MAT 347
Factorization in polynomial rings
January 20, 2020

Let us fix an integral domain R for this worksheet. We want to find out when $R[X]$ is a UFD. Our strategy is as follows. Let F be the field of fractions of R. We know that $F[X]$ is a Euclidean domain, hence a UFD. Let $f(X) \in R[X]$. We can factor $f(X)$ uniquely as a product of irreducibles in $F[X]$, but does this means we get a unique factorization in $R[X]$?

0. Show that R is a UFD iff every non-zero non-unit in R is a product of prime elements.

1. Let $I \trianglelefteq R$. Denote by $(I) \trianglelefteq R[X]$ the ideal of $R[X]$ generated by the subset I. Notice that I consists of the polynomials in $R[X]$ all of whose coefficients are in I. Prove that
$$R[X]/(I) \cong (R/I)[X].$$
(Hint: remember the isomorphism theorems.)

2. Continue with the notation of Question 1. Prove that $I \trianglelefteq R$ is a prime ideal iff $(I) \trianglelefteq R[X]$ is a prime ideal.

   Hint: Use the characterization of prime ideals in terms of the quotient they generate.

3. Let $p \in R$. Prove that p is prime in R iff p is prime in $R[X]$.

   Hint: Use the characterization of prime element in terms of the ideal it generates.

4. Prove that if $R[X]$ is a UFD, then R is a UFD.

   Hint: Remember Question 0.

For the rest of this worksheet, we will assume that R is a UFD.

Definitions. Let R be a UFD. Let $f(X) \in R[X]$ be non-zero. We define the *content* of $f(X)$, denoted C_f, as the GCD of all the coefficients of $f(X)$. We could also interpret C_f to be the "greatest" divisor of $f(X)$ among the elements in R. Notice that content is only defined up to associates. We say that f is *primitive* if its content is 1. Notice that every non-zero polynomial can be written as the product of its content and a primitive polynomial, and that this decomposition is unique up to multiplication by units.

5. Prove that the product of two primitive polynomials is a primitive polynomial.

   Hint: Assume that $f(X), g(X)$ are primitive and that $d = C_{fg}$. Let p be an irreducible factor of d in R. Use Question 3.

6. Prove that $C_{fg} = C_f C_g$ for every non-zero $f(X), g(X) \in R[X]$.

7. **Gauss' Lemma:** Let $f(X) \in R[X]$. Prove that if $f(X)$ is reducible in $F[X]$, then it is reducible in $R[X]$.

   More specifically, assume that

   $$f(X) = a(X)b(X)$$

   with $a(X), b(X) \in F[X]$, $\deg a(X) \geq 1$, $\deg b(X) \geq 1$. Then show that we can find $\lambda \in F^\times$ such that

   $$f(X) = A(X)B(X)$$

   with $A(X) = \lambda a(X) \in R[X]$ and $B(X) = \lambda^{-1} b(X) \in R[X]$.

   Give an example that shows that it is not possible to conclude that $a(X), b(X) \in R[X]$.

   Hint: first find a non-zero $d \in R$ such that $df(X) = a_1(X)b_1(X)$, where $a_1(X)$, $b_1(X)$ are in $R[X]$ and are scalar multiples of $a(X), b(X)$. Then try to get rid of d...

8. Let $f(X) \in R[X]$ be primitive. Show that $f(X)$ is irreducible in $R[X]$ if and only if it is irreducible in $F[X]$.

9. Suppose $f(X), g(X) \in R[X]$ are primitive. Show that $f(X), g(X)$ are associates in $R[X]$ if and only if they are associates in $F[X]$.

10. Prove that $R[X]$ is a UFD. How can you describe the irreducible elements in terms of the irreducibles of R and $F[X]$?