Recall Product topology

X_λ topological spaces ($\lambda \in \Lambda$)

$\prod_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} : x_\lambda \in X_\lambda\}$

$p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu$
$x_\mu : (x_\lambda)_{\lambda\in\Lambda} \to x_\mu$

Product topology on $\prod X_\lambda$ is the *coarsest* topology such that all the p_μ' s are continuous.

we saw the basis $\prod_{\lambda\in\Lambda} U_\lambda$ where $U_\lambda \subset X_\lambda$ open $\forall\ \lambda$ and $X_\lambda = U_\lambda$ for λ outside for some finite subset of Λ

$Y \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ continuous $\Leftrightarrow$ f_λ is continuous for all λ
$y \longmapsto (f_\lambda(y))$.

Box topology : Basis $\prod_{\lambda\in\Lambda} U_\lambda$, $U_\lambda \subset X_\lambda$ open $\forall\ \lambda$
* *finer than product topology*

§20, 21 Metric Topology

Recall Metric space: (X, d), X set, $d : X \times X \to \mathbb{R}_{\geq 0}$

(i) $d(x, y) = 0 \Leftrightarrow x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ – *triangular inequality*

Open ball (or ϵ–ball)

$B_{\epsilon, d}(x) = \{y \in X : d(x, y) < \epsilon\}$ or just $B_\epsilon(x)$

Definition The ***metric topology*** is the topology on X generated by the basis $\{B_{\epsilon, d}(x) : \epsilon > 0,\ x \in X\}$

Check basis:
(i) union in X : clear
(ii) $x \in B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$

$B_r(x) \subset B_{\epsilon_1}(x_1) \subset B_{\epsilon_2}(x_2)$, provided $r \leq \min(\epsilon_1 - d(x_1, x_2), \epsilon_2 - d(x_1, x_2))$.
This follows by triangular inequality.

Theorem 23

Let (X, d) be metric space $U \subset X$ open $\Leftrightarrow \forall\ x \in U\ \exists\ \epsilon > 0$ such that $B_\epsilon(x) \subset U$

Proof: "$\Leftarrow$" trivial. "$\Rightarrow$" By definition, $\exists\ \epsilon > 0$, $y \in X$ such that $x \in B_\epsilon(y) \subset U$. But $B_r(x) \subset B_\epsilon(y)$ provided $r \leq \epsilon - d(x, y)$. □

Examples

(1) $X = \mathbb{R}^n$ $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ (Euclidean metric)

metric topology = standard topology

(2) X arbitrary set

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

metric topology = discrete topology

If $|X| > 1$, $\nexists\, d$: metric s.t. metric topology of (X, d) is the trivial topology.

Why ?

Lemma 24

Any metric topology is T_2.
Proof: If $x \neq y$, then $B_\epsilon(x)$, $B_\epsilon(y)$ disjoint nbds provided $\epsilon \leq \frac{1}{2}\, d(x, y)$.

Definition A topological space (X, τ) is ***metrizable*** if $\exists$ metric d on X s.t. metric topology of (X, d) equals τ.

Other basic properties of the metric topology.

(1) X, Y metric spaces. $f : X \to Y$ in continuous for metric topology $\Leftrightarrow$ continuous in ϵ–δ sense. (as in lecture 1)
(2) If Y $\subset$ X subset of a metric space (X, d), then the two natural topologies on Y coincide.
- subspace topology in metric topology on X.
- metric topology of $(Y, d|_{Y\times Y})$

This justifies why $S^2 \setminus \{N\} \to \mathbb{R}^2$ continuous where $S^2 \setminus \{N\}$ has subspace topology in $\mathbb{R}^3$.

$$(a, b, c) \longmapsto \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

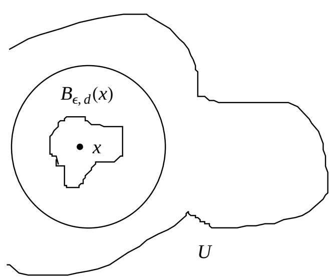
(3) If (X_i, d_i) for $1 \leq i \leq n$ metric spaces, then the product topology on $\prod X_i$ is the metric topology of $(\prod X_i, d)$, where $d((x_i)_{i=1}^n, (x_i)_{i=1}^n) = \max(d_i(x_i, y_i))$.
$\therefore$ the product topology on $\mathbb{R}^n$ is the metric topology of the metric $d_\square(\mathbf{x}, \mathbf{y}) = \max_{1\leq i\leq n} \{|x_i - y_i|\}$

Now we can see in a nicer way that product topology = standard topology on $\mathbb{R}^n$.

Thereom 25

Suppose d, d' are metrics on a set X inducing metric topology τ, τ'. Then $\tau \subset \tau' \Leftrightarrow \forall\, x \in X$, $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $B_{\delta, d'}(x) \subset B_{\epsilon, d}(x)$.

Proof:



Let $U \subset X$ open in τ. We need U open in τ'. U open in $\tau \Rightarrow \exists\, \epsilon > 0$ such that $B_{\epsilon, d}(x) \subset U$. By assumption, $\exists\, \delta > 0$, such that $B_{\delta, d'}(x) \subset B_{\epsilon, d}(x)$. □

Examples

1) $X = \mathbb{R}^2$, d = Euclidean metric, $d_\square((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|)$

2) $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum (x_i - y_i)^2}$ (standard topology), $d_\square(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i|\}$ (product topology)

We want to apply theorem 25 to see these are the same topology

$d_\square(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{y}) \le \sqrt{n}\, d_\square(\mathbf{x}, \mathbf{y}) \Rightarrow B_{\epsilon/\sqrt{n}, d_\square} \subset B_{\epsilon, d} \subset B_{\epsilon, d_\square}$

Theorem 26

Let (X, d) metric space then there exist a metric d' on X that induces the same topology as d. s.t. $d'(x, y) \le 1$. $\forall\, x, y$.

Examples

$d'(x, y) = \min(1, d(x, y)) = \overline{d}(x, y)$

$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$

Proof:

We need to show that d and $\overline{d}$ induce the same topology. $\overline{d}$ is a metric: clearly $0 \le \overline{d} \le 1$.

(i) $\overline{d}(x, y) = 0 \Rightarrow d(x, y) = 0 \Rightarrow x = y$

(ii) $\overline{d}(x, y) = \overline{d}(x, y)$

(iii) triangle inequailty, (we want $\min(1, d(x, z)) \le \min(1, d(x, y)) + \min(1, d(y, z))$)

This is true since if $d(x, y) \ge 1$, then LHS $\le 1 \le d(x, y) \le$ RHS, for $d(y, z) \ge 1$ is similarly done.

If $d(x, y) < 1$ and $d(y, z) < 1$. We need $\min(1, d(x, z)) \le d(x, y) + d(y, z)$. True b/c LHS $\le d(x, z) \le$ RHS.

Hence $\overline{d}$ is metric. Now we will show that d, $\overline{d}$ induce the same topology. This is true b/c $B_{\epsilon, \overline{d}}(x) = B_{\epsilon, d}(x)$, $\forall\, \epsilon \le 1$ and by Thm 25. □

Infinite Products

(X_λ, d_λ) metric spaces $(\lambda \in \Lambda)$. In general, $\prod_{\lambda \in \Lambda} X_\lambda$ not metrizable (counterexample in §21). But can at least define a natrual metric on it. $\overline{\rho}((x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda}) = \sup\{\overline{d}_\lambda(x_\lambda, y_\lambda) \mid \lambda \in \Lambda\}$: A real number in $[0, 1]$ b/c $0 \le \overline{d}_\lambda \le 1$, $\forall\, \lambda$. "uniform metric"

Just consider the case $X_\lambda = \mathbb{R}$, $\forall\, \lambda \Rightarrow \prod X_\lambda = \mathbb{R}^\Lambda$, $\overline{d}_\lambda(x, y) = \min(1, |x - y|)$, $\forall\, \lambda$.

Check this is a metric (exercise).

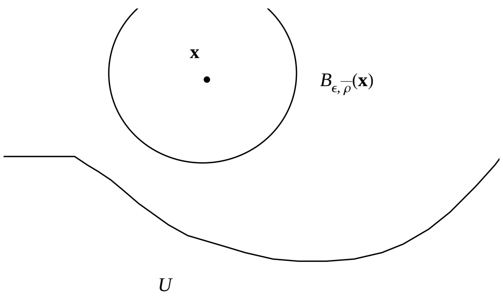
Theorem 27

On $\mathbb{R}^\Lambda$, this metric topology of $\overline{\rho}$ (uniform topology) is coarser than the box topology and finer than the product topology.

Proof:

Compare with box topology: Fix U open in the uniform topology, pick $x \in U$. U is open $\Rightarrow \exists\, \epsilon > 0$ s.t. $B_{\epsilon, \overline{\rho}}(\mathbf{x}) \subset U$.

$B_{\epsilon,\overline{\rho}}(\mathbf{x}) = \left\{y \in \mathbb{R}^{\Lambda} : \sup \overline{d}(x_\lambda, y_\lambda) < \epsilon\right\}.$



If $\overline{d}(x_\lambda, y_\lambda) < \frac{\epsilon}{2}\ \forall\, \lambda \Rightarrow \sup \overline{d}(x_\lambda, y_\lambda) \leq \frac{\epsilon}{2} < \epsilon \Leftrightarrow y \in \prod B_{\epsilon/2,\overline{d}}(x_\lambda)$ (open set in box topology.)
proof continued next lecture.