MAT 347
Quotient groups
September 30, 2019

Let G be a group and let $S \subseteq G$. We want to define an equivalence relation in G that will identify all the elements in S, and we want to maintain the group operation. Given $a, b \in G$, we say that $a \sim b$ if there exists $x \in S$ such that $b = ax$. In general, this relation will not be an equivalence relation.

1. Find necessary and sufficient conditions for $\sim$ to be an equivalence relation.

For the rest of this worksheet, let G be a group and let $H \leq G$. We will consider the equivalence relation defined above with $S = H$. Given $a \in G$, the *left coset* of a is the equivalence class of this relation, and we denote it aH. (Why do we use this notation?) The *quotient set* G/H is the set of all equivalence classes. The *index* of H on G, written $|G : H|$ is the number of equivalence classes.

2. Prove that $aH = bH$ iff [there exists $x \in H$ such that $b = ax$] iff $a^{-1}b \in H$
3. In general, what is the cardinality of each coset aH? What is the relation between $|G|$, $|H|$, and $|G : H|$?
4. Consider the group $G = D_8$ and consider the two subgroups $H_1 := \langle s \rangle$ and $H_2 := \langle r \rangle$. For each of them, write the complete list of cosets, and list which elements are in each coset.

Next, we want to try to use the operation on G to define an operation on the set G/H. Given $aH, bH \in G/H$, we can try to define their product by

$$(aH) \star (bH) = (ab)H$$

[*Note:* I am using $\star$ to emphasize that I am defining a new operation. As soon as we make sure this operation works and there is no ambiguity, we will drop the $\star$.]

5. In general, the operation $\star$ is not well-defined. Going back to the example in Question 4, show that with one of those subgroups, the operation is well-defined, but with the other subgroup, the operation is not well-defined.

Definition: We say that the subgroup H is a *normal subgroup* of G if the operation $\star$ in G/H is well-defined. We write $H \lhd G$ (the book writes $H \trianglelefteq G$).

6. Assume $H \lhd G$. In this case we know the operation in G/H is well-defined. What other conditions do we need to impose so that G/H is a group with this operation?

The big theorem about normal subgroups

Notation: Let G be a group. Given subsets A, B and elements x, y we will use the following notation:

$$\begin{aligned} xA &:= \{xa \mid a \in A\}, \\ xAy &:= \{xay \mid a \in A\}, \\ AB &:= \{ab \mid a \in A, b \in B\}, \text{ etc.} \end{aligned}$$

7. Let G be a group and let $H \leq G$. Explore the relation between the following statements (which ones imply which ones)?

   (a) $H \lhd G$.

   (b) There exists some group L and some group homomorphism $f : G \to L$ such that $H = \ker f$.

   (c) $aH = Ha$ for all $a \in G$.
   (Notice that this does not mean that a commutes with the elements of H. It only means that the sets aH and Ha are the same set.)

   (d) $aHa^{-1} = H$ for all $a \in G$.

   (e) $aHa^{-1} \subseteq H$ for all $a \in G$.

Another look at quotient groups

8. Suppose that G is a group and $\sim$ an equivalence relation on G such that the set $G/\sim$ of equivalence classes becomes a group in the natural way. In other words, $\overline{g} \cdot \overline{h} = \overline{gh}$ for all $g, h \in G$, where $\overline{x}$ denotes the equivalence class of any $x \in G$. Show that $\sim$ is one of the equivalence relations considered in Problem 1, where $S = N$ is a normal subgroup of G, i.e. $G/\sim$ is nothing but the quotient group G/N. (Hint: it may help to notice that the map $G \to G/\sim$ sending x to $\overline{x}$ is a homomorphism.)