§1. Introduction

Fix ℓ, p primes. $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ $\overline{\mathbb{Q}}_\ell$ fixed algebraic closure of $\mathbb{Q}_\ell$

Depends on $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$: $\iota \circ \mathrm{rec}_{\mathbb{Q}_p} = \overset{HT}{\mathrm{rec}} \circ \iota$ & $\mathrm{rec}_{\mathbb{Q}_p}(\_ \otimes |\det|^{-\frac{1}{2}})$ independent

$\boxed{\text{integral}}$

$\left\{ \begin{array}{l} \text{smooth irreducible} \\ \text{admissible rep}^{ns}\ \pi \\ \text{of } GL_2(\mathbb{Q}_p) \text{ over } \overline{\mathbb{Q}}_\ell \text{ v.s.} \end{array} \right\} \Big/ \cong \quad \xrightarrow[1:1]{\mathrm{rec}_{\mathbb{Q}_p}} \quad \left\{ \begin{array}{l} (\rho, N),\ \text{Frobenius semisimple} \\ \boxed{\text{bounded}} \\ \rho : W_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_\ell) \end{array} \right\} \Big/ \cong$

smooth

$\rho(\mathrm{Frob}_p) N \rho(\mathrm{Frob}_p^{-1}) = p^{-1} N$

Geometric Frobenius

smooth vectors

$\boxed{\ell \neq p}$

$\wr$

$WD(\_)$

$\left\{ \begin{array}{l} \text{Unitary } \overline{\mathbb{Q}}_\ell\text{-Banach space} \\ \text{rep}^{ns} \text{ of } GL_2(\mathbb{Q}_p),\ \text{admissible}, \\ \text{topologically irreducible} \end{array} \right\} \Big/ \cong \qquad \left\{ \begin{array}{l} \rho : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_\ell) \\ \text{continuous, Frobenius semisimple} \end{array} \right\} \Big/ \cong$

Main problem : $(\ \ )^{\text{smooth}}$ & $WD(-)$ loose information if $\ell = p$

class of commensurability of integral structures

Hodge filtration (p-adic Hodge theory)

cf. [Br 08] § 1.6

"À l'origine du programme de Langlands p-adique est la volonté de comprendre ce qu'il faut ajouter à π pour reconstruire ρ. Autrement dit, on veut élucider l'apparition de la théorie de Hodge p-adique côté Galois en termes de théorie des représentations côté automorphe"

C. Breuil

3

First Breakthrough:

Theorem (Breuil [Br03]): $\exists$ $\boxed{\text{explicit}}$ bijection

$$\left\{\begin{array}{l}\text{irreducible admissible}\\ \text{supersingular rep}^{\text{ns}} \text{ of } GL_2(\mathbb{Q}_p)/\overline{\mathbb{F}}_p\end{array}\right\}_{/\cong} \xrightarrow[1:1]{rec_{\mathbb{Q}_p}} \left\{\begin{array}{l}\text{irreducible}\\ \text{continuous}\\ G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{F}}_p)\end{array}\right\}_{/\cong}$$

compatible with the Serre weight conjecture

further breakthroughs in char 0: $St_{GL_2(\mathbb{Q}_p)} \supseteq$ Lattices parametrized by the $\mathcal{L}$-invariant of $D_{st}(\rho)$

Banach admissible $\mathcal{B}(2,\mathcal{L})$ <u>non-zero</u> (by mod p reduction)

[Br-04]

Dual of <u>distributions</u>

④

Second breakthrough:

- In fact, defined on Banach space rep^{ns} & "in families"
- Compatible with mod p reduction [Ber-10]
- Inducing an equivalence of categories [Pas-13]

Theorem (Colmez [Col10]): $\exists$ exact covariant functor

$$\left\{\begin{array}{l}\text{smooth admissible}\\ \text{finite length}\\ GL_2(\mathbb{Q}_p)\text{-rep}^{ns} / \overline{\mathbb{F}}_p\end{array}\right\} \longrightarrow \mathrm{Rep}_{\overline{\mathbb{F}}_p} G_{\mathbb{Q}_p}$$

Novelty: defined via $\boxed{(\varphi,\Gamma)\text{-modules}}$

§2 (φ,Γ)-modules & Galois representations (cf. [Be-10])

$$\mathbb{Z}_p^\times \xleftarrow{\chi_{cyc}} \mathrm{Gal}\left(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p\right) =: \Gamma$$

$$\left(\mathbb{F}_p(\!(X)\!)\right)^{sep} \circlearrowleft \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty})\right) = \ker(\chi_{cyc})$$

$\circlearrowleft$

$\varphi : X \longmapsto X^p$

via field of norms (Fontaine - Wintenberger)

φ-action

Diagonal action

Get a functor $\overline{\rho} \longmapsto M(\overline{\rho}) := \left(\mathbb{F}_p(\!(X)\!)^{sep} \otimes_{\mathbb{F}_p} \overline{\rho}\right)^{\ker(\chi_{cyc})}$

fin. dim. mod p, $G_{\mathbb{Q}_p}$-rep

finite dimensional $\mathbb{F}_p(\!(X)\!)$-vector space

commuting

+ semilinear endomorphism φ

+ semilinear continuous Γ-action

<u>Def</u>: A (φ,Γ)-module over $\mathbb{F}_p((X))$ is a f. d. $\mathbb{F}_p((X))$-vector space D with

$\boxed{\text{commuting}}$

- a semilinear Frobenius $\varphi_D : D \longrightarrow D$
- a continuous <u>semilinear</u> action of Γ

$$\downarrow$$

$$\gamma\cdot X := (1+X)^{\chi_{cyc}(\gamma)} - 1 \quad \text{for } \gamma\in\Gamma$$

$\boxed{\text{étale}}$ if

$$Mat(\varphi_D)\in GL_{(rk D)}(\mathbb{F}_p((X)))$$

Theorem (Fontaine) The functor $Rep_{\mathbb{F}_p}(G_{\mathbb{Q}_{p^f}}) \xrightarrow{M(\_)} \Phi\text{-}\Gamma\text{-}Mod^{et}_{\mathbb{F}_{p^f}}$

$\mathbb{F}$ ← coefficients (cf. lecture 1) $\mathbb{F}_{p^f}\otimes_{\mathbb{F}_p}\mathbb{F}$

↳ is an equivalence

<u>Rmk</u>: $\mathbb{F}_{p^f}\otimes_{\mathbb{F}_p}\mathbb{F} \xrightarrow{\sim} (\mathbb{F}_p\otimes_{\mathbb{F}_p}\mathbb{F})^f$

$(a\otimes b) \longmapsto (1\otimes Fr^{-j}(a)\cdot b)_{j=0,\dots,f-1}$

$Fr: x\mapsto x^p$

$\circlearrowleft$

$\mathbb{F}_{p^f} \xrightarrow{\iota} \mathbb{F}$ fixed embedding

hence $D \xrightarrow{\sim} D^{(0)}\oplus D^{(1)}\oplus\dots\oplus D^{(f-1)}$

$$\begin{tikzcd} D^{(0)} \arrow[loop above, "\Gamma"] \arrow[r, bend left, "\varphi"] & D^{(1)} \arrow[r, bend left] & \cdots & D^{(f-1)} \arrow[loop above, "\Gamma"] \arrow[lll, bend left, "\varphi"] \end{tikzcd}$$

Skip (6')

e.g. $\bar{\rho} = \omega^r \cdot \mu_\lambda$

mod p cyclotomic character

unramified character

$\mu_\lambda(\mathrm{Frob}_p^{-1}) = \lambda$

$M(\bar{\rho}) = \mathbb{F}_p((X)) \cdot (x \otimes e)$

where $x^{p-1} = \lambda$ $\quad e = \mathbb{F}_p$-basis of $\bar{\rho}$

$\left(x \in \overline{\mathbb{F}_p} \hookrightarrow \mathbb{F}_p((X))^{sep} \right)$

$\boxed{\text{Then}}$ $\varphi(x \otimes e) = \lambda \cdot (x \otimes e)$

$\gamma(x \otimes e) = \omega^r(\gamma) \cdot (x \otimes e)$

Further on Galois representations

Fix $\pi_f \in \overline{\mathbb{Q}}_p$ $\pi_f^{p^f-1} = -p$. Then $\frac{g(\pi_f)}{\pi_f} \in \mu_{p^f-1}(\mathbb{Q}_p)$ $\rightsquigarrow$ $G_{\mathbb{Q}_{p^f}} \xrightarrow{\omega_{\mathbb{Q}_{p^f}}} \mathbb{F}_{p^f}^\times$

$$g \longmapsto \frac{g(\pi_f)}{\pi_f} \bmod p$$

well defined, independent of π_f

Rmk: Set $\omega_f := \iota \circ \omega_{\mathbb{Q}_{p^f}}$. Then any

$$\begin{tikzcd} I_{\mathbb{Q}_p} \arrow[r] \arrow[dr] & \mathbb{F}^\times \\ & \mathbb{F}_{p^f}^\times \arrow[u, "\iota"] \end{tikzcd}$$

is of the form: ω_f^r, $r \in \{0, \dots, p^f-2\}$

If $g \in G_{\mathbb{Q}_p}$ $\rightsquigarrow$ $x \longmapsto \omega_2^h(gxg^{-1}) = \omega_2^{p^j \cdot h}(x)$

$h \in \mathbb{N}$, $(h, p+1) = 1$

$\boxed{\text{primitive}}$ character: $\omega_2^h \neq \omega_2^{ph}$

Upshot: $\mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^h$ is a 2 dim$^{\mathrm{d}}$ irreducible $G_{\mathbb{Q}_p}$-rep$^{\mathrm{n}}$

Proposition: If $\bar{\rho} : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$ is absolutely irreducible

then $\bar{\rho} \cong \left(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^h\right) \otimes \mu_\lambda$ for some $(h, p+1) = 1$, $\lambda \in \mathbb{F}$

Sketch: $\bar{\rho} \circlearrowleft I_{\mathbb{Q}_p}^{\mathrm{wild}}$ $\Rightarrow$ $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ is a $\underbrace{\left(I_{\mathbb{Q}_p} / I_{\mathbb{Q}_p}^{\mathrm{wild}}\right)}_{\cong \varprojlim \mathbb{F}_{p^m}^\times}$-rep

trivial since $I_{\mathbb{Q}_p}^{\mathrm{wild}}$ a pro-p group

Norm map

abelian prime-to-p order

up to unramified character

$\Rightarrow \bar{\rho}|_{I_{\mathbb{Q}_p}} \simeq \omega_2^h \oplus \omega_2^{h'}$ Hence $\omega_2^h \hookrightarrow \bar{\rho}|_{G_{\mathbb{Q}_{p^2}}}$ $\overset{\text{Frobenius reciprocity}}{\rightsquigarrow}$ $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^h \longrightarrow \bar{\rho}$

non-zero

irreducible

Proposition: The (φ,Γ)-module of $\operatorname{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$ is

$r \in \{0, \dots, p-1\}$

$$M\left(\operatorname{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}\right) \xrightarrow{\sim} \mathbb{F}_p(\!(X)\!)\, e_0 \oplus \mathbb{F}_p(\!(X)\!)\, e_1 \quad \text{with:}$$

$$\varphi(e_0) = e_1$$

$$\varphi(e_1) = \frac{1}{X^{(r+1)(p-1)}} \cdot e_0$$

& the unique Γ-action (semilinear, commuting with φ) s.t. $\gamma \cdot x \equiv x \bmod X \quad \forall \gamma \in \Gamma$, $x \in M\left(\operatorname{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}\right)$.

Idea: describe : $M(\mu_\lambda)$

$M\left(\operatorname{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} V\right)$ via $M(V)$

$M(\omega_2^{r+1}) \leftarrow$ non formal

9'

Skip

Lemma 1 : Let $Y \in \mathbb{F}_p(\!(X)\!)^{sep}$ s.t. $Y^{p+1} = X$.

Then $g(Y) = \omega_2^p(g)\left(\omega(g) \cdot \underbrace{\frac{X}{g(X)}}_{\in 1 + X\mathbb{F}_p[\![X]\!]}\right)^{-\frac{1}{p+1}} \cdot Y$

Use $\mathbb{F}_p(\!(X)\!)^{sep} \hookrightarrow \mathcal{O}^\flat_{\mathbb{C}_p}$

compute here!

Lemma 2 : $M(\omega_2^{r+1}) \xrightarrow{\sim} \underbrace{(\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})(\!(X)\!) \cdot e_0}_{=:D}$ with $\varphi(e_0) = \frac{1}{X^{(r+1)(p-1)}} e_0$

& unique Γ-action (semilinear commuting with φ ≡ id mod X)

$\gamma(e_0) = \left(\omega(\gamma) \frac{X}{\gamma(X)}\right)^{\frac{r+1}{p+1}} \cdot e_0$

Sketch : Check the $G_{\mathbb{Q}_{p^2}}$-repⁿ attached to D.

$\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}(\!(X)\!) \xrightarrow{\sim} \mathbb{F}(\!(X)\!) \oplus \mathbb{F}(\!(X)\!)$

$\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p(\!(X)\!)^{sep} \longrightarrow \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p(\!(X)\!)^{sep} \oplus \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p(\!(X)\!)^{sep}$

$\rightsquigarrow D \xrightarrow{\sim} D^{(0)} \oplus D^{(1)}$

with basis

$v^{(0)} = (Y^{r+1}, 0) \cdot e_0 \quad v^{(1)} = (0, Y^{r+1}) e_0$

Check via $(v^{(0)}, v^{(1)})$ & Lemma 1 that :

9''

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$$V(D)|_{I_{\mathbb{Q}_p}} \simeq \omega_2^{r+1}$$

$$\parallel$$

$$\left(\left(\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p(X)^{sep}\right) \otimes D\right)^{\varphi = 1}$$

3.§ Smooth $GL_2(\mathbb{Q}_p)$-representations over $\mathbb{F}$

$\mathrm{stab}_{GL_2(\mathbb{Q}_p)}(v) \underset{\text{open}}{\subseteq} GL_2(\mathbb{Q}_p)\ \forall v \in \pi$

Key: Spherical Hecke algebras

classification

- first attempt: Barthel-Livné [BL 94]: GL_2
- Generalized to GL_n by Herzig [Her-11]
- supersingular for $GL_2(\mathbb{Q}_p)$: Breuil [Br-03]

two fundamental operations:

characters of $\mathbb{Q}_p^\times$

$$\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2) = \left\{ \begin{array}{l} f: GL_2(\mathbb{Q}_p) \to \mathbb{F},\ f(b\cdot g) = \chi_1\otimes\chi_2(b)\cdot f(g) \\ \text{smooth} \end{array} \right\}$$

left action by right translation

$$\underset{\substack{GL_2(\mathbb{Z}_p) \\ \downarrow \\ \text{smooth } GL_2(\mathbb{Z}_p)\text{-rep}^n}}{\mathrm{ind}^{GL_2(\mathbb{Q}_p)}}\ \sigma = \left\{ \begin{array}{l} f: GL_2(\mathbb{Q}_p) \to \sigma,\ f(k\cdot g) = k\cdot f(g) \\ \mathrm{supp}(f) \text{ compact} \end{array} \right\}$$

$\boxed{\text{Def}}$: A Serre weight is an absolutely irreducible $GL_2(\mathbb{F}_p)$ rep^n $/\mathbb{F}$

i.e. abs. irreducible <u>smooth</u> $GL_2(\mathbb{Z}_p)$ rep^n $/\mathbb{F}$

Complete classification: $r \in \{0, \dots, p-1\}$, $s \in \{0, \dots, p-2\}$ $\rightsquigarrow (r) \otimes \det^s := (\mathrm{Sym}^r \mathbb{F}^2) \otimes \det^s$

$\left(= (V(\begin{smallmatrix} r+s \\ s \end{smallmatrix}))(\mathbb{F}) \Big|_{GL_2(\mathbb{F}_p)} \right)$

<u>algebraic</u> rep^n

Explicit action: $(r) \cong \mathbb{F}[\![x, y]\!]_{\mathrm{hom}, \deg r}$ $\quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f \right)(x, y) = f(ax + cy, bx + dy)$

Note π irred. smooth admissible $\rightsquigarrow 0 \neq \mathrm{Hom}_{GL_2(\mathbb{Z}_p)}(\sigma, \pi|_{GL_2(\mathbb{Z}_p)}) \cong \mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \sigma, \pi)$

$\exists\, \sigma$ Serre weight

Frobenius reciprocity

$$\mathcal{H}(\sigma) := \mathrm{End}_{GL_2(\mathbb{Q}_p)}\left(\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \sigma\right)$$

Theorem: For any Serre weight σ, have

$$\mathcal{H}(\sigma) \xrightarrow{\sim} \mathbb{F}[T_1, T_2^{\pm}]$$

$T_1 \longleftrightarrow$ projection to $\mathbb{F}\cdot x^r \hookrightarrow (r)$ as element of $\mathrm{End}_{\mathbb{F}}(\sigma)$

- Consequence of Cartan decomposition $GL_2(\mathbb{Q}_p) = \bigsqcup_{n \leq m} GL_2(\mathbb{Z}_p)\begin{pmatrix} p^n & \\ & p^m \end{pmatrix} GL_2(\mathbb{Z}_p)$
- when $\sigma = \mathbb{1}$, $\mathcal{H}(\sigma) \xrightarrow{\sim} \mathcal{C}_c\left(GL_2(\mathbb{Z}_p) \backslash GL_2(\mathbb{Q}_p) / GL_2(\mathbb{Z}_p), \mathbb{F}\right)$

  with convolution

13

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e.g. $f : \sigma \hookrightarrow \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2$ then $f * T_1 = \chi_2(p)^{-1} \cdot f$

$f * T_2 = (\chi_1(p)\chi_2(p))^{-1} \cdot f$

Corollary: Let π be [absolutely irreducible, admissible, smooth] rep^n of $GL_2(\mathbb{Q}_p)/\mathbb{F}$

Berger: have central character

Then $\exists\ \lambda_1, \lambda_2, \sigma$ s.t. $\dfrac{\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \sigma}{(T_1 - \lambda_1, T_2 - \lambda_2)} \twoheadrightarrow \pi$

$\lambda_1 \in \mathbb{F}$, $\lambda_2 \in \mathbb{F}^\times$, σ Serre weight

In fact, choose $f : \sigma \longrightarrow \pi|_{GL_2(\mathbb{Z}_p)}$ simultaneous eigenvector for T_1, T_2

get $\chi_f : \mathcal{H}(\sigma) \longrightarrow \mathbb{F}$ hence $\left(\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \sigma\right) \otimes_{\mathcal{H}(\sigma), \chi_f} \mathbb{F} \twoheadrightarrow \pi$

Theorem ([BL94]) The absolutely irreducible admissible smooth rep^{ns} of $GL_2(\mathbb{Q}_p)$ / $\mathbb{F}$ fall into

1. Irreducible principal series $\mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2$, $\chi_1 \neq \chi_2$ $\longleftarrow$ comparison with $\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)}$ & "change of weight" (Herzig)
2. Twist of St $\longleftarrow$ Thm: $0 \to \mathbb{1} \to \mathrm{Ind}_{\overline{B}(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \mathbb{1} \to \underset{\substack{\uparrow \\ \text{irreducible}}}{St} \to 0$
3. Characters
4. Supersingular $\longleftarrow$ $\mathrm{Hom}_{GL_2(\mathbb{Z}_p)}(\sigma, \pi|_{GL_2(\mathbb{Z}_p)}) \circlearrowleft T_1$ nilpotent $\forall \sigma$

Breuil : $\dfrac{\mathrm{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \sigma}{(T_1, T_2 - 1)} =: \pi(\sigma, 0)$ are irreducible.

Moreover the only iso. : $\pi(r,0) \xrightarrow{\sim} \pi(p-1-r, 0) \otimes \omega^r \circ \det$

$\xrightarrow{\sim} \pi(r, 0) \otimes \mu_{-1}$

Theorem (Breuil: the mod p local Langlands correspondence)

There exists a unique bijection

$$\left\{\text{supersingular rep}^{n} \text{ of } GL_2(\mathbb{Q}_p)\right\}_{/\cong} \xrightarrow[\text{rec}\ \bar{\mathbb{F}}_p]{1:1} \left\{\begin{array}{l}\text{absolutely irred.}\\ \bar{\rho}: G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{F}})\end{array}\right\}_{/\cong}$$

$$[\pi(r,0)] \longmapsto \left[\mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}\right]$$

$$\pi(\bar{\rho}) \longleftarrow\!\shortmid \bar{\rho}$$

<u>Note</u> 1. If $\bar{\rho}$ is **generic** $(0<r<p-1)$

then $\mathrm{soc}_{GL_2(\mathbb{Z}_p)}(\pi(\bar{\rho})) = W(\bar{\rho})$

2. Include <u>all</u> $\bar{\rho}$:

$$0 \to \chi_1 \to \bar{\rho} \to \chi_2 \to 0 \ \xmapsto[\text{if } \chi_1\chi_2^{-1} \neq 1, \omega^{\pm}]{} \ 0 \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2\omega^{-1} \to \pi(\bar{\rho}) \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1\omega^{-1} \to 0$$

Key : $\pi(r,0)|_{GL_2(\mathbb{Z}_p)}$ can be $\boxed{\text{completely}}$ computed

harmless: have central character

Theorem : Let $r\in\{0,\dots,p-1\}$. Then $\pi(r,0)|_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \simeq \underset{\substack{\uparrow\\ (r)}}{\pi^{even}} \oplus \underset{\substack{\uparrow\\ (p-1-r)\otimes\det^r}}{\pi^{odd}}$

&

$$0\to \overbrace{(\omega^r\otimes 1)}^{=:\chi} \longrightarrow \pi^{even,+}\oplus\pi^{even,-}\longrightarrow \pi^{even}|_{I} \to 0 \qquad I := \begin{pmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p\\ p\mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}$$

$$0\to \overbrace{(1\otimes\omega^r)}^{\chi^s} \longrightarrow \pi^{odd,+}\oplus\pi^{odd,-}\longrightarrow \pi^{odd}|_{I} \longrightarrow 0$$

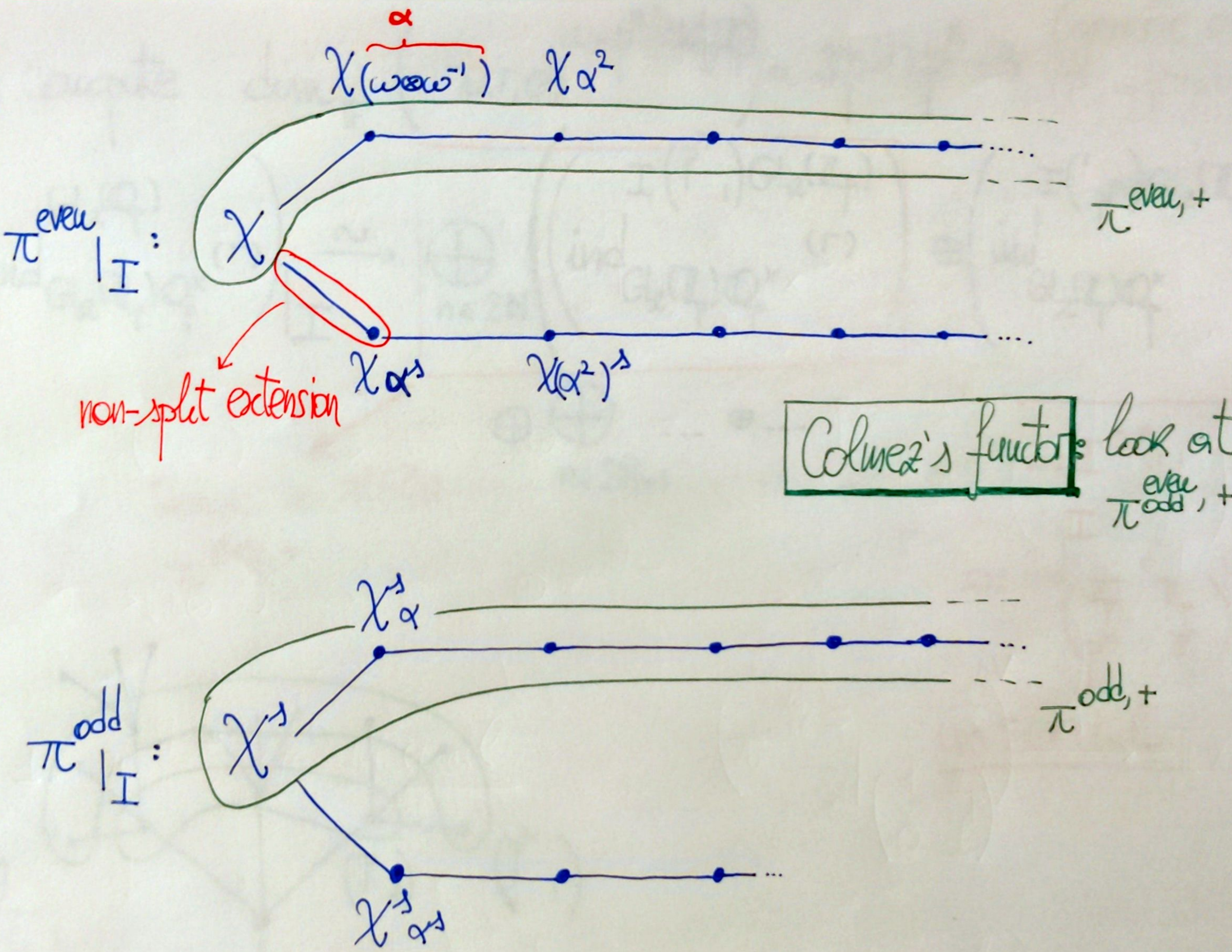
Rmk

1. $\pi^{\substack{even\\odd},\pm}$ are <u>uniserial</u> & $\left(\pi^{\substack{even\\odd},+}\Big|_{\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}}\right)^\vee \xrightarrow{\sim} \mathbb{F}[[X]]$

Specific to $GL_2(\mathbb{Q}_p)$!

2. $\left(\pi^{even,+} + \pi^{odd,+}\right) = \text{Image of } \sum_{n\in\mathbb{N}} \begin{pmatrix}p^n & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}\cdot(r)$ in $\pi(r,0)$

Upshot:



I-socle filtration gr^1 gr^2 gr^3 ....

$\rightsquigarrow$ Can compute $\dim_{\mathbb{F}}\left(\pi(r,0)^{1+p^{n+1}\mathrm{Mat}_2(\mathbb{Z}_p)}\right) = 2(p+1)p^n - 4$ (generic r)

($PS = p^n(p+1)$)

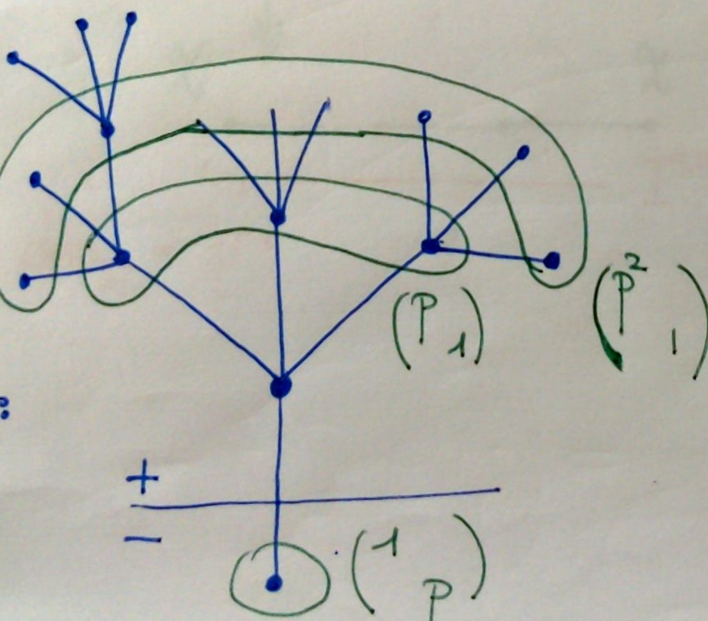
<u>Idea</u>: $\left(\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)}(r)\right)\Big|_I \xrightarrow{\sim} \boxed{\bigoplus_{n\in 2\mathbb{N}} \Bigg(\underbrace{\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{I\begin{pmatrix} p^n & \\ & 1\end{pmatrix}GL_2(\mathbb{Z}_p)}(r)}_{I^{+,n}_{(r)}}} \oplus \underbrace{\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{I\begin{pmatrix} & 1\\ p^{n+2} & \end{pmatrix}GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}(r)}_{=:\, I^{-,n+2}_{(r)}} \Bigg) \oplus \bigoplus_{n\in 2\mathbb{N}+1} \cdots \oplus \cdots$

Image in $\pi(r,0)$: $\pi^{\text{even},+}$

$I^{-,n+2}_{(r)} \simeq \mathrm{ind}_{\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^{n+2} & \mathbb{Z}_p^\times \end{pmatrix}}^{I}(r)$

<u>uniserial</u>

explictly compute I_1:



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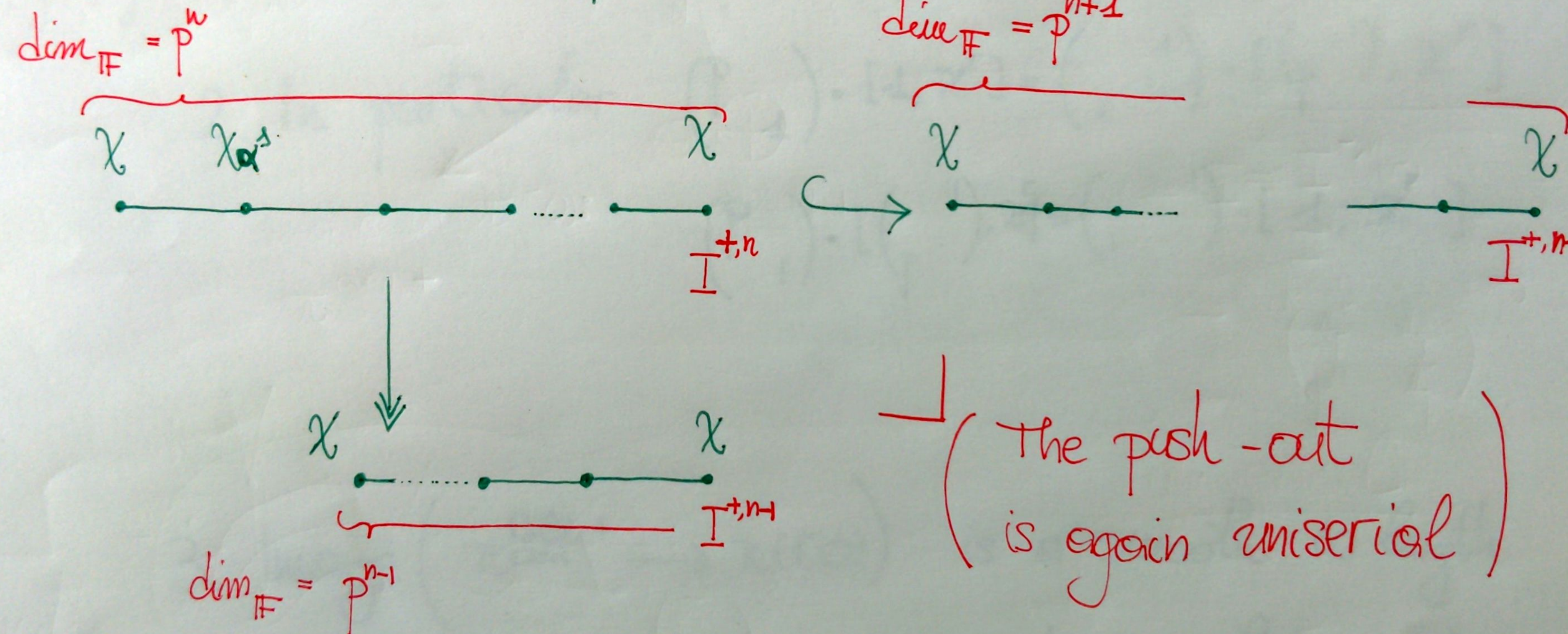
$$T\big|_{I^{+,n}_{(0)}} : I^{+,n}_{(0)} \longrightarrow I^{+,n+1}_{(0)} \oplus I^{+,n-1}_{(0)}$$ ← all uniserial

for r=0 $$\begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \longmapsto \left(\sum_{x\in\mathbb{F}_p} \begin{pmatrix} p^{n+1} & [x] \\ & 1 \end{pmatrix} , \begin{pmatrix} p^{n-1} & \\ & 1 \end{pmatrix} \right)$$

i.e.



(The push-out is again uniserial)

20

Upshot: 1. $\pi(r,0)^{\begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p & 1+p\mathbb{Z}_p \end{pmatrix} =: I_1} = \mathbb{F}\cdot[1, x^r] \oplus \mathbb{F}[\begin{pmatrix} & 1 \\ p & \end{pmatrix}, x^r]$

$(r)^{I_1} = \chi \qquad \chi^s = ((p-1-r)\otimes\det^r)^{I_1}$

2. In particular $\begin{pmatrix} p & \\ & 1 \end{pmatrix}\cdot[1, x^r] = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\cdot[\begin{pmatrix} & 1 \\ p & \end{pmatrix}, x^r]$

$\begin{pmatrix} p & \\ & 1 \end{pmatrix}\cdot[\begin{pmatrix} & 1 \\ p & \end{pmatrix}, x^r] = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\cdot[1, x^r]$

3. Image$\left(\pi^{even,+}_{odd} \longrightarrow \pi(r,0)\right)$ is a smooth ∞ length uniserial rep^n of $\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}$

Key phenomenon

$$\left(\pi^{\mathrm{even},+} \oplus \pi^{\mathrm{odd},+}\right) \hookrightarrow \pi(r,0)\big|_{I}$$

$\circlearrowleft \begin{pmatrix} p & \\ & 1 \end{pmatrix}$ $\quad$ $\circlearrowleft \begin{pmatrix} \mathbb{Z}_p^\times & \\ & 1 \end{pmatrix}$

smooth $\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}$-rep

i.e. $\boxed{\text{torsion } \mathbb{F}[\![X]\!]\text{-module}}$

Observation $\begin{pmatrix} p & \\ & 1 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_p^\times & \\ & 1 \end{pmatrix} \circlearrowleft \mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}]\!] \simeq \mathbb{F}[\![X]\!]$ by conjugation:

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix} \cdot X = X^p \quad \& \quad \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot X = (1+X)^{\gamma} - 1$$

$$\Rightarrow \text{If } M \circlearrowleft \overset{\text{continuous}}{} \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \text{ then } M\big|_{\mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}]\!]} \otimes_{\mathbb{F}[\![X]\!]} \mathbb{F}(\!(X)\!) \in \phi\text{-}\Gamma\text{-}\mathrm{Mod}_{\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}}$$

4 § Colmez's functor ([Col10], § 4.1)

Skip

$$D_r^\natural(\pi) := \left(\pi^{even,+} \oplus \pi^{odd,+}\right)^\vee \hookleftarrow D_r^+(\pi) := \left(\frac{\pi^{even,+}}{[1, x^r]} \oplus \frac{\pi^{odd,+}}{[\begin{pmatrix} & 1 \\ p & \end{pmatrix}, x^r]}\right)^\vee$$

$\pi = \pi(r,0)$

index 2

$\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}$-acts

free $\mathbb{F}[[X]]$-modules
rk 2
Γ-semilinear action

$\begin{pmatrix} p & \\ & 1 \end{pmatrix}$-acts

greatest φ-stable lattice

<u>Hence</u>:

$$D_r^+(\pi) \otimes_{\mathbb{F}[[X]]} \mathbb{F}((X)) \xrightarrow{\sim} D_r^\natural(\pi) \otimes_{\mathbb{F}[[X]]} \mathbb{F}((X)) =: D(\pi)$$

$$\cap$$

$$\Phi\text{-}\Gamma\text{-Mod}_{\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}}$$

formal properties

more subtle properties (exactness, image as $G_{\mathbb{Q}_p}$-rep)

Skip

Note: $D^{\natural}_{r}(\pi)\otimes_{\mathbb{F}[[X]]}\mathbb{F}((X))$ is independent of (r): functoriality of $\pi \mapsto D(\pi)$

finite dim^{al}

$\mathrm{Rep}_{\mathbb{F}}^{sm,\, f.l.,\, adm}\, GL_2(\mathbb{Q}_p)$

$\cup$

Def: Let $W \hookrightarrow \pi|_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}}$ s.t. $\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}}^{GL_2(\mathbb{Q}_p)} W \xrightarrow{pr_W} \pi$

say W gives a standard presentation of π if

$$\ker(pr_W) = \langle GL_2(\mathbb{Q}_p)\cdot W \cap \begin{pmatrix} p & \\ & 1\end{pmatrix} W \rangle_{\mathbb{F}}$$

$\to$ Get a Serre subcategory (even for $\mathbb{Q}_p \rightsquigarrow K/\mathbb{Q}_p < \infty$)

Theorem (Colmez): Any $\pi \in \mathrm{Rep}_{\mathbb{F}}^{sm, fl, adm}(GL_2(\mathbb{Q}_p))$ has a standard presentation

(24)

Skip

Define: $D^{\natural}_{W}(\pi) := \left(\mathrm{Image}\left(\bigoplus_{n\in\mathbb{N}} I^{+,n}(W) \longrightarrow \pi\right)\right)^{\vee}$

$D^{+}_{W}(\pi) := \left(\mathrm{Image}\left(\bigoplus_{\substack{n\in\mathbb{N}\\ n\geq 1}} I^{-,n}(W) \longrightarrow \pi\right)\right)^{\perp}$ $\circlearrowleft \begin{pmatrix} p & \\ & 1\end{pmatrix} \begin{pmatrix} \mathbb{Z}_p^{\times} & \\ & 1\end{pmatrix}$

$\boxed{\textbf{Lemma}}$ (i) $D^{+}_{W}(\pi) \hookrightarrow D^{\natural}_{W}(\pi)$ has finite cokernel

(ii) $D^{\natural}_{W'}(\pi) \twoheadrightarrow D^{\natural}_{W}(\pi)$ has finite kernel (for $W \subseteq W'$ std. presentation)

Upshot: Get a well defined functor

$$\begin{array}{ccc}\mathrm{Rep}_{\mathbb{F}}^{\mathrm{sm, f.l., adm}}\, GL_2(\mathbb{Q}_p) & \longrightarrow & \Phi\text{-}\Gamma\text{-}\mathrm{Mod}_{\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}} \\ \pi & \longmapsto & D_W^{\natural}(\pi) \otimes_{\mathbb{F}_p[\![X]\!]} \mathbb{F}(\!(X)\!) =: D(\pi)\end{array}$$

$\boxed{\text{any}}$ W giving a standard presentation

Theorem (Colmez) [Col10, IV.2.12, IV.1.8]: The functor $\pi \longmapsto D(\pi)$ is $\boxed{\text{exact}}$ & $\boxed{D(\pi) \text{ is étale}}$

SKP

Exactness : Show $0 \to \pi_1 \longrightarrow \pi \longrightarrow \pi_2 \longrightarrow 0$

$\Rightarrow \exists$ std presentation s.t.

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$$

&

$$0 \longrightarrow \mathrm{Ker}\left(I^+(W_1) \to \pi_1\right) \longrightarrow \mathrm{Ker}\left(I^+(W) \to \pi\right) \longrightarrow \mathrm{Ker}\left(I^+(W_2) \longrightarrow \pi\right) \longrightarrow 0$$

are both exact

Etaleness : Show

$$\mathrm{Coker}\left(\begin{array}{ccc} \bigoplus_{x \in \mathbb{F}_p} D^+_W(\pi) & \hookrightarrow & D^+_W(\pi) \\ (\mu_x)_x & \longmapsto & \sum_{x \in \mathbb{F}_p} \begin{pmatrix} p & [x] \\ & 1 \end{pmatrix} \cdot \mu_x \end{array}\right)$$

is finite dim^{al}

Difficulty: Computation of $D(\pi(r,0))$ is <u>indirect</u>

(i) prove it is irreducible

(ii) compute $\Gamma \curvearrowright G^{D^\natural / X \cdot D^\natural}$

More direct approach: torsion $\mathbb{F}[\![X]\!][F]$-modules (lecture 4)

$F \cdot X = X^p F$

$\boxed{\text{Recall}}$ $M := \pi^{\text{even},+} \oplus \pi^{\text{odd},+}$ is a torsion $\mathbb{F}[\![X]\!]$-module

i) $M[X] = \langle \underbrace{[1, x^r]}_{e_0}, \underbrace{[\begin{pmatrix} p & \\ & 1 \end{pmatrix}, x^r]}_{e_1} \rangle$

commute

ii) $F \curvearrowright M$ (induced by $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$)

iii) smooth $\mathbb{Z}_p^\times$-semilinear action

Lemma: Endow $M^\vee$ with: $\mathbb{F}[[X]]$-action $(X.f)(t) := f(X.t)$

F-action $(F.f)(t) := f(F.t)$

Then $M^\vee \otimes_{\mathbb{F}[[X]]} \mathbb{F}((X)) \in \Phi\text{-}\Gamma\text{-}\mathrm{Mod}^{\text{ét}}_{\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}}$

Reason

$$\mathbb{F}_p[[X]] \otimes_{\varphi} M \xrightarrow{id \otimes F} M \;\underset{(\_)^\vee \otimes_{\mathbb{F}_p[[X]]} \mathbb{F}_p((X))}{\rightsquigarrow}\; M^\vee \otimes_{\mathbb{F}_p[[X]]} \mathbb{F}_p((X)) \xhookrightarrow[\sim]{(id \otimes F)^\vee} \left(M^\vee \otimes \mathbb{F}((X))\right) \otimes_{\varphi} \mathbb{F}_p[[X]]$$

the inverse is $\varphi \otimes id$ by definition

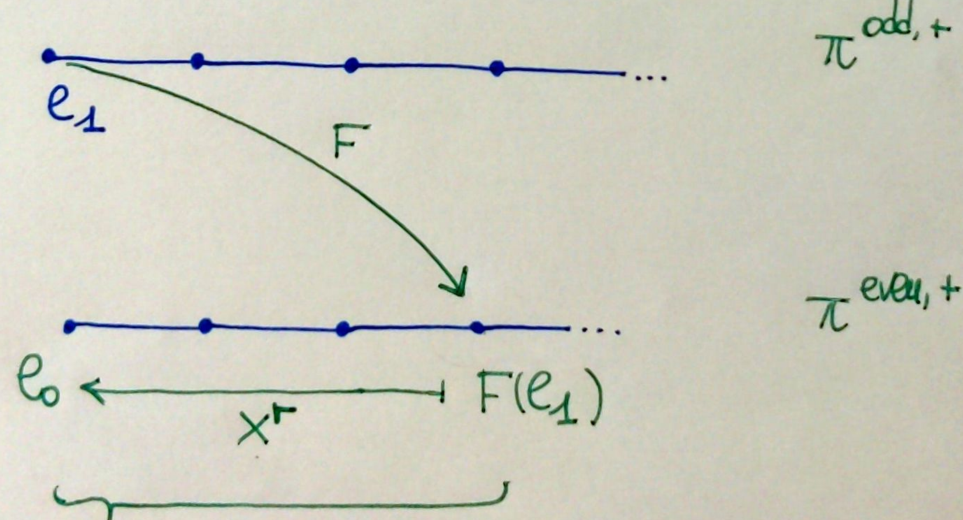
Key computation : $\begin{pmatrix} p & \\ & 1 \end{pmatrix} \cdot e_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot e_1$

this is $\left((p-1-r)\otimes \det^r\right)^{I_1}$

$$\begin{pmatrix} p & \\ & 1 \end{pmatrix} \cdot e_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot e_0$$

this is $(r)^{I_1}$

i.e.



Serre weight $(r) \hookrightarrow \mathrm{soc}_{GL_2(\mathbb{Z}_p)}\, \pi(r,0)$

Conclusion

$$M^\vee \otimes_{\mathbb{F}_p[[X]]} \mathbb{F}_p(X) \xrightarrow{\sim} \mathbb{F}(X).e_0^\vee \oplus \mathbb{F}(X).e_1^\vee$$

with $\varphi(e_0^\vee) = \dfrac{1}{X^{p-1-r}} \cdot e_1^\vee$

$\varphi(e_1^\vee) = \dfrac{1}{X^r} \cdot e_0^\vee$

i.e. (change of basis)

$$\boxed{\begin{aligned} \varphi(e_0'^\vee) &= \frac{1}{X^{(p-1)(r+1)}} \cdot e_1'^\vee \\ \varphi(e_1'^\vee) &= e_0'^\vee \end{aligned}}$$

Γ-action trivial mod X

$$\boxed{\text{This is } M\left(\mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}\right)}$$