MAT 347
Semidirect products
October 29, 2019

Semidirect products

Recall that if if A, B are subsets of a group G, we write $AB = \{ab : a \in A, b \in B\} \subset G$. *The first two problems are a review from Homework 5!*

1. Suppose that H, K are subgroups of G. Suppose that $H \cap K = \{1\}$. Prove that every element of HK can be written uniquely as hk for $h \in H, k \in K$.

2. Suppose that H, K are normal subgroups of G and $H \cap K = \{1\}$. Explain how to multiply h_1k_1 with h_2k_2. Prove that HK is isomorphic to $H \times K$.

3. Prove that $D_{4n} \cong D_{2n} \times \mathbb{Z}/2\mathbb{Z}$ if n is odd.

4. Suppose that N is normal in G, but K is not. (We will write N instead of H, to remember that N is normal.) Explain how to multiply n_1k_1 with n_2k_2, expressing your answer as nk for some $n \in N, k \in K$.

5. Suppose now that N, K are two abstract groups (i.e. not embedded as subgroups of a third group). Suppose that we are given a homomorphism $\varphi : K \to \operatorname{Aut} N$, so for each element $k \in K$, we are given an automorphism $\varphi(k) : N \to N$ of N. Explain how we can use this to define a new group structure on the set $N \times K$, motivated by your computation in Question 4. The set $N \times K$ with this group structure will be denoted $N \rtimes_\varphi K$ and is called the *semidirect product* of N and K with respect to φ.

6. Show that N, K are both (isomorphic to) subgroups of $N \rtimes_\varphi K$ and that N is a normal subgroup.

7. Show that D_{2n} is isomorphic to a semidirect product of $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

8. Let F be a field. Consider $N = F, K = F^\times$. Define a natural map $\varphi : K \to \operatorname{Aut} N$ (not the trivial one with kernel K) and form the semidirect product $N \rtimes_\varphi K$. How can you think about this group?

Remark: we write $N \rtimes_\varphi K$ (as opposed to $K \rtimes_\varphi N$) to remember that $N \lhd G$. In other words, the triangle faces the same way.

Isometries

Definition: An *isometry of the plane* is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(x)-f(y)| = |x-y|$ (where $|\cdot|$ denotes the length of a vector). The set of isometries of the plane forms a group $\mathrm{Isom}(\mathbb{R}^2)$ under composition.

1. Show that any translation is an isometry.
2. Show that any orthogonal linear operator on $\mathbb{R}^2$ is an isometry.
3. Show that any isometry is the composition of a translation and an orthogonal linear map. (You may use the following fact without proof: if f is an isometry such that $f(0) = 0$, then f is an orthogonal linear map.)
4. What can you say about the subgroup of translations inside $\mathrm{Isom}(\mathbb{R}^2)$?
5. Express $\mathrm{Isom}(\mathbb{R}^2)$ as a semidirect product.