§ Introduction

<u>Recall</u> $K = \mathbb{Q}_{p^f}$ $\quad \mathbb{F}/\mathbb{F}_p \twoheadleftarrow \mathcal{O}/\mathbb{Z}_p \longrightarrow E/\mathbb{Q}_p$ coefficients

Fix $\bar{\rho} : G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \longrightarrow GL_2(\mathbb{F})$ continuous

Mazur ([Ma89]): Have $R^{\square}_{\bar{\rho}}/\mathcal{O}$ complete local noetherian

$$\begin{tikzcd}
G_K \arrow[r, "\rho^{\square}"] \arrow[dr, "\rho"] \arrow[ddr, bend right, "\bar{\rho}"'] & GL_2(R^{\square}_{\bar{\rho}}) \arrow[d, dashed, "\exists!\, R^{\square}_{\bar{\rho}} \to A"] \\
 & GL_2(A) \arrow[d, two heads] \\
 & GL_2(\mathbb{F})
\end{tikzcd}$$

s.t.

framed deformation

local artinian $\mathcal{O}$-algebra, residue field $\mathbb{F}$

Problem: properties of subspaces of $\mathrm{Spec}(R^{\square}_{\bar\rho})$ cut out by

p-adic Hodge theory conditions

(i) No natural moduli interpretation

(Fontaine-Laffaille [FL

integral p-adic Hodge theory Br 97, 98, 99)

(ii) Needed in e.g. modularity lifting

(global nature)

(local/global compatibility gives conditions at $v \mid p$)

$$\mathrm{Hom}_{G_F}\left(\bar r, \varinjlim_{U_v} H^1_{\acute et}\left(X_{U^v U_v} \times_F \bar F, \mathcal{O}\right)\right)$$

$$\begin{tikzcd} R_{\bar r|_{G_{F_v}}} \arrow[r] & R_{\bar r} \arrow[r] & \overset{\circlearrowleft}{\overline{\mathbb{T}}} \end{tikzcd}$$

More precisely:

find $Y^{p\text{-HT}}_{\bar\rho} \subseteq \mathrm{Spec}(R^{\square}_{\bar\rho})$

s.t. $x \in Y^{p\text{-HT}}_{\bar\rho}(\bar E)$

$\Longleftrightarrow$

$G_K \xrightarrow{\rho^{\square}} GL_2(R^{\square}_{\bar\rho}) \xrightarrow{x} GL_2(\bar E)$ satisfies$_{p\text{-HT}}$

(local nature: Breuil-Mézard conjecture)

Theorem (Kisin [Ki08]) $\exists\ R^{\square}_{\bar\rho} \twoheadrightarrow R^{\square,\lambda,\tau}_{\bar\rho}$ p-flat s.t.

$$\begin{tikzcd} R^{\square}_{\bar\rho} \arrow[r, two heads] \arrow[d, "\rho_x"'] & R^{\square,\lambda,\tau}_{\bar\rho} \arrow[dl, dashed, "\exists"] \\ E & \end{tikzcd}$$

iff ρ_x is potentially crystalline,
Hodge-Tate weights λ
inertial type τ] cf. Lecture 6

Starting point: $\operatorname{Spec}(R^{\square}_{\bar\rho}[1/p]) \longrightarrow \operatorname{Spec}(R^{\square}_{\bar\rho})$

$$\begin{tikzcd} \operatorname{Spec}(R^{\square}_{\bar\rho}[1/p]) \arrow[r] \arrow[d, phantom, "\cup" rotate=90] & \operatorname{Spec}(R^{\square}_{\bar\rho}) \arrow[d, phantom, "\cup" rotate=90] \\ \left\{ \begin{array}{l} x : R^{\square}_{\bar\rho}[1/p] \to E \\ \rho_x \text{ is pcris, type}(\lambda,\tau) \end{array} \right\} \arrow[r, phantom, "\subseteq"] & \overline{\left\{ \quad \right\}} \end{tikzcd}$$

Zariski closure

Problem: are all points here of type(λ,τ)?

Idea: Construct

$$\begin{tikzcd} Y^{\lambda,\tau}_{\bar\rho} \arrow[r, "\text{proper (projective)}"'] & \operatorname{Spec}(R^{\square}_{\bar\rho}) \end{tikzcd}$$

same geometric points

$\boxed{\text{moduli problem}}$

Kisin : further study the geometric properties of $Y_{\bar{\rho}}^{\lambda,\tau}[1/p]$

$\rightsquigarrow$ the irr. components of $\mathrm{Spec}(R_{\bar{\rho}}^{\square,\lambda,\tau}[1/p])$ are formally smooth equidimensional of dim. $4+[K:\mathbb{Q}_p]$

Congruences of automorphic forms
Breuil-Mézard conjecture

geometric properties of $R_{\bar{\rho}}^{\square,\lambda,\tau}$ $\boxed{\text{integrally}}$

Skip maybe

e.g. (BM02)

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix} \otimes \omega$$

Computation of $\boxed{\text{moduli}}$ of φ-modules

$\left(\boxed{\text{matrices}}\right.$ with φ-conjugation$\left.\right)$

$$\rightsquigarrow R^{\square,\binom{2}{1},\tau} \cong \begin{cases} \dfrac{\mathcal{O}[[X,Y]]}{(XY-p)} & \text{if } \tau = [\omega]^r \oplus \mathbb{1} \\ \mathcal{O}[[X]] & \text{if } \tau = [\omega_2]^r \oplus [\omega_2^p]^r \\ & \text{or } [\omega_2]^{r+1-p} \oplus [\omega_2^p]^{(r+1-p)} \\ 0 & \text{else} \end{cases}$$

up to formally smooth variables

$$\text{e.g. (lecture 6)} \left(\frac{M_\infty(\sigma(\tau)^\circ)}{m} \right)^\vee \cong \mathrm{Hom}_{GL_2(\mathcal{O}_K)}\left(\sigma(\tau)^\circ \otimes_{\mathcal{O}} \mathbb{F}, \pi[\bar{r}]\right)$$

if $\sigma(\tau)^\circ$ is "rigid" enough w.r.t. $W(\bar{\rho})$ & $M_\infty(\sigma(\tau)^\circ)$ is cyclic $\rightsquigarrow$ get a "piece" of $\pi[\bar{r}]$.

[Br-14] [EGS], [LMS], [HW], [Le] Determine $\pi[\bar{r}]^{K_1}$

cyclycity of $M_\infty(\widetilde{P}_\sigma)$

structure of $\widetilde{P}_\sigma := \mathrm{Proj}_{GL_2(\mathbb{F}_q)}\sigma / \sigma$

Serre weight

$V\binom{1}{-1}$

$\cong$

Deeper? $\widetilde{P}_\sigma$ replaced by $\widetilde{P}_\sigma \otimes_{\mathbb{Z}_p} (sl_{2,\mathcal{O}_K})$ $\boxed{\text{locally}}$ tame

Def. (Multi Hodge type deformation)

$$\mathrm{Spec}\left(R_{\bar{\rho}}^{\square,\le\lambda,\sigma}\right) := \overline{\bigcup_{0\le\lambda'\le\lambda}\ \bigcup_{\substack{\tau \\ JH(\sigma(\tau)^\circ\otimes_{\mathcal{O}}\mathbb{F})\ni\sigma}} \mathrm{Spec}\left(R_{\bar{\rho}}^{\square,\lambda',\tau}[1/p]\right)}$$

Zariski closure

also the reduced union of $\mathrm{Spec}(R_{\bar{\rho}}^{\square,\lambda',\tau})$

Computation by $\boxed{\text{interpolation of matrices}}$

§ Integral p-adic Hodge theory

⑤

R : complete local noetherian $\mathcal{O}$-algebra

$\tau : I_K \longrightarrow GL_2(\mathcal{O})$ tame inertial type

Simplicity : $\tau \cong [\omega_f]^{\sum_j \mu_1^{(j)} p^j} \oplus [\omega_f]^{\sum_j \mu_2^{(j)} p^j}$

$\mu_1^{(j)} > \mu_2^{(j)} \quad \forall j$

$L = K(\sqrt[p^f-1]{-p})$ $\quad \Delta = Gal(L/K) \simeq \mathbb{F}_q^\times$

$\rightsquigarrow \mathcal{Y}^{[0,h],\tau}(R) \ni (\mathfrak{M}, \phi_{\mathfrak{M}})$ s.t.

(i) $\mathfrak{M}$ rk 2 projective over

$$\begin{matrix} \varphi \circlearrowright \\ \Delta \circlearrowright \end{matrix} \mathfrak{S}_{L,R} := (W(\overset{\circlearrowright^{Fr}}{\mathbb{F}_q}) \otimes_{\mathbb{Z}_p} R)[\![u]\!]$$

$u \mapsto u^p$

trivial $\quad u \mapsto (\omega_K(g) \otimes 1) u$

(ii) $\mathrm{coker}\left(\mathfrak{S}_{L,R} \otimes_{\varphi} \mathfrak{M} \xrightarrow{1 \otimes \phi_{\mathfrak{M}}} \mathfrak{M}\right)$ killed by $(\underbrace{\boxed{u^{p^f-1}}}_{=:v} + p)^h$

$\mathfrak{S}_{L,R}$-linear

(iii) $\mathfrak{M}^{(j)}/(u) \xrightarrow{\sim} (\tau \otimes_{\mathcal{O}} R)^\vee$

$\circlearrowright$ linear Δ-action, compatible with $\phi_{\mathfrak{M}}^{(j)}$

Note $\Delta \circlearrowright \mathfrak{M} \xrightarrow{\sim} \bigoplus (\underbrace{\mathfrak{M} \otimes_{\iota \circ Fr^{-j}} R}_{\mathfrak{M}^{(j)} :=})$

semilinear

$\iota : \mathcal{O}_K \hookrightarrow \mathcal{O}$

$\phi_{\mathfrak{M}}^{(j)} : \mathfrak{M}^{(j)} \longrightarrow \mathfrak{M}^{(j+1)}$

e.g. $\overline{\mathfrak{M}} \simeq \underset{\substack{\circlearrowright \\ \Delta \,:\, [\omega]^{-r}}}{\sigma_{L,\mathbb{F}}} \cdot f_1 \oplus \underset{\substack{\circlearrowright \\ \Delta \,:\, 1}}{\sigma_{L,\mathbb{F}}} f_2$

$K = \mathbb{Q}_p$

$$\mathrm{Mat}_{(f_1, f_2)}(\Phi_{\overline{\mathfrak{M}}}) = \begin{pmatrix} 0 & u^{(p-1)+r+1} \\ u^{2(p-1)-(r+1)} & 0 \end{pmatrix}$$

(5')

Def: $\beta = (\beta^{(j)})_j = ((f_1^{(j)}, f_2^{(j)}))_j$ is an eigenbasis if $\Delta \circlearrowright f_i^{(j)}$ by $[\omega_f]^{\sum_{k=0}^{f-1} \mu_i^{(k)} \cdot p^k}$

Define $\underset{\substack{\cap \\ \mathrm{Mat}_2\left((\sigma_{L,R}^{(j)})^{\Delta=1}\right) = R[\![v+p]\!]}}{\Delta^{(j)}_{\mathfrak{M},\beta}}$:= Matrix of the $[\omega_f^{-1}]^{\sum_j \mu_2^{(j)} p^j}$-part of $\Phi_{\mathfrak{M}}^{(j)}$

$$\left[\begin{pmatrix} u^{-\sum \mu_1^{(j)} p^j} & \\ & u^{-\sum \mu_2^{(j)} p^j} \end{pmatrix} \cdot \mathrm{Mat}_\beta(\Phi_{\mathfrak{M}}) \cdot \begin{pmatrix} u^{\sum_j \mu_1^{(j)} p^j} & 0 \\ 0 & u^{\sum_j \mu_2^{(j)} p^j} \end{pmatrix}\right]$$

e.g. $\Delta_{\overline{\mathfrak{M}},(f_1, f_2)} = \begin{pmatrix} 0 & v \\ v^2 & 0 \end{pmatrix}$

Note: (i) Can replace "$[0,h]$" with "$\leq\lambda$": $(v+p)^{\lambda_2^{(j)}} \mid$ entries of $A^{(j)}_{m,\beta}$

$$(v+p)^{\lambda_1^{(j)}+\lambda_2^{(j)}} \mid \det A^{(j)}_{m,\beta}$$

(ii) τ has attached combinatorial data : $\widetilde{\omega}(\tau)$ $\left(\text{e.g.} = t_{\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}} = t_{\begin{pmatrix}r\\ 0\end{pmatrix}}\right)$

$\overline{\rho}|_{I_K}$ ———————— : $\widetilde{\omega}(\overline{\rho})$ $\left(\text{e.g.} = (12)\,t_{\begin{pmatrix}r+2\\ 1\end{pmatrix}}\right)$ $\in \underline{\widetilde{W}}$

translation

extended affine Weyl group of $\mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}_p} GL_2$

(iii) Similar construction in niveau $2f$:

$$\tau = [\omega_{2f}]^{\left(\sum_j \mu_1^{(j)} p^j\right) + q\left(\sum_j \mu_2^{(j)} p^j\right)} \oplus [\omega_{2f}^{q}]^{(\text{same})}$$

& replace $\mathbb{F}_q$ by $\mathbb{F}_{q^2}$ in $\sigma_{\mathbf{L},R}$ ($\rightsquigarrow$ symmetries between $A^{(j)}_{m,\beta}$ & $A^{(j+f)}_{m,\beta}$)

A fundamental diagram

Functor to $G_{K_\infty} = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p \middle/ \bigcup_n K(\sqrt[p^n]{-p})\right)$: $T_{dd}^*: Y^{\le\lambda,\tau} \longrightarrow \Phi\text{-Mod}^{et} \xrightarrow[\sim]{\mathbb{V}_K^*} \mathrm{Rep}\, G_{K_\infty}$

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, mapsto] & \left(\mathfrak{M} \otimes_{\mathfrak{S}_L} \boxed{\widehat{\mathcal{O}_K[[u]][1/u]}}\right)^{\Delta=1} \arrow[d] \\
 & \mathcal{O}_{L,\xi}
\end{tikzcd}$$

Upshot: Given $\bar{\rho}$, at most <u>one</u> $\overline{\mathfrak{M}} \in Y^{\le\lambda,\tau}(\mathbb{F})$, $T_{dd}^*(\overline{\mathfrak{M}}) \cong \bar{\rho}|G_{K_\infty}$.

$$\begin{tikzcd}
 & \mathrm{Spf}(\quad) \arrow[ddl, "\text{formally smooth}"'] \arrow[ddr] & \\
 & \text{closed immersion followed by a formally smooth} & \\
\mathrm{Spf}(R_{\bar{\rho}}^{\square,\le\lambda,\tau}) & & \mathrm{Spf}(R^{\le\lambda,\tau}_{\overline{\mathfrak{M}},\bar{\beta}})
\end{tikzcd}$$

Parametrizes pairs $(\mathfrak{M}, \beta)$ where:

- $T_{dd}^*(\mathfrak{M} \otimes_R \mathbb{F}) \cong \bar{\rho}|G_{K_\infty}$
- β is a gauge basis for $\mathfrak{M}$

⑦'

$$A_{m,\beta} \longmapsto A_{m,\beta}\cdot\tilde{\omega}(\tau) \underset{\substack{\uparrow\\ \text{in our e.g.}}}{=} \begin{pmatrix}0 & v\\ v^2 & 0\end{pmatrix}\begin{pmatrix}v^r & 0\\ & 1\end{pmatrix} \longmapsto \left(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}\right)\otimes\omega$$

⑩'

e.g. $\bar{\rho}|_{I_{\mathbb{Q}_p}} = \begin{pmatrix}\omega_2^{r+1} & \\ & \omega_2^{p(r+1)}\end{pmatrix}\otimes\omega$

$\tau = [\omega]^r \oplus \mathbb{1}$

$$\rightsquigarrow \tilde{\omega}(\bar{\rho},\tau) = (12)\, t_{\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)}$$

$$= \tilde{\omega}(\bar{\rho})\left(\tilde{\omega}(\tau)\right)^{-1}$$

holds in greater generality

LLLM Prop. 5.5.7

the "combinatorial data": $\tilde{\omega}(\bar{\rho}) = (12)\, t_{\left(\begin{smallmatrix}2+r\\1\end{smallmatrix}\right)}$

$\tilde{\omega}(\tau) = t_{\left(\begin{smallmatrix}r\\0\end{smallmatrix}\right)}$

Upshot: Study the moduli problem $\mathrm{Spf}(R^{\leq\lambda,\tau}_{\overline{\mathfrak{M}},\bar{\beta}})$

Difficulty: which $(\mathfrak{M},\phi_{\mathfrak{M}})$ satisfy $T^*_{dd}(\mathfrak{M})\cong\rho|_{G_{K_\infty}}$, $\rho\in\mathrm{Spf}(R^{\square,\leq\lambda,\tau}_{\bar{\rho}})$?

Monodromy

(p-adically complete, flat/$\mathcal{O}$, top. f.t.)

$$\mathfrak{M}\otimes_{R[[u]]}\left(\varprojlim_{n} R[[u,\tfrac{u^{p^n}}{p}]][\tfrac{1}{p}]\right) \circlearrowleft \varphi \qquad \circlearrowleft N_\nabla = -\overbrace{\left(\prod_{n=0}^{\infty}\left(1+\frac{v^{p^n}}{p}\right)\right)}^{=:\lambda}\cdot u\frac{d}{du}$$

$$=: \mathfrak{M}^{\mathrm{rig}}$$

recursively approximate $N_{\mathfrak{M}}$

Theorem (Kisin, [Ki06]): $\exists!\ N_{\mathfrak{M}}/N_\nabla \circlearrowright \mathfrak{M}^{\mathrm{rig}}[\frac{1}{\lambda}]$ s.t. $N_{\mathfrak{M}}\circ\Phi_{\mathfrak{M}}=(v+p)\Phi_{\mathfrak{M}}\circ N_{\mathfrak{M}}$.

$N_{\mathfrak{M}}\equiv 0 \mod u$

Moreover $N_{\mathfrak{M}}(\mathfrak{M}^{\mathrm{rig}})\subseteq\mathfrak{M}^{\mathrm{rig}} \iff T^*_{dd}(\mathfrak{M})[\frac{1}{p}]\simeq\rho|_{G_{K_\infty}}$ $\exists\,\rho\in\mathrm{Rep}^{\leq\lambda,\tau}_{R[\frac{1}{p}]}(G_K)$

$\downarrow$

"monodromy condition"

Proposition: Assume $N < \mu_1^{(j)} - \mu_2^{(j)} < p-N$. The monodromy condition is equivalent to

$$\left[-(p^f-1)\frac{v\,d}{dv}A^{(j-1)}_{m,\beta} - A^{(j-1)}_{m,\beta}\begin{pmatrix}\sum_{i=0}^{f-1}\mu_1^{(i-j)}p^i & 0\\ 0 & \sum_{i=0}^{f-1}\mu_2^{(i-j)}p^i\end{pmatrix}\right]\left(A^{(j-1)}_{m,\beta}\right)^{adj} \in (v+p)^{h-1}\mathrm{Mat}_2(R[\![v+p]\!])$$

$$+\, p^{N-2(h)+3}\mathrm{Mat}_2(R[\![v+p]\!])$$

similar for τ of niveau $2f$
(& conjugate to make <u>dominant</u>)

get a condition
on the $(v+p)$-coefficients
of the matrix entries

$\rightsquigarrow$ ideal I^{∇} of R

define $\boxed{R^{\leq\lambda,\tau,\nabla}_{\overline{m},\overline{\beta}} := R^{\leq\lambda,\tau}_{\overline{m},\overline{\beta}}\big/\left(I^{\nabla}\right)^{p\text{-sat}}}$

Can complete the diagram

Compute I^{∇} via combinatorial data from $\bar\rho, \tau$.

$$\begin{tikzcd}
 & \mathrm{Spf}(\quad) \arrow[r, "f.s."] \arrow[dl, "f.s."'] \arrow[dr] & \mathrm{Spf}\, R^{\leq\lambda,\tau,\nabla}_{\bar{\mathfrak{M}},\bar\beta} \arrow[d, hook] \\
\mathrm{Spf}\, R^{\square,\leq\lambda,\tau}_{\bar\rho} & & \mathrm{Spf}\, R^{\leq\lambda,\tau}_{\bar{\mathfrak{M}},\bar\beta}
\end{tikzcd}$$

$\leftarrow$ closed immersion

Shapes $\bar\rho$ has shape $\widetilde{\omega}(\bar\rho,\tau) \in \underline{\widetilde{W}}$ w.r.t. τ if

$$\mathrm{Iw}_1\, \widetilde{\omega}(\bar\rho,\tau)\, \mathrm{Iw} = \mathrm{Iw}_1\, A_{\bar{\mathfrak{M}},\bar\beta}\, \mathrm{Iw}$$

(pro)-v Iwahori of $GL_2(\mathbb{F}[\![v]\!])$

<u>Unique</u> $\bar{\mathfrak{M}} \in Y^{\leq\lambda,\tau}(\mathbb{F})$ s.t. $T_{dd}^{*}(\bar{\mathfrak{M}}) = \bar\rho|_{G_{K_\infty}}$

$\bar\beta$ any eigenbasis

Key insight 1: $R_{\bar\rho}^{\square,\le\lambda,\tau} \neq 0 \iff \widetilde{w}(\bar\rho,\tau)\in \mathrm{Adm}(\lambda)$

[LLLM Cor. 5.5.8]

for $\lambda=\binom{3}{0}$ BDJ & M_∞ $\longrightarrow$ $\Updownarrow$

$W(\bar\rho)\cap JH\left(\sigma(\tau)^\circ\otimes_{\mathcal O}\mathbb F\otimes_{\mathbb F_p}\mathfrak{sl}_{2,\mathbb F_q}\right)\neq\emptyset$

upper bound on the possible shapes

Key insight 2: Gauge basis: If $\overline{\mathfrak M}\in Y^{\le\lambda,\tau}(\mathbb F)$ has shape $\widetilde{w}$

I_w

$$\Rightarrow \exists!\,\bar\beta \text{ s.t. } \widetilde{w}^{-1}A_{\overline{\mathfrak M},\bar\beta}\in \underbrace{\left(I_{w_1}\cap\widetilde{w}^{-1}I_w\widetilde{w}\right)}_{\text{matrices in } \mathbb F[v] \text{ with degree bounds}}$$

up to $T(\mathbb F)$-scaling

Fix $(\overline{\mathfrak M},\bar\beta)$ shape w.r.t. v

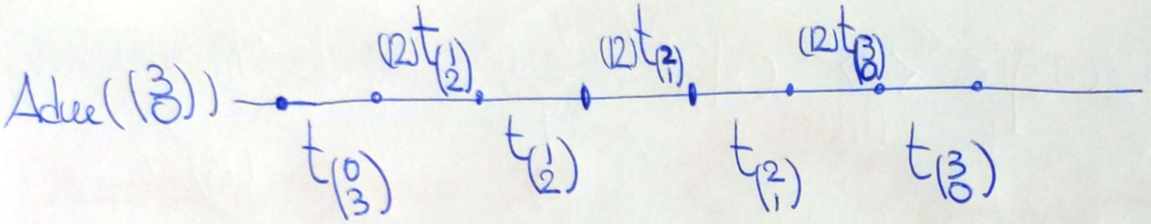
Prop.: Let $\mathfrak M\in Y^{\le\lambda,\tau}(R)$ with $\mathfrak M\otimes\mathbb F\simeq\overline{\mathfrak M}$.

$\exists!\,\beta$ lifting $\bar\beta$ s.t.
$$\begin{cases}\deg_v\left(A_{\mathfrak M,\beta}\right)_{ik}\le \nu_k-\delta_{i<w(k)} \\ A_{\mathfrak M,\beta}\equiv \begin{bmatrix} \diagdown & * \\ 0 & \diagdown\end{bmatrix} \bmod v\end{cases}$$

up to $\mathrm{Ker}(T(\mathcal O)\twoheadrightarrow T(\mathbb F))$-scaling

equality if $i=w(k)$)

column degree bounds



pivot entry

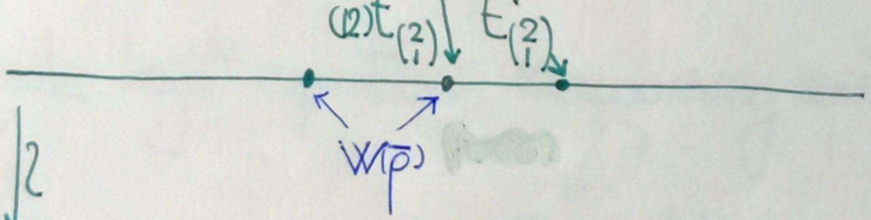
$$A_{m,\beta} = \begin{pmatrix} (v+p)d_{11} + c_{11} & (v+p) + c_{12} \\ v\big((v+p)+c_{21}\big) & (v+p)d_{22} + c_{22} \end{pmatrix}$$

$$\begin{pmatrix} (v+p)^2 + (v+p)d_{11} + c_{11} & c_{12} \\ v\big((v+p)d_{21} + c_{21}\big) & (v+p) + c_{22} \end{pmatrix}$$

$$\tilde{\varpi} := (12)t_{\binom{2}{1}}: \quad \frac{I_\omega}{I_{\omega_1} \cap \tilde{\varpi}^{-1} I_\omega \tilde{\varpi}} \cong \begin{pmatrix} \mathbb{F}^\times & 0 \\ 0 & \mathbb{F}^\times \end{pmatrix}$$

$$t_{\binom{2}{1}}: \quad \cong \begin{pmatrix} \mathbb{F}^\times & 0 \\ v\cdot\mathbb{F} & \mathbb{F}^\times \end{pmatrix}$$

$JH\big(\sigma(\tau)^\circ \otimes \mathbb{F} \otimes \det\big)$ if $\tilde{\varpi}(\bar{\rho}, \tau) = t_{\binom{2}{1}}$



Choose the useful types (extension graph for $GL_2(\mathbb{F}_q)$)

$\left\{\begin{matrix}\text{Serre} \\ \text{weights}\end{matrix}\right\}$ adjecent pairs $\longleftrightarrow$ JH of $\sigma(\tau)^\circ \otimes_{\mathcal{O}} \mathbb{F}$

§Computation of $R_{\bar\rho}^{\square,\leq\binom{3}{0},\tau}$

Diagram @ page ⑩ : $R_{\bar\rho}^{\square,\leq\binom{3}{0},\tau}[[X_1,\dots,X_{2f}]] \xrightarrow{\sim} R_{\overline{\mathfrak{M}},\bar\beta}^{\leq\binom{3}{0},\tau,\nabla}[[Y_1,\dots,Y_4]]$

and :

$$\begin{tikzcd}
\mathrm{Spf}\left(R_{\overline{\mathfrak{M}},\bar\beta}^{\leq\binom{3}{0},\tau}\right) & \mathrm{Spf}\left(R_{\overline{\mathfrak{M}},\bar\beta}^{\leq\binom{3}{0},\tau,\nabla}\right) \arrow[l, hook'] \\
\mathrm{Spf}\left(R_{\overline{\mathfrak{M}},\bar\beta}^{\leq\binom{3}{0},\tau}[\tfrac{1}{p}]\right) \arrow[u] & \mathrm{Spf}\left(R_{\overline{\mathfrak{M}},\bar\beta}^{\leq\binom{3}{0},\tau}[\tfrac{1}{p}]/I^{\nabla}\right) \arrow[l, hook'] \arrow[u, hook] \arrow[u, bend right=60, "p\text{-flat closure}"']
\end{tikzcd}$$

triples $(\mathfrak{M},\beta,j)$

$\downarrow$

gauge basis

$j : \mathfrak{M}\otimes\mathbb{F} \xrightarrow{\sim} \overline{\mathfrak{M}}$

i.e. compute $\left(\text{finite height equations, monodromy equations}\right)^{p\text{-saturation}} \subseteq \underbrace{\mathcal{O}\left[\left[\text{coefficients of } A_{\overline{\mathfrak{M}},\bar\beta}\text{-entries}\right]\right]}_{=:R}$

finite height equations $\swarrow$ 3 equations

monodromy equations $\swarrow$ 8 equations ("$h=3$")

Strategy: 1. Show $\underbrace{\langle \text{equations in row 4} \rangle}_{=: I} \subseteq \left(I^{\leq\binom{\beta}{2}} + I^{\nabla}\right)^{p\text{-sat}}$

2. Show $R/I \twoheadrightarrow R^{\leq\binom{\beta}{2}, \tau, \nabla}_{\overline{m}, \overline{\beta}}$ equidim^al, same dimension

same nb. of irr. components

$\left(\Rightarrow \operatorname{supp}(\ker(\mathrm{pr})) \subseteq \operatorname{rad}(R/I) = 0\right)$

1. Example
$\left(a = \frac{r}{p-1}\right)$

$$\frac{1}{p}\left(M_7 - \frac{1}{p}M_8\right) = \frac{1}{p}\Big((a-1)c_{21}c_{12} + c_{11}d_{22} - p\big((a-3)c_{12} + (a+1)c_{21} + d_{11}d_{22} + p\big)$$
$$- (a-1)c_{12}c_{21} - c_{11}d_{22} - pc_{12}\Big) + O(p^{N-5})$$

$$\ldots = c_{12} - (a-1)\underbrace{(c_{12}+c_{21})}_{\equiv d_{11}d_{22}+p \text{ via } FH_1} - (d_{11}d_{22}+p) + O(p^{N-5})$$

we have solved c_{12} !

$$= c_{12} - a(d_{11}d_{22}+p) + O(p^{N-5})$$

Other are similar $\Rightarrow$ $$R/I \cong \frac{\mathcal{O}[\![d_{11},d_{22}]\!]}{(d_{11}d_{22}+p)\left(\frac{a(a-1)}{(a-2)(a+1)}d_{11}d_{22}+p\right)}$$

"related to" $R_{\bar{\rho}}^{\binom{2}{1},\tau}$

$\mathfrak{p}^{\binom{2}{1}}$

$\mathfrak{p}^{\binom{3}{0}}$

"related to" $R_{\bar{\rho}}^{\binom{3}{0},\tau}$

2. From 1. get

$$\begin{tikzcd} R/I \arrow[r, two heads] & R_{\overline{\mathfrak{m}},\bar{\beta}}^{\leq\binom{3}{0},\tau,\nabla} \arrow[r, two heads] & R_{\bar{\rho}}^{\square,\lambda,\tau} \end{tikzcd}$$

up to formally smooth variables

direct check 2 irr. components

Need to exhibit at least 2 irr. components

Show this is $\neq 0$ for all $\lambda \leq \binom{3}{0}$ $\Leftarrow$ $W(\bar{\rho}) \cap \mathrm{JH}(\sigma(\tau)^{\circ}\otimes\det) \neq \emptyset$

By combinatorics of types & weight (page ⑪)

[GLS] (Serre weight conjecture)

Upshot : $\mathrm{Irr}\left(\mathrm{Spec}\left(R_{\bar{\rho}}^{\square,\leq\binom{3}{0},\tau}\right)\right) \xrightarrow{\sim} \left\{0 \leq \lambda \leq \binom{3}{0}\right\}$

$\mathcal{C} \longmapsto$ Hodge-Tate weights of any $x : \mathcal{C} \to E$

We can now label the components of R/I :

$$\mathcal{M} \in Y^{\leq\binom{2}{1},\tau} \quad \left(\subseteq Y^{\leq\binom{3}{0},\tau}\right) \iff \begin{cases} (v+p) \mid A_{\mathcal{M},\beta} \\ \det A_{\mathcal{M},\beta} = (v+p)^3 \end{cases}$$

e.g. $\iff A_{\mathcal{M},\beta} = \begin{pmatrix} (v+p)d_{11} & (v+p) \\ (v+p) & (v+p)d_{22} \end{pmatrix}$

$d_{11}d_{22} + p = 0$

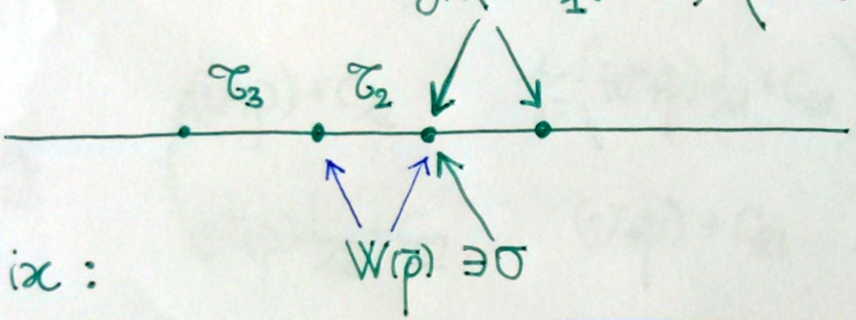
hence $\ker\left(R \twoheadrightarrow R_{\bar{\mathcal{M}},\bar{\beta}}^{\leq\binom{3}{0},\tau,\nabla} \twoheadrightarrow R_{\bar{\rho}}^{\square,\binom{2}{1},\tau} \cong R_{\bar{\mathcal{M}},\bar{\beta}}^{\leq\binom{2}{1},\tau,\nabla}\right) = (c_{11}, c_{22}, c_{12}, c_{21}, d_{11}d_{22} + p)$

$\searrow$ as in step 2

by the upshot $\left(\frac{a(a-1)}{(a-2)(a+1)} d_{11}d_{22} + p\right)$ corresponds to $R_{\bar{\rho}}^{\square,\binom{3}{0},\tau}$

§ Multi Hodge type deformations

$\mathrm{JH}(\sigma\otimes\overline{\tau}_1^{\circ}\otimes\det)$ (← shape $\widetilde{\omega}(\overline{\rho},\tau)$)



(Extension graph of $GL_2(\mathbb{F}_p)$)

fix:

Recall:
$$\mathrm{Spec}\left(R_{\overline{\rho}}^{\square,\le\binom{3}{0},\sigma}\right)=\overline{\bigcup_{\lambda\le\binom{3}{0}}\ \bigcup_{\substack{\tau,\\ \mathrm{JH}(\sigma\otimes\overline{\tau}^{\circ}\otimes\det)\ni\sigma}}\mathrm{Spec}\left(R_{\overline{\rho}}^{\square,\lambda,\tau}[1/p]\right)}^{\text{Zariski closure}}$$

&
$$R_{\overline{\mathfrak{m}},\overline{\beta}}^{\le\binom{3}{0},\tau_i,\nabla}=R\Big/\underbrace{\left(I^{\le\binom{3}{0}}+I^{\nabla}\right)^{p\text{-sat}}}_{=:\,I_i}$$

power series by "interpolation"

Theorem (BHHMS1)
$$R_{\overline{\rho}}^{\square,\le\binom{3}{0},\sigma}[[X_1,\dots,X_{2f}]]\xrightarrow{\sim}\underbrace{\left(\widehat{\bigotimes_{j=0}^{f-1}}S^{(j)}\Big/\left(I_2^{(j)}\cap I_{i_j}^{(j)}\right)\right)}_{\text{call this ring }S^{\mathrm{expl}}}[[Y_1,\dots,Y_4]]$$

$i_j\in\{1,3\}$ depending explicitly on σ

Idea: interpolation of matrices: $\mathrm{Mat}\left(\Phi_{\left(\mathfrak{M}\otimes_{\mathcal{O}_L}\mathcal{O}_{L,\xi}\right)^{\Delta=1}}\right)=A_{\mathfrak{M},\beta}\cdot\tilde{\omega}(\tau)$

Slope

e.g. $$\begin{pmatrix}(v+p)+c_{12} & \frac{1}{v}\left((v+p)d_{11}+c_{11}\right)\\ (v+p)d_{22}+c_{22} & (v+p)+c_{21}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}v^{r+1} & \\ & 1\end{pmatrix}$$ if $\tau=\tau_2=[\omega]^r\oplus\mathbb{1}$ $\quad (12)t_{\binom{2}{1}}$

$$\begin{pmatrix}\frac{1}{v}\left((v+p)^2+(v+p)d_{11}+c_{11}\right) & c_{12}\\ (v+p)d_{21}+c_{21} & (v+p)+c_{22}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}v^{r+1} & \\ & 1\end{pmatrix}$$ if $\tau=\tau_1=[\omega_2^{r+1-p}]\oplus[\omega_2^{p(r+1-p)}]$ $\quad t_{\binom{2}{1}}$

interpolate $\rightsquigarrow$ $$\begin{pmatrix}(v+p)+c_{12}+b_{12}/v & \frac{1}{v}\left((v+p)d_{11}+c_{11}\right)\\ (v+p)d_{22}+c_{22} & (v+p)+c_{21}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}v^{r+1} & \\ & 1\end{pmatrix}$$

étale φ-module

$$\mathcal{M}\,\Big/\,\mathcal{O}_{K,\xi}\otimes_{\mathbb{Z}_p}\overbrace{\mathcal{O}[\![c_{12},\dots,c_{22}]\!]}^{\text{the ring "}S^{(j)}\text{"}}$$

Theorem (continued): Moreover, each $I_2^{(j)} \cap I_{i_j}^{(j)}$ has 4 minimal primes

$$\mathfrak{p}_j^{\binom{3}{0},2} \quad \mathfrak{p}_j^{\binom{2}{1},2} \quad \mathfrak{p}_j^{\binom{3}{0},i_j} \quad \mathfrak{p}_j^{\binom{2}{1},i_j}$$

compatible with

$$\begin{tikzcd} R_{\bar\rho}^{\square,\leq\binom{3}{0},\sigma}[\![\underline{X}]\!] \arrow[r, "\sim"] \arrow[d] & S^{expl}[\![\underline{Y}]\!] \arrow[d] \\ R_{\bar\rho}^{\square,\lambda,\tau}[\![\underline{X}]\!] \arrow[r, "\sim"] & \left(\widehat{\bigotimes_{j=0}} S^{(j)}\right) \Big/ \sum_j \mathfrak{p}_j^{\lambda_j,\bullet} [\![\underline{Y}]\!] \end{tikzcd}$$

where $\bullet \in \{2, i_j\}$ depending explicitly on $\widetilde{\omega}(\bar\rho,\tau)_j$

Strategy: 1. Create

$$\begin{tikzcd} R^{\square}_{\bar\rho|G_{K_\infty}} \arrow[r, hook] \arrow[d, two heads] & R^{\square}_{\bar\rho|G_{K_\infty}}[\![\underline{X}]\!] \arrow[r, two heads] & S^{expl}[\![\underline{Y}]\!] \\ R^{\square}_{\bar\rho} \arrow[r, two heads] & R^{\square,\leq\binom{3}{0},\sigma}_{\bar\rho} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

← tangent space computation (CA)

2. Show ← comparison of Kisin modules

3. Describe $\mathrm{Irr}(\mathrm{Spec}(S^{expl}))$ ← $\boxed{\text{explicit}}$ check on ideals

1. $\left(f=1,\ \tau_1=[\omega_2^{r+1-p}]\oplus[\omega_2^{p(r+1-p)}]\right)$

Have
$$\begin{tikzcd} \mathbb{V}_K^*(\mathcal{M}[[\underline{Y}]])\equiv\bar{\rho}|_{G_{K_\infty}} \bmod c_{12},\dots,c_{22},\underline{Y} \arrow[r, rightsquigarrow, "\text{universal property}"] & R^{\square}_{\bar{\rho}|_{G_{K_\infty}}} \arrow[r] \arrow[d] & S^{\mathrm{expl}}[[\underline{Y}]] \\ & R^{\square}_{\bar{\rho}|_{G_{K_\infty}}}[[\underline{X}]] \arrow[ur, dashed, "\text{choose this extension}"'] & \end{tikzcd}$$

Injectivity on tangent spaces:

tangent vector

zero tangent vector

Assume
$$\mathcal{M}[[\underline{Y}]]\otimes_{S^{\mathrm{expl}}[[\underline{Y}]],t}\frac{\mathbb{F}[\varepsilon]}{(\varepsilon^2)}\xrightarrow[\boxed{\iota\ \equiv \mathrm{id} \bmod \varepsilon}]{\sim}\mathcal{M}[[\underline{Y}]]\otimes_{S^{\mathrm{expl}}[[\underline{Y}]],t_0}\frac{\mathbb{F}[\varepsilon]}{(\varepsilon^2)}$$

Deduce:

$$\mathrm{Mat}(\iota)\in\mathbb{1}_2+\varepsilon\,\mathrm{Mat}_2(\mathbb{F}(\!(v)\!))\quad\text{i.e.}\quad(1+\varepsilon M)\begin{pmatrix} & v\\ v^{r+2} & \end{pmatrix}(1-\varepsilon\varphi(M))=\begin{pmatrix} v+\varepsilon\left(c_{12}+\frac{b_{12}}{v}\right) & \varepsilon\left(d_{11}+\frac{c_{11}}{v}\right)\\ \varepsilon(v d_{22}+c_{22}) & v+\varepsilon c_{21}\end{pmatrix}\left(\begin{matrix}0 & 1\\ v^{r+1} & 0\end{matrix}\right.$$

Get successively :

i) $M \in \mathrm{Mat}(\mathbb{F}[\![v]\!])$ $\longleftarrow$ $\overbrace{\mathbb{1}-\varepsilon\varphi(M)}^{\text{"big" } v\text{-degrees}} = \overbrace{\begin{pmatrix} & v^{-r-2} \\ v^{-1} & \end{pmatrix}(\mathbb{1}-\varepsilon M)\begin{pmatrix} & \\ \text{- - - -} & \end{pmatrix}\begin{pmatrix} & 1 \\ v^{r+1} & \end{pmatrix}}^{\text{"small" } v\text{-degrees}}$

ii) $M \in \mathrm{Mat}(\mathbb{F})$ $\longleftarrow$ $M - \begin{pmatrix} & 1 \\ v^{r+1} & \end{pmatrix}\frac{1}{v}\varphi(M)\begin{pmatrix} & v^{-r-1} \\ 1 & \end{pmatrix} = \begin{pmatrix} & \\ \text{- - - -} & \end{pmatrix}$ $\longleftarrow$ deduce this is $=0$ via (i)

iii) $M=0$ $\longleftarrow$ irreducibility of $\bar{\rho}$ & previous item.

2. By φ-flatness, enough to prove after $\_\otimes_\sigma E$.

$$\begin{tikzcd}
R^{\square}_{\bar\rho|G_{K_\infty}} \arrow[rr] \arrow[dd, two heads] & & S/(I_2\cap I_1)[[Y]] \arrow[rr, "x", "\text{closed point}"'] \arrow[dr] & & E & \Leftarrow \operatorname{Ker}(x) \supseteq I_1 \text{ or } I_2 \\
 & & & S/I_i[[Y]] \arrow[ur, dashed] & & \\
R^{\square}_{\bar\rho} \arrow[r, two heads] & R^{\square,\le\binom{3}{0},\sigma}_{\bar\rho} \arrow[r, two heads] & R^{\square,\le\binom{3}{0},\tau_i}_{\bar\rho} \arrow[ur, "\sim"] & & &
\end{tikzcd}$$

$\downarrow$

$\exists$ by definition of $R^{\square,\le\binom{3}{0},\sigma}_{\bar\rho}$

by change of variables & the result of §3 :

in fact $\mathcal{M}[[Y]]\otimes_x E$ is one of the étale φ-modules of §3 (& satisfy the monodromy condition)

i.e. is the étale φ-mod of $S_{pf}\left(R^{\le\binom{3}{0},\tau_i,\nabla}_{\overline{\mathfrak{M}},\bar\beta}\right)$ [

3. Show $|\mathrm{Irr}(S^{expl})| = 4^f$ (& have the correct dimension)

$$\begin{tikzcd}
 & S^{expl} \arrow[r, two heads] \arrow[d, two heads] & S/I_2 & \simeq \arrow[d] & R_{\bar\rho}^{\square, \leq \left(\begin{smallmatrix}3\\0\end{smallmatrix}\right), \tau_2} \arrow[r] & \text{p flat, 2 irr. cmpts} \\
 & S/I_1 & & \text{by §3} & & \\
\text{by §3} \arrow[r] & \wr & & & & \\
 & R_{\bar\rho}^{\square \leq \left(\begin{smallmatrix}3\\0\end{smallmatrix}\right), \tau_1} & & & & \\
 & \text{p-flat, 2 irr. cmpts} & & & &
\end{tikzcd}$$

$\Rightarrow$ ETS $\mathrm{Irr}(S/I_2) \cap \mathrm{Irr}(S/I_1) = \phi$

i.e. (p-flatness) $p \in I_1 + I_2$

G_{1,τ_1} up to ↘ change of variables

Direct computation : $I_1 + I_2 \ni (C_{12} - p + (a_1 - 2) d_{11} d_{22}) - (C_{12} - a_2(d_{11} + p)) + \mathcal{O}(p^{N-5})$

with $a_1 = \frac{r+2}{p+1}$

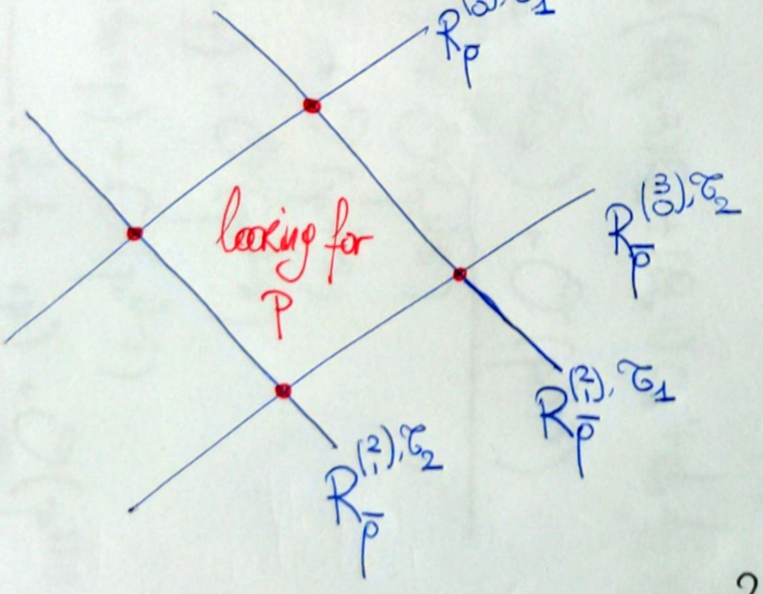
$a_2 = r/(p-1)$

$= (a_2 - 1)p + (a_1 + a_2 - 2) d_{11} d_{22} + \mathcal{O}(p^{N-5})$

Key alignment !! $a_1+a_2-2\equiv 0 \mod p$, hence

$$p\big(\underbrace{(a_2-1)+(*)\cdot d_{11}d_{22}}_{\text{unit}}\big)+\mathcal{O}(p^{N-5}) \in I_1+I_2$$

Future use: $\forall\, j=0,\dots,f-1$, have $\boxed{p\in\left(\bigcap_{\bullet\in\{2,i_j\}} \wp_j^{\binom{2}{1},\bullet}\right)+\left(\bigcap_{\bullet\in\{2,i_j\}}\wp_j^{\binom{3}{0},\bullet}\right)}$



Proof: explicit computation on the generators of $\wp^\lambda$.

Again a key alignment: get

$$\frac{2}{a_2}\cdot p-2c_{12}\,\frac{a_1\underbrace{\boxed{(a_1+a_2-2)}}_{\equiv 0 \mod p}}{(a_1-2)^2\cdot a_2}+\mathcal{O}(p^{N-7})\in \dots$$

Shape $(12)\, t_{\binom{2}{1}}$ $\qquad a_2 = v/_{p-1}$

$$A \quad \begin{pmatrix} (v+p)\, d_{11} + c_{11} & (v+p) + c_{12} \\ v\big((v+p) + c_{21}\big) & (v+p)\, d_{22} + c_{22} \end{pmatrix}$$

$I^{\leq(3)}$ $\qquad$ FH$_1$

$$d_{11} \cdot d_{22} - (c_{12} + c_{21}) + p$$

$$c_{12} c_{21} - d_{11} c_{22} - c_{11} d_{22} - p(c_{12} + c_{21})$$

$$c_{11} c_{22} + p\, c_{12} c_{21}$$

I^{∇}

M_1 $$(a_2 - 1)\, d_{11} c_{22} + a_2 c_{11} d_{22} + p(d_{11} d_{22} - 2c_{21} + p) + \mathcal{O}(p^{N-3})$$

M_2 $$a_2\, c_{11} c_{22} + p(d_{11} c_{22} + p c_{21}) + \mathcal{O}(p^{N-3})$$

$$\vdots$$

M_7 $$(a_2 - 1)\, c_{12} c_{21} + c_{11} d_{22} - p\big((a_2 - 3) c_{12} + (a_2 - 1) c_{21} + d_{11} d_{22} + p\big) + \mathcal{O}(p^{N-3})$$

M_8 $$p\big((a_2 - 1)\, c_{12} c_{21} + c_{11} d_{22} + p c_{12}\big) + \mathcal{O}(p^{N-3})$$

I

$$c_{21} + (a_2 - 1)(d_{11} d_{22} + p) + \mathcal{O}(p^{N-5})$$

$$c_{12} - a_2 (d_{11} d_{22} + p) + \mathcal{O}(p^{N-5})$$

$$c_{11} + \frac{a_2(a_2 - 1)}{a_2 + 1}\, d_{11}(d_{11} d_{22} + p) + \mathcal{O}(p^{N-5})$$

$$c_{22} - \frac{a_2 (a_2 - 1)}{a_2 - 2}\, d_{22}(d_{11} d_{22} + p) + \mathcal{O}(p^{N-5})$$

$$(d_{11} d_{22} + p) \cdot \left(\frac{a_2 (a_2 - 1)}{(a_2 - 2)(a_2 + 1)} \cdot d_{11} d_{22} + p \right) + \mathcal{O}(p^{N-5})$$

$a_1 = \frac{r+2}{p+1}$

Shape	$t_{\left(\substack{2\\1}\right)}$	
A	$\begin{pmatrix} (v+p)^2 + (v+p)\,d_{11} + c_{11} & c_{12} \\ v\left((v+p)\,d_{21} + c_{21}\right) & (v+p) + c_{22} \end{pmatrix}$	
$I^{\leq \left(\substack{3\\5}\right)}$		
I	$d_{11} + (a_1 - 2)\, c_{12} d_{21} + \mathcal{O}(p^{N-5})$ $c_{22} + (a_1 - 1)\, c_{12} d_{21} + \mathcal{O}(p^{N-5})$ $c_{21} + \frac{(a_1-1)(a_1-2)}{a_1} \cdot c_{12}\, d_{21}^2 + \mathcal{O}(p^{N-5})$ $c_{11} - c_{12} d_{21} \left(\frac{(a_1-1)^2 (a_1-2)}{a_1} c_{12} d_{21} - p \right) + \mathcal{O}(p^{N-5})$ $c_{12} \cdot \left((a_1-1)(a_1-2)\, c_{12}\, d_{21} - 2p + \mathcal{O}(p^{N-5}) \right)$	G_1, τ_1