**MAT 347**
**An example of the FTGT**
**March 17, 2020**

# The Fundamental Theorem of Galois Theory

**Definition.** A finite field extension $K/F$ is *Galois* if it is normal and separable.

**Theorem.** *Let $K/F$ be a finite Galois extension. Let $G = \mathrm{Gal}(K/F)$. Consider the maps $\widehat{I}$ and $\widehat{G}$ from Section 4 of Alfonso's notes.*

*(1) The maps $\widehat{I}$ and $\widehat{G}$ are inverses of each other. In other words:*

- $\widehat{G}(\widehat{I}(H)) = \mathrm{Gal}(K/\mathrm{Inv}(H)) = H$ *for every subgroup* $H \leq G$.
- $\widehat{I}(\widehat{G}(M)) = \mathrm{Inv}(\mathrm{Gal}(K/M)) = M$ *for every intermediate field* $F \subseteq M \subseteq K$.

*(2) Let $H \leq G$ and let $M = \widehat{I}(H)$, so that $H = \widehat{G}(M)$.*
*Then $|H| = [K : M]$. In particular $|G| = [K : F]$.*
*Equivalently, $(G : H) = [M : F]$.*

*(3) Under the same conditions as in Part (2), $K/M$ is always Galois. In addition, TFAE:*

- *$M/F$ is Galois.*
- *$M/F$ is a normal field extension.*
- *$H$ is a normal subgroup of $G$.*

*In that case,* $\mathrm{Gal}(M/F) \cong \mathrm{Gal}(K/F)/\mathrm{Gal}(K/M)$.

# An Example

Let $f(X) = X^4 - 2$. Let $K \subset \mathbb{C}$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$.

1. Find all roots of $f(X)$ in $\mathbb{C}$.
2. Find a (nice) set of two elements that generate the field extension $K/\mathbb{Q}$.
3. Calculate $[K : \mathbb{Q}]$.

4. Find a basis for $K$ as a $\mathbb{Q}$-vector space.

5. List all the elements of $G$ by showing how they act on a set of generators. (*Hint:* the easy part is to give an upper bound $|G| \leq \ldots$ by considering the action on generators. On the other hand, use the FTGT to compute $|G|$ to see that all such actions are possible.)

6. Determine the group $G$ up to isomorphism. (*Hint:* study the group operation by using the action on generators.)

7. Find all intermediate fields of $K/\mathbb{Q}$.

8. Which intermediate fields are normal over $\mathbb{Q}$? Each one of them has to be the splitting field of some polynomial. Find such polynomials.

9. Find a primitive element for the field extension $K/\mathbb{Q}$, i.e. an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. (*Hint:* find an $\alpha$ such that $\widehat{G}(\mathbb{Q}(\alpha)) = 1\ldots$)