sigma} \otimes \text{St}_{P \cap M^\alpha}^{M^\alpha} = I(P \cap M^\alpha, \bar{\sigma}, P \cap M^\alpha)$  have parameters  $(\psi', I)$  where  $\alpha \notin I$ .  $\square$

**Lemma 3** Let  $P_1 = M_1 N_1$  and  $P_2 = M_2 N_2$  be parabolic subgroups of  $G$  containing  $P$ , and assume  $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$ . Let  $A, w, A'$  be as in V.7, and assume that  $w^{-1}(\beta) < 0$

for all  $\beta \in \Delta_{P_2} - \Delta_M$ . Then

$$\pi_{P_2,A} \subset (\tau_\alpha - 1)\pi_{P_1,A} + \pi_{P,A'}.$$

**Proof** Let  $f \in \pi_{P_2,A}$  and let  $\bar{f}$  be its image in  $\text{Ind}_P^G \bar{\sigma}$ . As  $\pi_{P_2} \subset \pi_{P^\alpha}$ , we get  $\bar{f} \in \text{Ind}_{P^\alpha}^G \bar{\sigma}$  by Lemma 2. If  $\bar{f}$  does not vanish on  $PwB$ , its support, being  $P^\alpha$  invariant, contains  $P s_\alpha w B$ . But  $w^{-1}(\alpha) < 0$  means  $s_\alpha w < w$  and  $w$  being minimal in  $A$ , that contradicts  $f \in \pi_{P,A}$ . Hence  $\bar{f}$  vanishes on  $PwB$  and there exist  $f_1 \in \pi_P$ ,  $f_2 \in \pi_{P,A'}$  with  $f = (\tau_\alpha - 1)f_1 + f_2$ . The point is to prove that we can take  $f_1$  in  $\pi_{P_1,A}$ . View  $\pi_{P_1}$  as  $\text{Ind}_{P_1}^G \sigma_1$  with  $\sigma_1 = \mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \text{ind}_{M_1^0}^{M_1} V_{N_1^0}$  and  $\pi_P$  as  $\text{Ind}_{P_1}^G \text{Ind}_{P \cap M_1}^{M_1} \sigma$ , the inclusion  $\pi_{P_1} \hookrightarrow \pi_P$  being induced by the natural intertwiner  $\text{ind}_{M_1^0}^{M_1} V_{N_1^0} \rightarrow \text{Ind}_{P \cap M_1}^{M_1} \sigma$ .

**Sublemma** For  $v \in A'$  we have  $P_1 w B \cap P v B = \emptyset$ .

**Proof** Indeed if  $P_1 w B \cap P v B \neq \emptyset$  there exists  $v'$  in  $W_{M_1}$  with  $v' w = v$ . Since  $\Delta - \Delta_M$  is orthogonal to  $\Delta_M$ ,  $W_{M_1}$  is the direct product of  $W_M$  and the subgroup  $W_1$  generated by the  $s_\beta$  for  $\beta \in \Delta_{P_1} - \Delta_M$ . For such a  $\beta$  we have  $s_\beta w < w$  and it follows (using [Deo] as in IV.9 Lemma 2), by induction on length, that  $v_1 w \leq w$  for any  $v_1 \in W_1$ . Writing  $v'$  as  $v_2^{-1} v_1$  with  $v_1 \in W_1$  and  $v_2 \in W_M$  we get  $v_1 w = v_2 v$ . But  $v_2 v \geq v$  and  $v_1 w \leq w$  so  $v \leq w$  contrary to the assumption  $v \in A'$ .  $\square$

Let us pursue the proof of Lemma 3.

Since  $f_2 \in \pi_{P,A'}$ , it follows from the sublemma that, seen as an element of  $\text{Ind}_P^G \sigma$ , it vanishes on  $P_1 w B$ ; but then, seen as an element of  $\text{Ind}_{P_1}^G \text{Ind}_{P \cap M_1}^{M_1} \sigma$ , it also vanishes on  $P_1 w B$ . So for any  $x \in P_1 w B$ ,  $f(x) = (\tau_\alpha - 1)f_1(x)$  in  $\text{Ind}_{P \cap M_1}^{M_1} \sigma$ . Now  $\dim P_1 < \dim G$  so V.8 Proposition and all its corollaries are true for  $M_1$ . As  $v_{M_1}(a_\alpha) \neq 0$  we conclude from V.9 Corollary 4 that there exists  $y \in \sigma_1$  with  $(\tau_\alpha - 1)f_1(x) = (\tau_\alpha - 1)y$ . But  $\tau_\alpha - 1$  does not kill any element of  $\text{Ind}_{P \cap M_1}^{M_1} \sigma$ , by V.4 Lemma (i), so  $f_1(x) = y$  belongs to  $\sigma_1$ . We can choose  $f'_1$  in  $\text{Ind}_{P_1}^G \sigma_1 \cap \pi_{P_1,A}$  with the same restriction as  $f_1$  on  $P_1 w B$  (use V.5 Remark). Put  $f'_2 = f - (\tau_\alpha - 1)f'_1 = (\tau_\alpha - 1)(f_1 - f'_1) + f_2$ . Then  $f_1 - f'_1$  vanishes on  $P_1 w B$ ,  $f_2$  vanishes on  $PwB$  so  $f'_2$  belongs to  $\pi_{P,A'}$  and  $f = (\tau_\alpha - 1)f'_1 + f'_2$  belongs to  $(\tau_\alpha - 1)\pi_{P_1,A} + \pi_{P,A'}$ .  $\square$

We now finish the proof of V.8 Proposition. Let  $R$  be the parabolic subgroup between  $P$  and  $Q$  with  $\Delta_R - \Delta_M = \{\alpha \in \Delta_Q, w^{-1}(\alpha) < 0\}$ . Applying Lemma 3 successively we get  $\pi_{R,A} \subset c_{R,w} \pi_{P,A} + \pi_{P,A'}$ , hence the result since  $\pi_{Q,A} \subset \pi_{R,A}$  and  $c_{R,w} = c_{Q,w}$ .  $\square$

We can get more out of that:

**Lemma 4** Let  $A$ ,  $w$ ,  $A'$  be as in V.7. Then  $\pi_{Q,A} \subset c_{Q,w} \pi_{P_w,A} + \pi_{Q,A'}$ .

**Proof** V.10 Lemma gives  $\pi_{P,A} \subset \pi_{P_w,A} + \pi_{P,A'}$  so from V.8 Proposition we get  $\pi_{Q,A} \subset c_{Q,w} \pi_{P_w,A} + \pi_{P,A'}$ . But  $\pi_{P_w} \subset \pi_{Q \cap P_w}$  and  $c_{Q,w} \pi_{Q \cap P_w} \subset \pi_Q$  by V.6 Proposition so  $c_{Q,w} \pi_{P_w,A} \subset \pi_{Q,A}$  and the result follows.  $\square$

## B) Filtration theorem for $\chi \otimes \text{ind}_K^G V$

**V.12.** We now turn to the filtration theorem (I.6). For that, as before, an irreducible representation  $V$  of  $K$  is fixed, with parameter  $(\psi, \Delta(V))$ , but we also fix a character  $\chi$  of  $\mathcal{Z}_G = \mathcal{Z}_G(V)$ . We let  $P = MN$  be the parabolic subgroup with  $\Delta_P = \Delta_0(\chi)$ , so  $P$  is the

smallest parabolic subgroup of  $G$  containing  $B$  such that  $\chi$  extends to a character – still written  $\chi$  – of  $\mathcal{Z}_M = \mathcal{Z}_M(V_{N^0})$ , and that character further factors through  $\mathcal{Z}_M \rightarrow R_M$ .

**Notation** For a  $\mathcal{Z}_M$ -module  $W$ , we put  $W^\chi = \chi \otimes_{\mathcal{Z}_M} W$ .

Recall that for each parabolic subgroup  $Q$  of  $G$  containing  $P$ ,  $V_Q$  denotes the irreducible representation of  $K$  of parameter  $(\psi, \Delta_Q \cap \Delta(V))$ ; we make the same identifications as in V.5. In particular we get a  $\mathcal{Z}_M[G]$ -submodule  $\pi_Q$  of  $\pi_P = \text{Ind}_P^G \sigma$  – we keep writing  $\sigma = \text{ind}_{M^0}^M V_{N^0}$ . Our main interest is in  $\pi_G^\chi$ , but its analysis goes through the  $\pi_Q^\chi$ , in particular  $\pi_P^\chi$ .

As  $\sigma^\chi$  satisfies property (H) of III.15, the maximal parabolic subgroup of  $G$  to which  $\sigma^\chi$  extends, trivially on  $N$ , has associated set of roots  $\Delta_M \sqcup \Theta_{\max}$  where  $\Theta_{\max}$  is the set of  $\alpha \in \Delta - \Delta_M$ , orthogonal to  $\Delta_M$  and such that  $\psi(Z^0 \cap M'_\alpha) = 1$  and  $\chi(\tau_\alpha) = 1$  (III.17 Corollary).

**Notation** We let  $\Theta = \Theta_{\max} \cap \Delta(V)$ ,  $P_e = P_{\Delta_M \sqcup \Theta}$  and write  ${}^e\sigma^\chi$  for the extension of  $\sigma^\chi$  to  $P_e$ , trivial on  $N$ . (Note that III.22 Lemma 2 gives an identification of  $\pi_P^\chi$  with  $\text{Ind}_{P_e}^G ({}^e\sigma^\chi \otimes \text{Ind}_{P_e}^{P_e} 1)$ .)

**Lemma** *The inclusion  $\pi_G \rightarrow \pi_{P_e}$  induces an isomorphism  $\pi_G^\chi \rightarrow \pi_{P_e}^\chi$*

**Proof** It suffices to show that for  $P_e \subset P_1 \subset P_2$  with  $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$ , the natural map  $\pi_{P_2}^\chi \rightarrow \pi_{P_1}^\chi$  is an isomorphism. If  $\alpha \notin \Delta(V)$  or if  $\psi$  is not trivial on  $Z^0 \cap M'_\alpha$  or  $\alpha$  not orthogonal to  $\Delta_M$ , then by V.6 we even have an isomorphism  $R_M \otimes_{\mathcal{Z}_M} \pi_{P_2} \xrightarrow{\sim} R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$ . Otherwise,  $\chi(\tau_\alpha) \neq 1$  and since  $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2} \subset \pi_{P_1}$  by V.6, we have an isomorphism  $\pi_{P_2}^\chi \xrightarrow{\sim} \pi_{P_1}^\chi$ .  $\square$

**V.13. Notation** Let  $\mathcal{D}$  be the set of parabolic subgroups of  $G$  between  $P$  and  $P_e$ .

- For  $Q, Q_1$  in  $\mathcal{D}$ ,  $Q \supset Q_1$ , put  $c_{Q, Q_1} = \prod_{\alpha \in \Delta_Q - \Delta_{Q_1}} (\tau_\alpha - 1)$  (then  $c_{Q, Q_1} \pi_{Q_1} \subset \pi_Q$  by V.6

Proposition).

- For  $Q \in \mathcal{D}$ , let  $\tau_Q$  be the image of  $\pi_Q \xrightarrow{c_{P_e, Q}} \pi_{P_e} \rightarrow \pi_{P_e}^\chi (= \pi_G^\chi)$ , and let  $\rho_Q$  be the image of  $\pi_Q \hookrightarrow \pi_P \rightarrow \pi_P^\chi$ .

- For  $Q \in \mathcal{D}$ , let  $Q^c \in \mathcal{D}$  be the parabolic subgroup such that  $\Delta_{Q^c} - \Delta_M = \Delta_{P_e} - \Delta_Q$ , and let  $\Phi_Q, \Psi_Q$  be the  $G$ -equivariant maps

$$\begin{aligned} \Phi_Q &: \tau_Q \hookrightarrow \pi_{P_e}^\chi \longrightarrow \pi_{Q^c}^\chi, \\ \Psi_Q &: \rho_Q \hookrightarrow \pi_P^\chi \xrightarrow{c_{Q^c, P}} \pi_{Q^c}^\chi. \end{aligned}$$

Here the last map is obtained from  $\pi_P \xrightarrow{c_{Q^c, P}} \pi_{Q^c}$  by tensoring by  $\chi$ .

- Let  $I_Q$  the submodule  $\text{Ind}_{P_e}^G ({}^e\sigma^\chi \otimes \text{Ind}_{Q^c}^{P_e} 1)$  of  $\pi_{P_e}^\chi$ . In particular,  $I_P = \rho_P = \pi_P^\chi$ . Note also that  $\tau_{P_e} = \pi_{P_e}^\chi$ .

**Remark 1** The maps  $\pi_Q \xrightarrow{c_{P_e, Q}} \pi_{P_e} \hookrightarrow \pi_{Q^c}$  and  $\pi_Q \hookrightarrow \pi_P \xrightarrow{c_{Q^c, P}} \pi_{Q^c}$  are equal because  $c_{Q^c, P} = c_{P_e, Q}$ . Therefore  $\text{Im } \Phi_Q = \text{Im } \Psi_Q$ .

**Remark 2** For  $Q, Q_1$  in  $\mathcal{D}$ ,  $Q \supset Q_1$ , we have  $\tau_{Q_1} \subset \tau_Q$  and  $\rho_{Q_1} \supset \rho_Q$ .

Our second main result in this chapter is:

**Theorem** *Let  $Q \in \mathcal{D}$ .*

- (i)  $\rho_Q = I_Q$ .  
 (ii)  $\text{Ker } \Psi_Q = \sum_{Q_1 \in \mathcal{D}, Q_1 \supseteq Q} \rho_{Q_1}$ .  
 (iii)  $\text{Ker } \Phi_Q = \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$ .  
 (iv) Let  $\mathcal{P} \subset \mathcal{D}$ ; then  $\tau_Q \cap \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} = \sum_{Q_1 \in \mathcal{P}} \tau_{Q \cap Q_1}$ .

It implies I.6 Theorem 6:

**Corollary 1** For  $Q \in \mathcal{D}$ ,  $\tau_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$  is isomorphic to  $I_e(P, \sigma^\chi, Q)$ .

**Proof** By Remark 1 we have that  $\tau_Q / \text{Ker } \Phi_Q$  is isomorphic to  $\rho_Q / \text{Ker } \Psi_Q$ . But  $\rho_Q = I_Q$  by (i) so we get by (ii) and (iii) a  $G$ -isomorphism between  $\tau_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$  and  $I_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \supseteq Q} I_{Q_1}$  which is  $I_e(P, \sigma^\chi, Q)$ .  $\square$

**Corollary 2** Enumerate the parabolic subgroups in  $\mathcal{D}$  as  $P = Q_1, \dots, Q_r = P_e$ , so that  $i \leq j$  if  $Q_i \subset Q_j$ . For  $i = 0, \dots, r$ , put  $I_i = \sum_{1 \leq j \leq i} \tau_{Q_j}$ . Then for  $i = 1, \dots, r$ ,  $I_i / I_{i-1} \simeq I_e(P, \sigma^\chi, Q_i)$ .

**Proof** For  $i = 1, \dots, r$   $I_i / I_{i-1} = \tau_{Q_i} / (\tau_{Q_i} \cap \sum_{1 \leq j < i} \tau_{Q_j})$  is also  $\tau_{Q_i} / \sum_{1 \leq j < i} \tau_{Q_i \cap Q_j}$  by (iv).

The assertion follows from Corollary 1.  $\square$

**Remark 3** The proofs below are in fact valid more generally: it would suffice, for a given parabolic subgroup  $P = MN$  of  $G$  containing  $B$ , to tensor  $\text{ind}_K^G V$  with the quotient of  $R_M$  in which all  $\tau_\alpha - 1$  for  $\alpha \in \Theta$  are killed.

Since we consider only parabolic subgroups in  $\mathcal{D}$ , and all the representations we consider are parabolically induced from analogously defined representations of the Levi quotient of  $P_e$ , it is enough to prove the theorem when  $P_e = G$ , i.e.  $\Delta = \Delta_M \sqcup \Theta$ , which we assume from now on.

**V.14.** Under that assumption  $\Delta = \Delta_M \sqcup \Theta$ , we prove V.13 Theorem in a succession of lemmas.

We fix  $Q \in \mathcal{D}$  and let  $M_Q$  be its Levi subgroup containing  $M$ .

**Lemma 1**  $\rho_Q \subset I_Q$ .

**Proof** Equality is clear when  $Q = P$ , so we assume  $Q \supsetneq P$ . For each  $\alpha \in \Delta_Q - \Delta_P$ , let  $P^\alpha$  be as in V.11. By V.11 Lemma 2,  $\rho_{P^\alpha}$  is included in  $I_{P^\alpha}$  so a fortiori  $\rho_Q \subset I_{P^\alpha}$ . But the subgroup of  $G$  generated by the  $P^\alpha$ 's for  $\alpha \in \Delta_Q - \Delta_P$  is  $Q$ , so  $\bigcap_{\alpha \in \Delta_Q - \Delta_P} I_{P^\alpha} = I_Q$ , and  $\rho_Q \subset I_Q$ .  $\square$

To prove equality in Lemma 1, we resort to filtration arguments. In the following  $A, w, A'$  are as in V.7 and  $\pi_{P,A}, \pi_{Q,A}$  as in V.7, V.8.

**Remark 1** We can also filter  $\pi_P^\chi$  by support yielding  $(\pi_P^\chi)_A \subset \pi_P^\chi$ . But from V.7 Lemma we get, after tensoring with  $\chi : R_M \rightarrow C$ , that  $\pi_{P,A} \rightarrow \pi_P^\chi$  induces an isomorphism  $(\pi_{P,A})^\chi \simeq (\pi_P^\chi)_A$ . We let  $\pi_{P,A}^\chi$  denote  $(\pi_P^\chi)_A$ .

We put  $\rho_{Q,A} = \rho_Q \cap \pi_{P,A}^\chi$ ,  $I_{Q,A} = I_Q \cap \pi_{P,A}^\chi$ , so  $\rho_{Q,A} = \rho_Q \cap I_{Q,A}$ .

**Remark 2** By V.8 Corollary 2 and Corollary 3,

$$0 \rightarrow R_M \otimes \pi_{Q,A} \rightarrow R_M \otimes \pi_Q \rightarrow R_M \otimes (\pi_Q/\pi_{Q,A}) \rightarrow 0$$

is an exact sequence of free  $R_M$ -modules (an extension of free  $R_M$ -modules is free and by induction  $R_M \otimes \pi_{Q,A}$  and  $R_M \otimes \pi_Q$  are free  $R_M$ -modules). Therefore the map  $(\pi_{Q,A})^\chi \rightarrow \pi_Q^\chi$  is injective.

**Lemma 2** (i) If  $w \notin W(M_Q)$  then  $I_{Q,A} = I_{Q,A'}$ , and  $\rho_{Q,A} = \rho_{Q,A'}$ .

(ii) If  $w \in W(M_Q)$  the maps  $\pi_Q \rightarrow \rho_Q \rightarrow I_Q \rightarrow \pi_P^\chi$  induces isomorphisms

$$(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \simeq \rho_{Q,A}/\rho_{Q,A'} \simeq I_{Q,A}/I_{Q,A'} \simeq \pi_{P,A}^\chi/\pi_{P,A'}^\chi.$$

(iii)  $\rho_{Q,A}$  is the image of  $\pi_{Q,A}$  in  $\pi_P^\chi$ .

**Note**  $w \in W(M_Q)$  means that for  $\alpha \in \Delta_Q$ ,  $w^{-1}(\alpha) > 0$ ; it is equivalent to  $c_{Q,w} = 1$  (V.8).

**Proof** (i) Let  $f \in I_{Q,A} - I_{Q,A'}$ ; then  $f$  is not identically 0 on  $PwB$ , but its support is left  $Q$ -equivariant, so for any  $v \in W_{M_Q}$ ,  $f$  is not identically 0 on  $PvwB$ . If  $w \notin W(M_Q)$  we can choose  $v \in W_{M_Q}$  so that  $vw < w$ . That implies  $vw \notin A$  by minimality of  $w$ , a contradiction. So  $I_{Q,A} = I_{Q,A'}$  and  $\rho_{Q,A} = \rho_{Q,A'}$  follows by intersecting with  $\rho_Q$ .

(ii) Let  $w \in W(M_Q)$ . Then  $c_{Q,w} = 1$  and V.8 Proposition gives that the map  $\pi_{Q,A} \rightarrow \pi_{P,A}$  induces an isomorphism  $\pi_{Q,A}/\pi_{Q,A'} \simeq \pi_{P,A}/\pi_{P,A'}$ . Tensoring with  $\chi$  gives an isomorphism of  $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi$  onto  $(\pi_{P,A})^\chi/(\pi_{P,A'})^\chi$  which is  $\pi_{P,A}^\chi/\pi_{P,A'}^\chi$  by Remark 1; since the image of that isomorphism is contained in  $\rho_{Q,A}/\rho_{Q,A'}$ , itself contained in  $I_{Q,A}/I_{Q,A'}$ , we get (ii).

(iii) We prove it by descending induction on  $\#A$ , the case  $A = W(M)$  being true by definition of  $\rho_Q$ . We assume that the result is true for  $A$  and prove it for  $A'$ . By V.11 Lemma 4 we have

$$\pi_{Q,A} \subset c_{Q,w}\pi_{P_w,A} + \pi_{Q,A'}.$$

If  $w \notin W(M_Q)$  then  $\chi(c_{Q,w}) = 0$ . Hence  $\pi_{Q,A}$  and  $\pi_{Q,A'}$  have the same image in  $\pi_P^\chi$ , which is  $\rho_{Q,A}$  by induction and  $\rho_{Q,A'}$  by (i). If  $w \in W(M_Q)$  we use the isomorphism  $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \simeq \rho_{Q,A}/\rho_{Q,A'}$  in (ii). Since  $(\pi_{Q,A})^\chi \rightarrow \rho_{Q,A}$  is surjective by induction,  $(\pi_{Q,A'})^\chi \rightarrow \rho_{Q,A'}$  has to be surjective too.  $\square$

**Lemma 3**  $\rho_Q = I_Q$ .

**Proof** By induction on  $\#A$ : if  $\rho_{Q,A'} = I_{Q,A'}$ , then Lemma 2 (i), (ii), and Lemma 1 give  $\rho_{Q,A} = I_{Q,A}$ .  $\square$

**Lemma 4** For  $Q_1 \in \mathcal{D}$ ,  $Q_1 \supsetneq Q$ ,  $\text{Ker } \Psi_Q$  contains  $\rho_{Q_1}$ .

**Proof** It enough to show that the composite map  $\pi_{Q_1}^\chi \rightarrow \pi_Q^\chi \rightarrow \rho_Q \xrightarrow{\Psi_Q} \pi_{Q^c}^\chi$  is 0. But it factors as  $\pi_{Q_1}^\chi \rightarrow \pi_Q^\chi \xrightarrow{c_{G,Q}} \pi_G^\chi \rightarrow \pi_{Q^c}^\chi$  since  $c_{Q^c,P} = c_{G,Q}$ . From  $c_{G,Q} = c_{Q_1,Q}c_{G,Q_1}$  we get  $c_{G,Q}\pi_{Q_1}^\chi = c_{Q_1,Q}c_{G,Q_1}\pi_{Q_1}^\chi \subset c_{Q_1,Q}\pi_G^\chi$  which is 0 since  $\chi(c_{Q_1,Q}) = 0$ .  $\square$

**Lemma 5**  $\text{Ker } \Psi_Q \subset \sum_{Q_1 \in \mathcal{D}, Q_1 \supsetneq Q} \rho_{Q_1}$ .

**Proof** We show by induction on  $\#A$  that

$$(*) \quad \text{Ker } \Psi_Q \cap \pi_{P,A}^\chi \subset \sum_{Q_1 \in \mathcal{D}, Q_1 \supsetneq Q} \rho_{Q_1}.$$



We assume that (\*) is true for  $A'$  and prove it for  $A$ . Note that  $\text{Ker } \Psi_Q \subset \rho_Q \subset \pi_P^\chi$  so  $\text{Ker } \Psi_Q \cap \pi_{P,A}^\chi = \text{Ker } \Psi_Q \cap \rho_{Q,A}$ . If  $w \notin W(M_Q)$  then  $\rho_{Q,A} = \rho_{Q,A'}$  by Lemma 2 (i), so the result is immediate. Assume  $w \in W(M_Q)$ . On  $\rho_{Q,A}/\rho_{Q,A'}$ ,  $\Psi_Q$  induces  $\bar{\Psi}_Q : \rho_{Q,A}/\rho_{Q,A'} \rightarrow \pi_{P,A}^\chi/\pi_{P,A'}^\chi \xrightarrow{c_{Q^c,P}} (\pi_{Q^c,A})^\chi/(\pi_{Q^c,A'})^\chi$ . By Lemma 2(ii), the first map is an isomorphism, so we focus on the second map.

**Notation** Put  $d_w^Q = \prod_{\alpha \in \Delta - \Delta_Q, w^{-1}(\alpha) > 0} (\tau_\alpha - 1)$ , so that  $c_{Q^c,P} = d_w^Q c_{Q^c,w}$  because  $\Delta - \Delta_Q = \Delta_{Q^c} - \Delta_M$ .

By V.8 Proposition and the remark before it,  $c_{Q^c,w}$  gives an isomorphism  $\pi_{P,A}/\pi_{P,A'} \xrightarrow{\sim} \pi_{Q^c,A}/\pi_{Q^c,A'}$ . If  $d_w^Q = 1$  then  $\bar{\Psi}_Q$  is injective and  $\text{Ker } \Psi_Q \cap \pi_{P,A}^\chi = \text{Ker } \Psi_Q \cap \pi_{P,A'}^\chi$  so (\*) follows from the induction hypothesis. Let  $d_w^Q \neq 1$ , choose  $\alpha \in \Delta - \Delta_Q$  with  $w^{-1}(\alpha) > 0$  and let  $Q^\alpha$  be the parabolic subgroup of  $G$  corresponding to  $\Delta_Q \cup \{\alpha\}$ . Then  $w \in W(M_{Q^\alpha})$  and Lemma 2 (ii) gives the isomorphism

$$\rho_{Q^\alpha,A}/\rho_{Q^\alpha,A'} \xrightarrow{\sim} \pi_{P,A}^\chi/\pi_{P,A'}^\chi.$$

Let  $f \in \text{Ker } \Psi_Q \cap \pi_{P,A}^\chi$ , and choose  $f' \in \rho_{Q^\alpha,A}$  with  $f - f' \in \pi_{P,A'}^\chi$ . As  $f' \in \text{Ker } \Psi_Q$  by Lemma 4,  $f - f' \in \text{Ker } \Psi_Q$  so  $f - f'$  belongs to  $\sum_{Q_1 \in \mathcal{D}, Q_1 \supseteq Q} \rho_{Q_1}$  by induction; as  $f'$  also belongs to that space, the result follows.  $\square$

**V.15.** We have proved (i) and (ii) in V.13 Theorem, and now we turn to part (iii). Describing  $\text{Ker } \Phi_Q$  is analogous to describing  $\text{Ker } \Psi_Q$ . We let  $A, w, A'$  be as before, and let  $\tau_{Q,A} \subset \tau_Q$  be the image of  $\pi_{Q,A}$  (or  $(\pi_{Q,A})^\chi$ ) in  $\pi_G^\chi = \tau_G$ , via the map  $\pi_Q^\chi \xrightarrow{c_{G,Q}} \pi_G^\chi$ . We observe that  $\tau_{Q,A'} \subset \tau_{Q,A}$  and  $\tau_{Q_1,A} \subset \tau_{Q,A}$  if  $Q_1 \subset Q$  in  $\mathcal{D}$ . We note also that by V.14 Remark 2 we have  $(\pi_{G,A})^\chi = \tau_{G,A} \subset \pi_G^\chi$ .

**Lemma 6** (i) *If for some  $\alpha \in \Delta - \Delta_Q$ ,  $w^{-1}(\alpha) > 0$  then  $\tau_{Q,A} = \tau_{Q,A'}$ . Otherwise the natural maps  $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \rightarrow \tau_{Q,A}/\tau_{Q,A'} \rightarrow \tau_{G,A}/\tau_{G,A'}$  are isomorphisms.*

(ii)  $\tau_{Q,A} = \tau_{G,A} \cap \tau_Q$ .

**Proof** (i) Let  $\phi \in \pi_{Q,A}$ . With  $P_w$  as in V.10, V.11 Lemma 4 implies that we can write  $\phi = c_{Q,w}\phi_w + \phi'$  with  $\phi_w \in \pi_{P_w,A}$  and  $\phi' \in \pi_{Q,A'}$ . Since  $d_w^Q c_{G,w} = c_{G,Q} c_{Q,w}$  we get  $c_{G,Q}\phi = d_w^Q(c_{G,w}\phi_w) + c_{G,Q}\phi'$ . But  $c_{G,w} = c_{G,P_w}$  so  $c_{G,w}\phi_w$  belongs to  $\pi_G$  by V.6 Proposition. In the first case of (i),  $\chi(d_w^Q) = 0$ , so  $\phi$  has the same image as  $\phi'$  in  $\tau_Q$ ; this implies  $\tau_{Q,A} = \tau_{Q,A'}$ . Let us assume we are in the second case of (i), so  $d_w^Q = 1$ . Consider the natural inclusions

$$\pi_{G,A}/\pi_{G,A'} \hookrightarrow \pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

By V.8 Proposition, the first space is  $c_{G,w}(\pi_{P,A}/\pi_{P,A'})$  and the second is  $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ . Consequently  $c_{G,Q}(\pi_{Q,A}/\pi_{Q,A'}) = \pi_{G,A}/\pi_{G,A'}$  since  $d_w^Q = 1$ . Thus  $c_{G,Q}$  induces a surjective map of  $\pi_{Q,A}/\pi_{Q,A'}$  onto  $\pi_{G,A}/\pi_{G,A'}$ . But by V.4 Lemma (i) (applied to  $M$ )  $c_{G,Q}$  acts injectively on  $\sigma$  hence on  $\pi_{P,A}/\pi_{P,A'}$ , so we actually get an isomorphism. Tensoring with  $\chi$  we get an isomorphism  $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \rightarrow \tau_{G,A}/\tau_{G,A'}$ ; but this factors as in the statement of (i), so (i) follows again.

(ii) We proceed by descending induction on  $\#A$ , the case  $A = W(M)$  being obvious. The containment  $\tau_{Q,A'} \subset \tau_{G,A'} \cap \tau_Q$  is clear, and we have  $\tau_{Q,A} = \tau_{G,A} \cap \tau_Q$  by induction. In the first case of (i)  $\tau_{Q,A'} = \tau_{Q,A} = \tau_{G,A} \cap \tau_Q \supset \tau_{G,A'} \cap \tau_Q$  so  $\tau_{Q,A'} = \tau_{G,A'} \cap \tau_Q$ . In the second

case of (i),  $\tau_{Q,A}/\tau_{Q,A'} \rightarrow \tau_{G,A}/\tau_{G,A'}$  is an isomorphism; as moreover  $\tau_{G,A'} \cap \tau_Q \subset \tau_{Q,A}$  by induction, the result follows.  $\square$

**Lemma 7** For  $Q_1 \in \mathcal{D}$ ,  $Q_1 \subsetneq Q$ , then  $\tau_{Q_1} \subset \text{Ker } \Phi_Q$ .

**Proof** Let  $P_1$  be the parabolic subgroup corresponding to  $\Delta_{Q_1} \sqcup (\Delta - \Delta_Q) = \Delta_{Q_1} \cup \Delta_{Q^c}$ . Since  $Q_1 \subsetneq Q$ , we get  $P_1 \subsetneq G$ . We have  $c_{P_1, Q_1} \pi_{Q_1} \subset \pi_{P_1} \subset \pi_{Q^c}$  so  $c_{G, Q_1} \pi_{Q_1} \subset c_{G, P_1} \pi_{Q^c}$ . As  $\chi(c_{G, P_1}) = 0$  the image of  $\pi_{Q_1} \xrightarrow{c_{Q^c, Q_1}} \pi_{Q^c} \rightarrow \pi_{Q^c}^\chi$  is 0; but that image is  $\Phi_Q(\tau_{Q_1})$ .  $\square$

**Lemma 8**  $\text{Ker } \Phi_Q \subset \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$ .

**Proof** We prove that  $\text{Ker } \Phi_Q \cap \tau_{G,A}$  is contained in the right-hand side, by induction on  $\#A$ . In the first case of Lemma 6 (i),  $\tau_{Q,A} = \tau_{Q,A'}$ , so  $\tau_{G,A} \cap \tau_Q = \tau_{G,A'} \cap \tau_Q$  by loc. cit. (ii). Consequently  $\text{Ker } \Phi_Q \cap \tau_{G,A} = \text{Ker } \Phi_Q \cap \tau_{G,A'}$  and we are done. So we assume that for all  $\alpha \in \Delta - \Delta_Q = \Delta_{Q^c} - \Delta_P$  we have  $w^{-1}(\alpha) < 0$ . On  $\tau_{Q,A}/\tau_{Q,A'}$ ,

$\Phi_Q$  induces  $\bar{\Phi}_Q : \tau_{Q,A}/\tau_{Q,A'} \rightarrow (\pi_{G,A})^\chi/(\pi_{G,A'})^\chi \rightarrow (\pi_{Q^c,A})^\chi/(\pi_{Q^c,A'})^\chi$ , where the first map is an isomorphism by Lemma 6 (i), and the second comes, upon tensoring with  $\chi$ , from the inclusion of  $\pi_{G,A}/\pi_{G,A'}$  into  $\pi_{Q^c,A}/\pi_{Q^c,A'}$ . By V.8 Proposition, we have, inside  $\pi_{P,A}/\pi_{P,A'}$ ,  $\pi_{G,A}/\pi_{G,A'} = c_{G,w}(\pi_{P,A}/\pi_{P,A'})$ , and  $\pi_{Q^c,A}/\pi_{Q^c,A'} = c_{Q^c,w}(\pi_{P,A}/\pi_{P,A'})$ . If for all  $\alpha \in \Delta - \Delta_{Q^c}$  we have  $w^{-1}(\alpha) > 0$ , then  $c_{G,w} = c_{Q^c,w}$ , and  $\pi_{G,A}/\pi_{G,A'} = \pi_{Q^c,A}/\pi_{Q^c,A'}$ ; thus  $\text{Ker } \Phi_Q \cap \tau_{G,A} = \text{Ker } \Phi_Q \cap \tau_{G,A'}$ , so we conclude by induction. In the opposite case, choose  $\alpha \in \Delta - \Delta_{Q^c} = \Delta_Q - \Delta_P$  with  $w^{-1}(\alpha) < 0$ , and let  $Q_\alpha$  correspond to  $\Delta_Q - \{\alpha\}$ . Then  $\tau_{Q_\alpha,A}/\tau_{Q_\alpha,A'} \rightarrow \tau_{G,A}/\tau_{G,A'}$  is an isomorphism by Lemma 6 (i). If  $f \in \text{Ker } \Phi_Q \cap \tau_{Q,A}$ , there is  $f' \in \tau_{Q_\alpha,A}$  with  $f - f' \in \tau_{G,A'}$ . As  $\tau_{Q_\alpha} \subset \tau_Q$  we have  $f - f' \in \tau_{G,A'} \cap \tau_Q = \tau_{Q,A'}$  by loc. cit. (ii). Lemma 7 gives  $\Phi_Q(f') = 0$ , so  $\Phi_Q(f - f') = 0$  and by induction  $f - f'$  belongs to the right-hand side of Lemma 8; since  $f' \in \tau_{Q_\alpha}$  also belongs to that space, so does  $f$ .  $\square$

**V.16.** It remains to prove (iv) of V.13 Theorem.

**Lemma 9** Let  $\mathcal{P} \subset \mathcal{D}$ . Then  $\left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} \right) \cap \tau_{G,A} = \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A}$ .

**Proof** The containment  $\supset$  is clear; we prove the other direction by descending induction on  $\#A$ . Let  $\mathcal{P}^- = \{Q_1 \in \mathcal{P} \mid w^{-1}(\alpha) < 0 \text{ for any } \alpha \in \Delta - \Delta_{Q_1}\}$ . If  $\mathcal{P}^-$  is empty then  $\tau_{Q_1,A} = \tau_{Q_1,A'}$  for any  $Q_1 \in \mathcal{P}$  (Lemma 6, (i)), and we have nothing to prove. Assume  $\mathcal{P}^-$  is not empty, and put  $Q_\cap = \bigcap_{Q_1 \in \mathcal{P}^-} Q_1$ . Then for  $\alpha \in \Delta - \Delta_{Q_\cap}$  we have  $w^{-1}(\alpha) < 0$

so by loc. cit. the map  $\tau_{Q_\cap,A} \rightarrow \tau_{G,A}/\tau_{G,A'}$  is surjective. For  $Q_1 \in \mathcal{P}$  let  $f_{Q_1} \in \tau_{Q_1}$  be chosen so that  $\sum_{Q_1 \in \mathcal{P}} f_{Q_1} \in \tau_{G,A'}$ ; by the inductive hypothesis we may assume that all

$f_{Q_1} \in \tau_{Q_1,A}$ . For  $Q_1 \in \mathcal{P} - \mathcal{P}^-$ , we even have  $f_{Q_1} \in \tau_{Q_1,A'}$  by loc. cit. Fix  $Q_2 \in \mathcal{P}^-$ ; for  $Q_1 \in \mathcal{P}^-$ ,  $Q_1 \neq Q_2$  choose  $f'_{Q_1} \in \tau_{Q_\cap,A}$  with  $f_{Q_1} - f'_{Q_1} \in \tau_{G,A'}$ . Since  $\tau_{Q_\cap,A} \subset \tau_{Q_1,A}$ ,  $f_{Q_1} - f'_{Q_1}$  belongs to  $\tau_{G,A'} \cap \tau_{Q_1} = \tau_{Q_1,A'}$ . So  $\sum_{Q_1 \in \mathcal{P}} f_{Q_1}$  appears as  $f_{Q_2} + \sum_{Q_1 \in \mathcal{P}^-, Q_1 \neq Q_2} f'_{Q_1}$

plus terms in  $\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}$ . But for  $Q_1 \in \mathcal{P}^-$ ,  $Q_1 \neq Q_2$ ,  $f'_{Q_1}$  belongs to  $\tau_{Q_\cap,A} \subset \tau_{Q_2,A}$  so

$f_{Q_2} + \sum_{Q_1 \in \mathcal{P}^-, Q_1 \neq Q_2} f'_{Q_1}$  belongs to  $\tau_{Q_2} \cap \tau_{G,A'} = \tau_{Q_2,A'} \subset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}$ .  $\square$

We finally prove (iv) of V.13 Theorem. Fix  $Q \in \mathcal{D}$  and let  $\mathcal{P} \subset \mathcal{D}$ . It is clear that

$$\left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} \right) \cap \tau_Q \supset \sum_{Q_1 \in \mathcal{P}} (\tau_{Q_1} \cap \tau_Q) \supset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q}.$$

We prove now

$$\left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} \right) \cap \tau_{Q,A} \subset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q} \text{ by induction on } \#A.$$

If there is  $\alpha \in \Delta - \Delta_Q$  with  $w^{-1}(\alpha) > 0$  then  $\tau_{Q,A} = \tau_{Q,A'}$  (Lemma 6 (i)) and there is nothing to prove, so we assume the contrary. By Lemma 9

$$\left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} \right) \cap \tau_{Q,A} = \left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A} \right) \cap \tau_{Q,A}.$$

Let  $\mathcal{P}^- \subset \mathcal{P}$  be the same subset as in the proof of Lemma 9. If  $\mathcal{P}^-$  is empty, then  $\tau_{Q_1,A} = \tau_{Q_1,A'}$  for any  $Q_1$  in  $\mathcal{P}$ . Hence

$$\left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} \right) \cap \tau_{Q,A} = \left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'} \right) \cap \tau_{Q,A} = \left( \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'} \right) \cap \tau_{Q,A'},$$

and the result follows from Lemma 9 and the induction hypothesis. Now assume  $\mathcal{P}^- \neq \emptyset$ , and write  $Q_\cap = Q \cap \bigcap_{Q_1 \in \mathcal{P}^-} Q_1$ ; then for  $\alpha \in \Delta - \Delta_{Q_\cap}$ ,  $w^{-1}(\alpha) < 0$  and again  $\tau_{Q_\cap,A} \rightarrow$

$\tau_{G,A}/\tau_{G,A'}$  is surjective. For  $Q_1 \in \mathcal{P}$  let  $f_{Q_1} \in \tau_{Q_1}$  be chosen so that  $\sum_{Q_1 \in \mathcal{P}} f_{Q_1} \in \tau_{Q,A}$ .

By Lemma 9 we may assume  $f_{Q_1} \in \tau_{Q,A}$ . For  $Q_1 \in \mathcal{P}^-$ , choose  $f'_{Q_1} \in \tau_{Q_\cap,A}$  with  $f_{Q_1} - f'_{Q_1} \in \tau_{G,A'}$  (then  $f_{Q_1} - f'_{Q_1} \in \tau_{Q_1,A'}$ ). Write

$$\sum_{Q_1 \in \mathcal{P}} f_{Q_1} = \sum_{Q_1 \in \mathcal{P}^-} (f_{Q_1} - f'_{Q_1}) + \sum_{Q_1 \in \mathcal{P} - \mathcal{P}^-} f_{Q_1} + \sum_{Q_1 \in \mathcal{P}^-} f'_{Q_1}.$$

We examine the right hand side. The last term belongs to  $\tau_{Q_\cap,A} \subset \tau_{Q,A}$ , so the sum of the first two belongs to  $\tau_Q$ . As each summand in those two terms indexed by  $Q_1$  is in  $\tau_{Q_1,A'}$ , their sum belongs to  $(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}) \cap \tau_Q$ , which is in  $\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q}$  by the induction hypothesis. But for  $Q_1 \in \mathcal{P}^-$ ,  $f'_{Q_1} \in \tau_{Q_\cap,A}$ , and  $\tau_{Q_\cap} \subset \tau_{Q_1 \cap Q}$  since  $Q_\cap \subset Q_1 \cap Q$ . Thus the third term also belongs to  $\sum_{Q_1 \in \mathcal{P}^-} \tau_{Q_1 \cap Q}$ .  $\square$

## VI. CONSEQUENCES OF THE CLASSIFICATION

**VI.1.** The representation theory of irreducible admissible representation of  $G$  over a field of characteristic  $p$  is very different from the theory over a field of characteristic  $\ell \neq p$ . The following proposition is a new example of these differences.

**Proposition** *Any irreducible representation  $\pi$  of  $G$  is a subquotient of  $\text{Ind}_B^G \sigma$  for some representation  $\sigma$  of  $Z$ .*

**Proof** The smoothness of  $\pi$  implies that  $\pi$  has a weight  $V$ . The irreducibility of  $\pi$  implies that  $\pi$  is a quotient of  $\text{ind}_K^G V$ . The representation  $\text{ind}_K^G V$  embeds in  $\text{Ind}_B^G(\text{ind}_{Z^0}^Z V_{U^0})$  by the intertwiner  $\mathcal{I}$  of III.13.  $\square$

We recall from I.3 that a representation of  $G$  is supercuspidal if it is irreducible, admissible, and does not appear as a subquotient of a parabolically induced representation  $\text{Ind}_P^G \sigma$ , where  $P$  is a proper parabolic subgroup of  $G$  and  $\sigma$  an irreducible admissible representation of the Levi quotient of  $P$ .

It is well known [BL1, Br] that there exist supercuspidal representations when  $G = \text{GL}_2(\mathbb{Q}_p)$ , therefore the proposition shows that we cannot drop the condition that  $\sigma$  is irreducible admissible in the definition of supercuspidality, unlike over a field of characteristic  $\ell \neq p$ .

**VI.2.** We derive the desired consequences of I.5 Theorem 4. Mostly we follow the pattern of [He2].

We now prove I.5 Theorem 5, which we recall.

**Theorem** *Let  $\pi$  be an irreducible admissible representation of  $G$ . Then  $\pi$  is supercuspidal if and only if  $\pi$  is supersingular.*

As noticed in the introduction, this theorem shows that the notion of supersingularity, for an irreducible admissible representation of  $G$ , is independent of the choices of  $\mathbf{S}, \mathbf{B}, K$ .

**Proof** Let  $\pi$  be supercuspidal. By I.5 Theorem 4, there is a supersingular  $B$ -triple  $(P, \sigma, Q)$  such that  $\pi \simeq I(P, \sigma, Q)$ . By III.24 Proposition,  $I(P, \sigma, Q)$  is a component of  $\text{Ind}_P^G \sigma$ , so  $P = G$  and  $\pi \simeq \sigma$  is supersingular.

Let  $\pi$  be supersingular. Assume it occurs as a subquotient of  $\text{Ind}_P^G \sigma$  for a parabolic subgroup  $P$  of  $G$  and an irreducible admissible representation  $\sigma$  of the Levi quotient  $M$  of  $P$ ; we may and do assume that  $P$  contains  $B$ . By I.5 Theorem 4, III.24 Proposition, and transitivity of parabolic induction, we may assume that  $\sigma$  is supersingular. By III.24 Proposition,  $\pi$  is isomorphic to some  $I(P, \sigma, Q)$  and I.5 Theorem 4 implies that  $P = G$ , so that  $\pi$  is indeed supercuspidal.  $\square$

Theorems 1 to 3 in Section I.3 are now rather immediate. They follow from I.5 Theorem 4 and the following elementary observations:

- (i) Any triple is  $G$ -conjugate to a  $B$ -triple.
- (ii) A  $B$ -triple is supersingular if and only if it is supercuspidal (by the theorem).
- (iii)  $I(P, \sigma, Q) \simeq I(P', \sigma', Q')$  if the triples  $(P, \sigma, Q), (P', \sigma', Q')$  are  $G$ -conjugate.

**VI.3.** We also have the desired consequence about supercuspidal support.

**Proposition** *Let  $\pi$  be an irreducible admissible representation of  $G$ . Then there is a parabolic subgroup  $P$  of  $G$  and a supercuspidal representation  $\sigma$  of the Levi quotient of  $P$  such that  $\pi$  is a subquotient of  $\text{Ind}_P^G \sigma$ . If  $P_1$  is a parabolic subgroup of  $G$  and  $\sigma_1$  a supercuspidal representation of the Levi quotient of  $P_1$  such that  $\pi$  is a subquotient of  $\text{Ind}_{P_1}^G \sigma_1$ , then there is  $g$  in  $G$  such that  $P_1 = gPg^{-1}$  and that  $\sigma_1$  is equivalent to  $x \mapsto \sigma(g^{-1}xg)$ .*

**Proof** By I.3 Theorem 3,  $\pi$  has the form  $I(P, \sigma, Q)$  for some supercuspidal triple  $(P, \sigma, Q)$  and the first assertion comes from III.24 Proposition. The uniqueness assertion is derived in the same way from I.3 Theorem 2.  $\square$

We say that the **supercuspidal support** of  $\pi$  is the class of  $(P, \sigma)$  for the equivalence relation appearing in the proposition.

**VI.4.** We give one more consequence mentioned in the introduction.

**Proposition** *Let  $(P, \sigma, Q)$  be a  $B$ -triple. Assume that  $\sigma$  is a supercuspidal (or equivalently, supersingular) representation of  $M$ . Then  $I(P, \sigma, Q)$  is finite-dimensional if and only if  $P = B$  and  $Q = G$ .*

**Proof** As  $Z$  is compact mod centre, any irreducible representation  $\tau$  of  $Z$  is finite dimensional [Hn, Vig2] and consequently supercuspidal. If  $P(\tau) = G$  then  $I(B, \tau, G) = {}^e\tau$  is finite dimensional. Conversely, let  $\pi$  be a finite-dimensional irreducible representation of  $G$ . Then its kernel is an open normal subgroup of  $G$ . Considering  $\iota : G^{\text{is}} \rightarrow G$  as in Chapter II,  $\text{Ker}(\sigma \circ \iota)$  is an open normal subgroup of  $G^{\text{is}}$  which by II.3 Proposition has to be  $G^{\text{is}}$  itself. Thus  $\pi$  is trivial on  $G'$  and since  $G = ZG'$ ,  $\pi$  restricts to an irreducible (supercuspidal) representation  $\tau$  of  $Z$ ; we have  $P(\tau) = G$  and  ${}^e\tau = \pi$ ,  $\pi = I(B, \tau, G)$ .  $\square$

**VI.5.** It is worth noting that our results recover the classifications obtained previously in special cases. Keep the notation of Chapter III. When  $\mathbf{G}$  is split, then for  $\alpha \in \Delta$ ,  $Z \cap M'_\alpha$  is simply the image in  $Z = S$  of the coroot  $\alpha^\vee$ , so our classification is the same as that of [Abe]; it also gives the classification of [He2] for  $\mathbf{G} = \text{GL}_n$ . Other special cases are worth mentioning: groups of semisimple rank 1 and inner forms of  $\text{GL}_n$ . Of course if  $\mathbf{G}$  has relative rank 0, all irreducible representations of  $G$  are finite dimensional and supercuspidal, and our classification theorem says nothing. If  $\mathbf{G}$  has relative semisimple rank 1, the classification is rather simple (see also [BL1, BL2, Abd, Che, Ko, Ly2]). An irreducible admissible representation  $\pi$  of  $G$  falls into one (and only one) of the following cases:

- 1)  $\pi$  is supercuspidal (hence infinite dimensional), i.e.  $\pi \simeq I(G, \pi, G)$ .
- 2)  $\pi$  is finite dimensional; it is then trivial on  $G'$  and restricts to an irreducible representation  $\tau$  of  $Z$ , trivial on  $Z \cap G'$ , and  $\pi \simeq I(B, \tau, G)$ .
- 3)  $\pi \simeq \sigma \otimes \text{St}_B^G$  where  $\sigma$  is as in 2), i.e.  $\pi \simeq I(B, \sigma|_Z, B)$ .
- 4)  $\pi \simeq I(B, \tau, B)$  where  $\tau$  is an irreducible representation of  $Z$  (hence finite dimensional and supercuspidal) which is not trivial on  $Z \cap G'$ .

**VI.6.** Let us briefly consider the case of inner forms of general linear groups. Thus  $\mathbf{G} = \text{GL}_{n/D}$  where  $D$  is a central division algebra of finite degree over  $F$ . We take for  $S$  the diagonal subgroup  $(F^\times)^n$  (so that  $Z$  is the diagonal subgroup  $(D^\times)^n$ ), and for  $B$  the upper triangular subgroup. We can take  $K = \text{GL}_n(\mathcal{O}_D)$  where  $\mathcal{O}_D$  is the ring of integers of  $D$ ; all other special parahoric subgroups of  $G$  are conjugate to  $K$ .

A parabolic subgroup  $P$  of  $G$  containing  $B$  is an upper triangular block subgroup, and the corresponding Levi subgroup  $M$  is the block diagonal subgroup. If the successive blocks down the diagonal have size  $n_1, \dots, n_r$ , then  $M$  appears as  $M_1 \times \dots \times M_r$ ,  $M_i = \text{GL}_{n_i}(D)$  and an irreducible admissible representation of  $M$  factors as a tensor product  $\pi_1 \otimes \dots \otimes \pi_r$ , where  $\pi_i$  is an irreducible admissible representation of  $M_i$  for  $i = 1, \dots, r$  determined up to isomorphism. (Conversely such a tensor product is an irreducible admissible representation of  $M$ : the reader can devise a proof as suggested in [He2], perhaps using [HV2, 7.10 Lemma].) Note that the group  $G'$  is the kernel of the non-commutative determinant  $\det : G \rightarrow F^\times$ . Parameters for the irreducible admissible representations of  $G$  can then be described in a way entirely parallel to the case  $D = F$  obtained in [He2]. (The cases of  $\text{GL}_n(D)$  where  $n \leq 3$  are treated in T. Ly's Ph.D. thesis [Ly2], [Ly3, Chapter 3].)

We simply state the results, leaving to the reader the translation from our classification in this paper.

For  $i = 1, \dots, r$  let  $\pi_i$  be a representation of  $M_i$  which is either supercuspidal or of the form  $\chi_i \circ \det$  for some character  $\chi_i : F^\times \rightarrow C^\times$ ; if for two consecutive indices  $i, i + 1$  we have  $\pi_i = \chi_i \circ \det$  and  $\pi_{i+1} = \chi_{i+1} \circ \det$ , assume  $\chi_i \neq \chi_{i+1}$ .

For each index  $i$  such that  $\pi_i = \chi_i \circ \det$ , choose an upper (block) triangular parabolic subgroup  $Q_i$  of  $M_i$ , and put  $\sigma_i = (\chi_i \circ \det) \otimes \text{St}_{Q_i}^{M_i}$ ; for other indices  $i$  put  $\sigma_i = \pi_i$ . Then  $\text{Ind}_P^G(\sigma_1 \otimes \dots \otimes \sigma_r)$  is irreducible and admissible. Conversely any irreducible admissible representation of  $G$  has such a shape, where the integers  $n_1, \dots, n_r$ , the parabolic subgroups  $Q_i$  of  $M_i = \text{GL}_{n_i}(D)$ , and the isomorphism classes of the  $\pi_i$ , are determined.

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