Lecture 9
Finiteness results in the non-semisimple case

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1. GK dimension of $\pi_v(\bar{r})$
2. Generation by $D_0(\bar{\rho})$
3. Finite length when $f = 2$

Notation. Keep (mostly) the notation in previous lectures.

- K = unramified extension over $\mathbb{Q}_p$ of degree f;
- $\mathcal{O}_K$ = integers of K, $\mathbb{F}_q \cong \mathcal{O}_K/p$;
- $G = \mathrm{GL}_2(K)$, Z =center;
- $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\overline{B} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$;
- I =Iwahori, I_1 =pro-p-Iwahori, $H := \left(\begin{smallmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{smallmatrix}\right) \cong I/I_1$;
- $K_1 = \mathrm{Ker}(\mathrm{GL}_2(\mathcal{O}_K) \to \mathrm{GL}_2(\mathbb{F}_q))$, $Z_1 = Z \cap K_1$;
- $(E, \mathcal{O}, \mathbb{F})$: for coefficients of representations.

Assumptions

Throughout the lecture, $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ will be reducible non-split

$$\bar{\rho} \cong \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & 1 \end{pmatrix}$$

with $3 \leq r_i \leq p-6$. Set

$$\sigma_0 := (r_0, r_1, \dots, r_{f-1})$$

called "ordinary" Serre weight. From Lecture 3, $\sigma_0 \in W(\bar{\rho})$.

Let $\pi_v(\bar{r})$ = admissible smooth $\mathbb{F}$-representation of G in mod p cohomology (cf. Lecture 1) with $\bar{r}|_{F_v} \cong \bar{\rho}$. Assume $r = 1$ (i.e. minimal case). Keep global technical conditions in Lecture 8.

Some useful facts

weight cycling. $\sigma \in W(\bar{\rho})$

$\langle G.\sigma\rangle \supset \langle G.\sigma_0\rangle = \pi_0$ $\quad \sigma_0$

(1) $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_{\sigma_0}) \cap W(\bar{\rho}) = \{\sigma_0\}$. $\quad \sigma^{I_1}$

(2) let $\pi_0 := \langle G.\sigma_0\rangle$, then π_0 is principal series and $\mathrm{soc}_G\pi_v(\bar{r}) = \pi_0$.

(3) (**Le**) $\pi_v(\bar{r})^{K_1} \cong D_0(\bar{\rho})$. $\leftarrow$ BP

(4) If $\mathrm{Ext}^1_K(\sigma, \pi_v(\bar{r})) \neq 0$ for some Serre weight σ, then $\sigma \in W(\bar{\rho})$.

$\pi(\bar{r}) = \tilde{H}^0_m[m]$, $\quad m \subseteq \mathbb{T}$

$m = (x_1, \ldots, x_n)$

$0 \to \pi(\bar{r}) \to \tilde{H}^0_m \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}} (\tilde{H}^0_m)^{\oplus n}$

$\tilde{H}^0_m \twoheadrightarrow Q \to 0$ (injective)

$\Rightarrow \mathrm{Hom}(\sigma, Q) \to \mathrm{Ext}^1(\sigma, \pi(\bar{r})) \to 0$

$\mathrm{Hom}(\sigma, Q) \hookrightarrow \mathrm{Hom}(\sigma, \tilde{H}^0_m)^{\oplus n}$

$\Rightarrow \sigma \in W(\bar{\rho})$.

Some useful facts

(1) $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_{\sigma_0}) \cap W(\bar{\rho}) = \{\sigma_0\}$.

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(i) (5) **(H., Breuil-Ding)** $\mathrm{Ord}_B(\pi_v(\bar{r}))$ is semisimple (as T-rep.).

$\mathrm{Ord}_B : \mathrm{Rep}_G^{sm} \longrightarrow \mathrm{Rep}_T^{sm}$

(Emerton)

- $\mathrm{Hom}_G(\mathrm{Ind}_{\bar{B}}^G \tau, \pi) \simeq \mathrm{Hom}_T(\tau, \mathrm{Ord}_B \pi)$
- $\mathrm{Ord}_B(\mathrm{Ind}_{\bar{B}}^G \tau) \simeq \tau$. $\Rightarrow \mathrm{Ord}_B(\mathrm{s.s}) = 0$

$\pi_0 \hookrightarrow \pi(\bar{r})$

excludes

$\left(\pi_0 \text{ — } \pi_0\right) \hookrightarrow \pi(\bar{r})$

Theorem 1 (H.-Wang)

The Gelfand-Kirillov dimension of $\pi_v(\bar{r})$ is f.

As in Lecture 6, can deduce that $\pi_v(\bar{r})^\vee$ is Cohen-Macaulay module of grade $2f$ (over $\Lambda := \mathbb{F}[\![I_1/Z_1]\!]$), and essentially self-dual.

Theorem 1 (H.-Wang)

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Strategy of the proof : (lecture 8).

$\mathfrak{m}_{K_1}$ = max ideal of $\mathbb{F}[\![K_1/Z_1]\!]$

$\mathfrak{m}_{I_1}$ = - - - - Λ

(1) Show $[\pi_v(\bar{r})[\mathfrak{m}_{K_1}^2] : \sigma_0] = 1$.

(2) Show $[\pi_v(\bar{r})[\mathfrak{m}_{K_1}^2] : \sigma] = 1$ for any $\sigma \in W(\bar{\rho})$.

(2') Show $[\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3] : \chi] = 1$ for any $\chi \in \pi_v(\bar{r})^{I_1}$.

(3) As in Lecture 6, deduce $\mathrm{GK}(\pi_v(\bar{r})) \leq f$ (need to use results of [BHHMS1]). $\rightsquigarrow$ $\mathrm{gr}(\Lambda)$, $J = (y_i z_i, z_i y_i)$, $\mathrm{gr}(\pi(F)^\vee)$ is killed by J.
Conclude by Gee-Newton.

Steps (2), (2')

These steps are purely representation theoretic.

Proposition 2

Let $\bar{\rho}$ be as above, π be an admissible smooth $\mathbb{F}$-rep. of G satisfying :

(a) $\pi^{K_1} \cong D_0(\bar{\rho})$; $\Rightarrow \mathrm{Soc}_{GL_2(\mathcal{O}_K)}\pi = \bigoplus_{\sigma\in W(\bar{\rho})} \sigma$

(b) if $\mathrm{Ext}^1_K(\sigma,\pi) \neq 0$ for some Serre weight σ, then $\sigma \in W(\bar{\rho})$;

Step (1) (c) there exists one $\sigma_0 \in W(\bar{\rho})$ such that $[\pi[\mathfrak{m}_{K_1}^2] : \sigma_0] = 1$.

Then the following hold :

(i) $[\pi[\mathfrak{m}_{K_1}^2] : \sigma] = 1$ for any $\sigma \in W(\bar{\rho})$; (2)

(ii) $[\pi[\mathfrak{m}_{I_1}^3] : \chi] = 1$ for any $\chi \in \pi^{I_1}$. (2')

$\pi: \pi^{I_1} \hookrightarrow \pi^{K_1}$ $= \pi^{K_1} = \bigoplus_{\sigma \in W(\bar{\rho})} D_{0,\sigma}(\bar{\rho})$

Proof. Recall : the diagram $(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$ is indecomposable by Lecture 3.

Define $\Sigma_0 := \{\sigma \in W(\bar{\rho}) : [\pi[\mathfrak{m}_{K_1}^2] : \sigma] = 1\}$

$\Sigma_1 := \{\chi \in \pi^{I_1} : [\pi[\mathfrak{m}_{I_1}^3] : \chi] = 1\} \subseteq \pi^{I_1} \circlearrowleft \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$

$\circlearrowleft \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$

use (a), (b) to show that:

if $\chi \in D_{0,\sigma}(\bar{\rho})^{I_1}$, then $\chi \in \Sigma_1$ iff $\sigma \in \Sigma_0$.

$$\Rightarrow \left(\bigoplus_{\chi \in \Sigma_1} \chi \hookrightarrow \bigoplus_{\sigma \in \Sigma_0} D_{0,\sigma}(\bar{\rho}) \right)$$

Sub. diag.

$$\begin{array}{ccc} \bigoplus_{\chi \in \Sigma_1} \chi & \hookrightarrow & \bigoplus_{\sigma \in \Sigma_0} D_{0,\sigma}(\bar{\rho}) \\ \cap & & \cap \\ D_1(\bar{\rho}) & \hookrightarrow & D_0(\bar{\rho}) \end{array}$$

direct summand!

(c) subdiag $\neq 0$

$\Rightarrow$ get the whole diagram. $\square$

Step (1) : Show $[\pi_v(\overline{r})[\mathfrak{m}_{K_1}^2] : \sigma_0] = 1.$ ordinary

Let $\Gamma := \mathrm{GL}_2(\mathbb{F}_q)$ so that $\mathbb{F}[\Gamma] \cong \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}$. Let

$$\widetilde{\Gamma} := \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2.$$ not a group alg.

Let $\mathrm{Proj}_\Gamma\sigma_0$, resp. $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$ be a projective envelope of σ_0 for Γ-representations, resp. $\widetilde{\Gamma}$-representations. over $\mathbb{F}$

Have

$$\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0 \twoheadrightarrow \mathrm{Proj}_\Gamma\sigma_0.$$ (Lecture 8)

Need to show

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}\big(\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0, \pi_v(\overline{r})\big) \overset{?}{=} 1.$$

Let M_∞ (and R_∞) be a *minimal* patching functor for $\bar{\rho}$ (cf. Lecture 8), e.g. take

$$M_\infty(-) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_2(\mathcal{O}_K)}(\underline{\mathbb{M}_\infty}, -^d)^d \quad \leftarrow \text{dual}$$

for a minimal patched module $\mathbb{M}_\infty$.

Recall that $\mathbb{M}_\infty/\mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$, so we have ← max ideal of R_∞

$$M_\infty(\Theta)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta, \pi_v(\bar{r}))^\vee.$$

$= \dim 1$

Equiv. to show

Theorem 3

The R_∞-module $M_\infty(\mathrm{Proj}_{\tilde{\Gamma}}\sigma_0)$ is cyclic.

Recall gluing lemma 2 of Lecture 8 :

Given finite dim. $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-modules Θ_1, Θ_2 which admit a common quotient Θ_0, form the fiber product

$$\Theta_1 \times_{\Theta_0} \Theta_2 := \{(x_1, x_2) : x_1 = x_2 \text{ in } \Theta_0\}$$

← exact

Apply $M_\infty(-)$ to get

$$0 \to M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2) \to M_\infty(\Theta_1) \times M_\infty(\Theta_2) \to M_\infty(\Theta_0) \to 0.$$

Assume both $M_\infty(\Theta_1)$, $M_\infty(\Theta_2)$ are cyclic R_∞-modules with annihilator I_1, I_2 (hence so is $M_\infty(\Theta_0)$ with annihilator I_0), then

$$M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2) \text{ is cyclic } \iff I_1 + I_2 = I_0.$$

Roughly, we glue $\mathrm{Proj}_\Gamma \sigma_0$ with an ordinary part of $\mathrm{Proj}_{\tilde{\Gamma}} \sigma_0$:

- $\Theta_1 := \mathrm{Proj}_\Gamma \sigma_0$.

  Theorem (Le) The R_∞-module $M_\infty(\mathrm{Proj}_\Gamma \sigma_0)$ is cyclic.

$\Theta_1 \twoheadrightarrow$

- $\Theta_0 := \mathrm{Ind}_{B(\mathbb{F}_q)}^{\Gamma} \chi_{\sigma_0}$ (a quotient of Θ_1). $\rightsquigarrow$ $\underline{M_\infty(\Theta_0) \text{ cyclic}}$

  $\twoheadrightarrow \sigma_0$

  Fact (1). $JH(\Theta_0) \cap W(\bar{\rho}) = \{\sigma_0\}$

  $\Rightarrow M_\infty(\Theta_0) = M_\infty(\sigma_0)$

- $\Theta_2 :=$ **ordinary part** of $\mathrm{Proj}_{\tilde{\Gamma}} \sigma_0$.

  Fact. There exists a (unique) quotient Θ_2 of $\mathrm{Proj}_{\tilde{\Gamma}} \sigma_0$ such that :

$$0 \to \sigma_0^{\oplus f} \to \Theta_2 \to \Theta_0 \to 0. \quad \text{not } \Gamma\text{-ext}$$

Θ_0 non-split

$\sigma_0 \quad \sigma_0 \quad \sigma_0$

Frob: $\mathrm{Ext}^1_K(\Theta_0, \sigma_0) \simeq \mathrm{Ext}^1_{I/Z_1}(\chi_{\sigma_0}, \sigma_0) = \dim f.$

Structure of Θ_i $(f=1)$

$\sigma_0 = \mathrm{Sym}^r = (r)$

Θ_1 : $\mathrm{Proj}_{GL_2(\mathbb{F}_p)}(r)$,

$$\begin{tikzcd} & (r) \arrow[dl, dash] \arrow[dr, dash] & \\ (p-1-r) \arrow[dr, dash] & & (p-3-r) \arrow[dl, dash] \\ & (r) & \end{tikzcd}$$

$\leftarrow$ cosocle.

Socle. $\longrightarrow$ (r)

Θ_0 :

$$\begin{tikzcd} (r) \arrow[d, dash] \\ (p-1-r) \end{tikzcd}$$

Θ_2 :

$$\begin{tikzcd} (r) \arrow[d, dash] \\ (p-1-r) \arrow[d, dash] \\ (r) \end{tikzcd}$$

Θ_0

Rk: is smaller than $\mathrm{Proj}_{\tilde{\Gamma}}\sigma_0$

but $M_\infty(\mathrm{Proj}_{\tilde{\Gamma}}\sigma)$ cyclic

$\Updownarrow$

$\boxed{M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2)}$ cyclic.

$\Theta_1 \times_{\Theta_0} \Theta_2$:

$$\begin{tikzcd} & (r) \arrow[dl, dash] \arrow[dr, dash] & \\ (p-1-r) \arrow[d, dash] & & (p-3-r) \arrow[d, dash] \\ (r) & & (r) \end{tikzcd}$$

remind: I-rep:

$$\begin{tikzcd} & \chi \arrow[dl, dash] \arrow[dr, dash] & \\ \chi\alpha \arrow[d, dash] & & \chi\alpha^{-1} \arrow[d, dash] \\ \chi & & \chi \end{tikzcd}$$

Cyclicity of $M_\infty(\Theta_2)$

Lemma

Let π be admissible $\mathbb{F}$-rep. of G. Assume

- $\mathrm{JH}(\Theta_0) \cap \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi = \{\sigma_0\}$ $\leftarrow$ $\pi = \pi(\bar{r})$ ok
- $\mathrm{Ord}_B(\pi)$ is semisimple. $\leftarrow$ (5)

Then the projection $\Theta_2 \twoheadrightarrow \sigma_0$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_0, \pi) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_2, \pi).$$

$\mathrm{Hom}(\sigma_0, \pi(\bar{r})) = \dim 1$ $\qquad$ $= \dim 1$

$\underline{f=1}$: $\left[\Theta_2 := \begin{matrix} & \Theta_0 \\ \sigma_0 & \end{matrix}\right] \xrightarrow{\beta} \pi$

(a) β is not injective. β factor through $\Theta_0 \to \pi$, $\hookrightarrow$

$\Theta_0 \to \sigma_0 \to \pi$,

$\langle G.\Theta_2\rangle \simeq \left(\begin{matrix} & \pi_0 \\ \pi_0 & \end{matrix}\right)$ non-split

$\not\subset$ semisimple of $\mathrm{Ord}_B(\pi)$. $\square$

(b) β is injective. $\begin{matrix} & & \sigma_0 \\ & * & \\ \sigma_0 & & \end{matrix} \hookrightarrow \pi \ldots$,

Proof of $I_1 + I_2 = I_0$

Can work locally : replace R_∞ with $R_{\bar{\rho}}$. $\leftarrow$ univ. def ring (Mazur).

$= \mathrm{Ann}(M_\infty(\Theta_0)) = \mathrm{Ann}(M_\infty(\sigma_0)) = \mathrm{Ann}(M_\infty(\mathrm{Proj}_\Gamma \sigma_0))$

- have an explicit description of I_0 (Fontaine-Laffaille) and I_1 (Le) ;
- the action of R_∞ on $M_\infty(\Theta_2)$ factors through $R^{\mathrm{red}}_{\bar{\rho}}$ (:=reducible deformation ring), i.e. $I^{\mathrm{red}} \subset I_2$. $\rightarrow$ parametrizing reducible def of $\bar{\rho}$.
- show $I^{\mathrm{red}} + I_1 = I_0$.

Lemma: π loc adm, s.t. $JH(\Theta_0) \cap \mathrm{Soc}_{GL_2(\mathcal{O}_K)} \pi = \{\sigma_0\}$

then $\pi^{\mathrm{ord}} \hookrightarrow \pi$ induces an isom.

$$\mathrm{Hom}_{GL_2(\mathcal{O}_K)}(\Theta_2, \pi^{\mathrm{ord}}) \xrightarrow{\sim} \mathrm{Hom}(\Theta_2, \pi)$$

$\pi^{\mathrm{ord}} := $ Image of

$$\mathrm{Ind}_B^G \mathrm{Ord}_B \pi \longrightarrow \pi.$$

Apply $\pi = M_\infty^\vee \rightsquigarrow$. $\mathrm{Hom}(\Theta_2, (M_\infty^\vee)^{\mathrm{ord}}) \simeq \mathrm{Hom}(\Theta_2, M_\infty^\vee)$.

$R_\infty \twoheadrightarrow R_{\bar{\rho}}^{\mathrm{red}}[[x]]$ $\quad R_\infty$,

R_∞ on $\mathrm{Ord}(M_\infty^\vee)$ factors through $R_{\bar{\rho}}^{\mathrm{red}}[[x]]$.

Proof of $I_1 + I_2 = I_0$

Can work locally : replace R_∞ with $R_{\bar{\rho}}$.

- have an explicit description of I_0 (Fontaine-Laffaille) and I_1 (Le) ;
- the action of R_∞ on $M_\infty(\Theta_2)$ factors through $R_{\bar{\rho}}^{\mathrm{red}}$ (:=reducible deformation ring), i.e. $I^{\mathrm{red}} \subset I_2$.
- show $I^{\mathrm{red}} + I_2 = I_0$.

Example. $R_{\bar{\rho}}^{\psi} = \mathcal{O}[[X_i, Y_i, Z_i]]_{1\le i\le f} \leftarrow 3f\ /\mathcal{O}$

$W(\bar{\rho}) = \{\sigma_0\}$

$\Updownarrow$

$\bar{\rho}$. max non-split

$I_0 = (\varpi, Y_i, Z_i)$

$I_1 = (\varpi, Y_i, Z_i(Z_i - p)) \leftarrow$ Kisin. dim $f\ /\mathcal{O}$

$I^{\mathrm{red}} = (\varpi, Z_i)$. | Krull-dim $R^{\mathrm{red}} = 2f\ /\mathcal{O}$.

$\rightsquigarrow I_1 + I^{\mathrm{red}} = I_0$.

1. GK dimension of $\pi_v(\bar{r})$
2. Generation by $D_0(\bar{\rho})$
3. Finite length when $f = 2$

The main result of this section is :

Theorem 4 (H.-Wang)

As a G-representation, $\pi_v(\bar{r})$ is generated by $D_0(\bar{\rho})$.

Corollary

We have $\mathrm{End}_G(\pi_v(\bar{r})) = \mathbb{F}$.

corresp to $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$.

Pf: $\mathrm{End}_G(\pi(\bar{r})) \longrightarrow \mathrm{End}_{\text{Diag.}}((D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho})))$.

is injective.

$= \mathbb{F}$

[BP] : indecomp + multi free.

$\square$

Example/Motivation

input: $\pi[\mathfrak{m}_{K_1}^2]$ is multi.-free (G_0)

Take $f = 1$, so $W(\bar\rho) = \{\sigma_0\}$ $(= \mathrm{Sym}^r)$, $\pi_v(\bar r) \cong (\pi_0 — \pi_1)$, with π_i PS.

$$\begin{tikzcd} D_0(\bar\rho) = (p-1-r) \arrow[d, dash] \arrow[r, dash] & \underline{(p-3+r)} \arrow[r, hook] & \pi_1 \arrow[d, dash] \\ (r) \arrow[rr, hook] & & \pi_0 \end{tikzcd}$$

$\Rightarrow \pi(\bar r) = \langle G \cdot D_0(\bar\rho) \rangle$

Let $\Omega \cong \mathrm{Inj}_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}\sigma_0$ together with a smooth action of G and assume $\pi_v(\bar r) \hookrightarrow \Omega$ (cf.[**BP**]). or. $\Omega = \tilde{H}^0_{\mathfrak{m}}$

Paškūnas : if $\pi_v(\bar r) \subset \pi \subset \Omega$ with $\pi[\mathfrak{m}_{K_1}^2]$ mulitiplicity free, then

$$\pi = \pi_v(\bar r).$$

Structure of Ω :

$$\mathcal{E} = \left[\begin{array}{c} \pi(\bar r) \oplus \pi(\bar r) \\ \pi(\bar r) \end{array}\right] \hookrightarrow \Omega/\pi(\bar r).$$

$$\underset{\text{2-dim}}{\mathcal{E}} \hookrightarrow \mathrm{Ext}^1(\pi(\bar r), \pi(\bar r))$$

Fact: $\pi = \pi(\bar r)$ iff $\pi \cap \mathcal{E} = \pi(\bar r)$.

$G_0 \hookrightarrow \pi(\bar{r})$

$$\mathrm{Ext}^1_G(\pi(\bar r), \pi(\bar r)) \longrightarrow \mathrm{Ext}^1_{GL_2(O_K)}(G_0, \pi(\bar r)) \xleftarrow{\sim} \mathrm{Ext}^1(G_0, D_0(\bar\rho)) \leftarrow \cdots$$

$D_0(\bar\rho)$

$\mathcal{E} \overset{\sim}{\longleftrightarrow}$

$\mathcal{E}$ = 2-dim

= dim 2. (Paskunas)

= 2-dim

$\mathrm{Hom}(G_0, \pi(\bar r)/D_0(\bar\rho))$

$= \mathrm{Hom}(G_0, \pi(\bar r)[\mathfrak{m}_{K_1}^2]/D_0(\bar\rho))$

no G_0

$= 0$!

(condition $\underline{[\pi(\bar r)[\mathfrak{m}_{K_1}^2] : G_0] = 1}$)

need $\pi \cap \mathcal{E} = \pi(\bar r)$

If not, $\begin{pmatrix} & G_0 \\ D_0(\bar\rho) & \end{pmatrix} \hookrightarrow \pi(\bar r)$

$\hookrightarrow \pi(\bar r)[\mathfrak{m}_{K_1}^2]$.

$\square$

The proof of Theorem 4

The starting point is :

Lemma 5

The *G*-cosocle of $\pi_v(\bar{r})$ is an irreducible PS, say π_f.

Proof. $\mathrm{GK}(\pi(\bar{r})) = f \rightsquigarrow \pi(\bar{r})^\vee$ is ess. self-dual.

$\pi_0 \hookrightarrow \pi(\bar{r})$ is socle

$\rightsquigarrow \pi(\bar{r})$ has cosocle. $E^{2f}(\pi_0^\vee)^\vee$ (up to twist).

$$\begin{array}{ll} \| & \\ \pi_f, & \text{(Kohlhaase)} \\ & \hookrightarrow \underline{\underline{\mathrm{PS}}} \end{array}$$

$\square$

Criterion

Let $\tau \subset \pi_v(\bar{r})|_I$. If for some i, some $\chi : I \to \mathbb{F}^\times$, the composition

$$\begin{tikzcd} & \mathrm{Ext}^i_I(\chi, \tau) \arrow[dl, dashed] \arrow[d, "\beta_i"] & \\ \mathrm{Ext}^i_I(\chi, V) \arrow[r] \arrow[rr, bend right, "0"'] & \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) \arrow[r, "\gamma_i"] & \mathrm{Ext}^i_I(\chi, \pi_f) \end{tikzcd}$$

$\Rightarrow \gamma_i \circ \beta_i = 0$

is non-zero, then $\pi_v(\bar{r})$ can be generated by τ as G-representation.

We will find some χ, i, τ such that "Criterion" applies.

$0 \to V \to \pi(\bar{r}) \twoheadrightarrow \pi_f$,

$\uparrow$ radical of $\pi(\bar{r})$.

subrep $\langle G.\tau \rangle \subsetneq \pi(\bar{r})$. iff $\langle G.\tau \rangle \subseteq V$.

How to choose χ, i and τ ?

Assume $W(\bar{\rho}) = \{\sigma_0\}$ for simplicity. Know the following information :

- $\mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) \neq 0$ if and only if $\chi \in \pi_v(\bar{r}))^{I_1}$ and

$$\dim_{\mathbb{F}} \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) = \binom{2f}{i}.$$

if $f=1$:

χ_{σ_0} , $\chi_{\sigma_0}^s$

This suggests to take : $\chi = \chi_{\sigma_0}$ (the ordinary character).

roughly: injective resolution of $\pi(\bar{r})$ using M_∞. $/R_\infty$ flat. fiber is $\pi(\bar{r})^\vee$

$m_\infty = (x_i)$

$K_\bullet(\underline{x}, M_\infty)$: $\cdots \to M_\infty^n \to M_\infty \to \pi_v(\bar{r})^\vee \to 0$

$\rightsquigarrow$ not minimal. (a_∞ of S_∞)

How to choose χ, i and τ ?

Assume $W(\bar{\rho}) = \{\sigma_0\}$ for simplicity. Know the following information :

- $\mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) \neq 0$ if and only if $\chi \in \pi_v(\bar{r}))^{I_1}$ and
$$\dim_{\mathbb{F}} \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) = \binom{2f}{i}.$$
This suggests to take : $\chi = \chi_{\sigma_0}$ (the ordinary character).
- π_f has injective dimension $2f$, and
$$\dim_{\mathbb{F}} \mathrm{Ext}^i_I(\chi, \pi_f) = \begin{cases} 0 & i < f \\ \binom{f}{2f-i} & f \leq i \leq 2f \end{cases}$$
This suggests to take $i = 2f$. if $i=2f$, 1-dim.

Can't take $i=1$.

- The multiplicity-freeness of $\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3]$ suggests : if take $\tau = \pi_v(\bar{r})[\mathfrak{m}_{I_1}^2]$ then

$\hookrightarrow \pi(\bar{r})$

$$\dim_{\mathbb{F}} \mathrm{Ext}_I^1(\chi, \tau) = 2f$$

and the map

$\gamma_i \circ \beta_i \neq 0.$

$$\beta_1 : \underset{2f}{\mathrm{Ext}_I^1(\chi, \tau)} \to \underset{2f}{\mathrm{Ext}_I^1(\chi, \pi_v(\bar{r}))}$$

$i = 2f$

$\gamma_{2f} \circ \beta_{2f} \neq 0.$

is an isomorphism.

$\mathrm{Hom}_I(\chi, \pi(\bar{r})/\tau) = 0$

impl β_1 is an isom.

In summary, in the diagram of "Criterion"

$$\mathrm{Ext}^i_I(\chi,\tau) \xrightarrow{\beta_i} \mathrm{Ext}^i_I(\chi,\pi_v(\bar{r})) \xrightarrow{\gamma_i} \mathrm{Ext}^i_I(\chi,\pi_f)$$

take

- $\chi = \chi_{\sigma_0}$
- $i = 2f$
- $\tau =$ a variant of $\pi_v(\bar{r})[\mathfrak{m}^2_{I_1}]$.

Show

(1) γ_{2f} is an isomorphism (easier);

(2) β_{2f} is a surjection ~~for any $0 \le i \le 2f$~~. Actually, inductively show β_i is surjective for any $0 \le i \le 2f$.

Step (2) : β_i is surjective

To deduce $\overset{\text{sur.}}{\cancel{\text{in}}}$jectivity of β_{2f} from that of $\overset{\text{isom.}}{\underline{\beta_0, \beta_1}}$, need :

Key ingredient : $\pi_v(\bar{r})^\vee|_I$ admits a Koszul complex projective resolution, as M_∞ is flat over R_∞ (which is regular) and $M_\infty/\mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$.

Example. when $f = 1$, $\pi_v(\bar{\rho}) = (\pi_0 — \pi_1)$, Paškūnas shows :

$$0 \to \Omega^\vee \xrightarrow{(-y,x)} \Omega^\vee \oplus \Omega^\vee \xrightarrow{\binom{x}{y}} \Omega^\vee \to \pi_v(\bar{r})^\vee \to 0.$$

$$\mathrm{End}_G(\Omega^\vee) \cong \mathbb{F}[\![x, y]\!]$$

Consider the following situation : $(R,\mathfrak{m})$ = noetherian local ring, $\underline{x} := (x_1,\ldots,x_n)$ with $x_i \in \mathfrak{m}$. Assume

$$\begin{tikzcd} \cdots \arrow[r] & K_2 \arrow[r] \arrow[d, "\widetilde{\beta}_2"] & K_1 \arrow[r] \arrow[d, "\widetilde{\beta}_1"] & K_0 \arrow[r] \arrow[d, "\widetilde{\beta}_0"] & 0 \\ \cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0 \end{tikzcd}$$

where

- $K_\bullet = K_\bullet(\underline{x}, R)$ = is Koszul complex, with $K_i \cong R^{\oplus\binom{n}{i}}$
- $F_\bullet$ = complex of free R-modules.

Lemma (Serre)

Assume

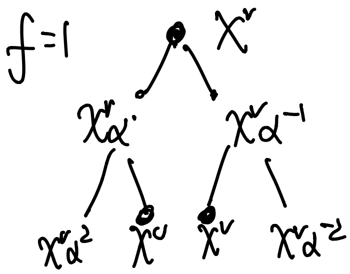
(a) $x_1,\ldots,x_n$ are linearly independent $\underline{\text{mod } \mathfrak{m}^2}$; $\longrightarrow$ our condition on β,

(b) $\widetilde{\beta}_0 : K_0 \to F_0$ is a $\underline{\text{direct summand.}}$

Then $\widetilde{\beta}_i : K_i \to F_i$ is a direct summand for all $0 \le i \le n$.

In practice, can not take $R = R_\infty$ in Serre's lemma, as R_∞ does not act on an injective resolution of τ.

To solve this, let $\lambda := (\mathrm{Proj}_I \chi^\vee)/\mathfrak{m}_{I_1}^3$ so that



$$\mathrm{End}_I(\lambda) \cong \mathbb{F}[x_i, y_i; 0 \le i \le f-1]/(x_i, y_i)^2.$$

Choose minimal projective resolutions :

but $R_\infty \not\circlearrowright$ $\quad$ $\circlearrowright R_\infty$ $\quad$ $\pi(\bar{r})^\vee$

$$Q_\bullet \twoheadrightarrow \tau^\vee, \quad K_\bullet \twoheadrightarrow \pi_v(\bar{r})^\vee$$

$\downarrow$ $\tau^\vee$ $\quad$ Hom$(-, \lambda)$

with $K_\bullet$ being Koszul, get morphisms $\mathrm{Hom}_I(K_\bullet, \lambda)^\vee \to \mathrm{Hom}_I(Q_\bullet, \lambda)^\vee$ of $\mathrm{End}_I(\lambda)$-modules, and

R $\quad$ injective. $\quad$ R

- Serre's lemma applies with $R := \mathrm{End}_I(\lambda)$. Actually get $\widetilde{\beta}_i$ are isomorphisms.
- $\mathbb{F} \otimes_R \mathrm{Hom}_I(Q_\bullet, \lambda)^\vee$ recovers $\boxed{\mathrm{Hom}_I(Q_\bullet, \chi^\vee)^\vee}$

$K_\bullet$ $\quad$ $K_\bullet$

input: $\widetilde{\beta}_0, \widetilde{\beta}_1$ isom

$\Updownarrow$

$(x_1, \ldots, x_n)$ lin. indep mod $\mathfrak{m}^2$.

1. GK dimension of $\pi_v(\bar{r})$

2. Generation by $D_0(\bar{\rho})$

3. Finite length when $f = 2$

Theorem 6 (H.-Wang)

If $\bar{\rho}$ is reducible non-split, then $\pi_v(\bar{r})$ has the form

Socle $\pi_0 - \pi' - \pi_f$ cosocle.

with π_0, π_f principal series. If moreover $f = 2$, then π' is irreducible and supersingular.

Already know : the G-socle of $\pi_v(\bar{r})$ is π_0 and G-cosocle is π_f.

Assume $f = 2$. Need to show π' is irreducible and supersingular.

Proof. $f=2$.

look at $\pi(\bar{r})/\pi_0$. adm. $\Rightarrow$ always have an irred sub-rep. say π'.

$\Rightarrow \mathrm{Ext}^1_G(\pi', \pi_0) \neq 0$.

① claim π' is s.s.

Fact [BP]. if π' is non-s.s, and if $\pi' \neq \pi_0$. then $\mathrm{Ext}^1_G(\pi', \pi_0) = 0$!

only $\begin{matrix}\pi_0 \\ | \\ \pi_0\end{matrix}$ but $\mathrm{Ord}_B(\pi(\bar{r}))$ is semi-simple.

② Show. π' is G-socle of $\pi(\bar{r})/\pi_0$.

determine $\mathrm{Soc}_{GL_2(O_K)}(\pi(\bar{r})/\pi_0)$. Assume $W(\bar{\rho}) = \{\sigma_0\}$

$W(\bar{\rho}^{ss}) = \{\sigma_0, \sigma_1 \leftrightarrow \sigma_1^{[s]}, \sigma_2\}$

$D_0(\bar{\rho})$:

$$\begin{matrix} & \sigma_2 & \\ \sigma_1 & & \sigma_1^{[s]} \\ & \sigma_0 & \end{matrix}$$

socle $\sigma_0 \hookrightarrow \pi_0$, $\sigma_1, \sigma_1^{[s]} \hookrightarrow \pi'$

$\| \quad \sigma_1 \oplus \sigma_1^{[s]}$; $\begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix}$

$\Rightarrow \pi' := \langle G.(\sigma_1 \oplus \sigma_1^{[s]}) \rangle \subseteq \pi(\bar{r})/\pi_0$.

$\begin{pmatrix}\pi' \\ | \\ \pi_0\end{pmatrix} \hookrightarrow \pi(\bar{r})$, s.t. $D_0(\bar{\rho}) / D_0(\bar{\rho}) \cap \begin{pmatrix}\pi' \\ | \\ \pi_0\end{pmatrix} = \sigma_2$

$\twoheadrightarrow Q$

Lemma

Let Q be an admissible quotient of $I(\sigma_2) := \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)Z}^{G}\,\sigma_2$. Assume the G-cosocle of Q is irreducible and isomorphic to

$$\pi_2 := I(\sigma_2)/(T-\lambda)$$

for some $\lambda \in \mathbb{F}^\times$. Then

$$\boxed{Q \cong I(\sigma_2)/(T-\lambda)^n}, \quad \text{some } n \geq 1.$$

$\simeq \underbrace{\pi_2 - \pi_2 - \pi_2 - \cdots - \pi_2}_{n \text{ copy.}}$

Know: $\pi(\bar r)$ is generated by $D_0(\bar\rho)$.

$$0 \to \begin{pmatrix} \pi' \\ | \\ \pi_0 \end{pmatrix} \hookrightarrow \pi(\bar r) \to \underline{\underline{Q}} \to 0$$

$\sigma_1 \quad \sigma_1^{[s]}$

σ_0

quotient will be generated by $\underline{\underline{\text{the image of } D_0(\bar\rho)}}$ in Q

$= \sigma_2$.

need $Q = \pi_2$. ie $n=1$.

$\pi(\bar r)$ is self-dual.

$\underbrace{\pi_0 - \pi_0 - \cdots \pi_0}_{n} \hookrightarrow \pi(\bar r)$.

$\square$

$\bar{\rho}$ non-split $\quad \pi_0 - \pi' - \pi_2$

$\bar{\rho}^{ss} \quad \pi_0 \oplus \pi' \oplus \pi_2$

G-N: $M_\infty \circlearrowleft R_\infty$: $M_\infty / \mathfrak{m}_\infty = \pi(\bar{r})^{\vee}$.

$\text{GK-dim}(M_\infty) - \text{Krull-dim}\, R_\infty = f$

and M_∞ is flat over R_∞ iff $M_\infty / (x_1, \dots, x_n)$ is flat over $R_\infty / (x_1, \dots, x_n) = \mathbb{F}$, $(x_1, \dots, x_n) = \mathfrak{m}_\infty$

R_∞ = regular ring: $= (x_1, \dots, x_n) = \mathfrak{m}_\infty$

regular sequence:

$\text{GK}(M_\infty / x_1) \geq \text{GK}(M_\infty) - 1$

with equality if x_1 is regular for M_∞

$\Rightarrow \text{GK}(M_\infty / (x_i)) \geq f$

$\text{GK}(M_\infty / (x_i)) = \text{GK}(\pi(\bar{r})^{\vee})$

CM-module $=$ then $(x_1, \dots, x_n)$ is reg. sequence for M_∞.

Thank you !