# MAT 1100, Algebra I, Fall 2019
# Homework 5, due on Friday December 6

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**All rings in these problems are commutative (except in the very last problem).**

1. Suppose $R$ is a domain and $M$ an $R$-module. Recall that $M_{\text{tor}} = \{x \in M : rx = 0 \text{ for some } r \in R - \{0\}\}$, a submodule of $M$. We say that $M$ is *torsion* if $M_{\text{tor}} = M$.

   (a) Let $\operatorname{Ann}_R(M) := \{r \in R : rm = 0\ \forall m \in M\}$, the *annihilator* of $M$. Show that $\operatorname{Ann}_R(M)$ is an ideal of $R$.

   (b) Suppose that $I$, $J$ are ideals of $R$ such that $R/I \cong R/J$ as $R$-modules. Show that $I = J$. (Hint: consider annihilators.)

   (c) Aside (for $R$ any commutative ring): show that an $R$-module $M$ is *simple* (i.e. it's nonzero and its only submodules are $\{0\}$ and $M$) if and only if $M$ is isomorphic to $R/I$ with $I$ a maximal ideal of $R$.

   (d) If $M$ is a finitely generated torsion $R$-module show that $\operatorname{Ann}_R(M) \neq 0$.

   (e) Give an example of a domain $R$ and a *torsion* $R$-module $M$ such that $\operatorname{Ann}_R(M) = 0$.

2. (a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. (Hint: construct maps in both directions.)

   (b) Find $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

   (c) Consider $V := \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$, and let $e_1$, $e_2$ be the standard basis of $\mathbb{R}^2$. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V$ isn't a pure tensor, i.e. is not of the form $v \otimes w$ for some $v, w \in \mathbb{R}^2$.

3. Suppose $R$ is a commutative ring, $M$ an $R$-module, and $I, J \lhd R$ ideals.

   (a) Show that $(R/I) \otimes_R M \cong M/IM$, where $IM$ is the submodule of $M$ generated by the elements $rm$ ($r \in I$, $m \in M$). (Hint: construct maps in both directions.)

   (b) Deduce that $(R/I) \otimes_R (R/J) \cong R/(I + J)$ as $R$-modules.

(c) Determine $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n$ as abelian group for $m, n \geq 1$.

(d) Let $f(x) := (x^2+1)^5(x^4-1)$, $g(x) := (x-i)^3(x^2-1)^2$. Express the $\mathbb{C}[x]$-module $\mathbb{C}[x]/(f(x)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(g(x))$ in both canonical forms that we discussed in class.

(e) Now redo part (d) for the module $\mathbb{C}[x]/(f(x)) \oplus \mathbb{C}[x]/(g(x))$.

4. (a) Find all abelian groups of order $720 = 2^4 \cdot 3^2 \cdot 5$, up to isomorphism. How many are there?

   (b) Determine all integers $n \geq 1$ with the property that any abelian group of order $n$ is cyclic. (Hint: look at the way in which $n$ factors into prime powers...)

5. (a) Suppose $K$ is a field. For any monic irreducible polynomial $f(x) \in K[x]$ of degree $d > 0$ and any integer $e > 0$ determine the possible $K[x]$-linear homomorphisms $K[x]/(f(x)^e) \to K[x]/(f(x)^e)$, and describe which of them are invertible. When $K = \mathbb{F}_p$, how many are there of each kind?

   (b) Suppose that $V$ is a finite-dimensional $K$-vector space and that $S : V \to V$ is a $K$-linear map. Recall that $V_S$ denotes $V$ considered as $K[x]$-module with scalar multiplication determined by $xv = S(v)$ (and same scalar multiplication of $K$ as before). Show that the $K[x]$-linear homomorphisms $V_S \to V_S$ are precisely the $K$-linear maps $V \to V$ that commute with $S$.

   (c) Determine the size of the centraliser and of the conjugacy class for the following matrices in $\mathrm{GL}_3(\mathbb{F}_p)$:

$$\begin{pmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}, C_{g(x)},$$

   where $\alpha, \beta, \gamma$ are pairwise distinct elements of $\mathbb{F}_p^\times$ and $g(x) \in \mathbb{F}_p[x]$ is a monic irreducible polynomial of degree 3. (You may use that $|\mathrm{GL}_3(\mathbb{F}_p)| = (p^3-1)(p^3-p)(p^3-p^2)$. Hint: try to see how the previous parts help to determine centralisers...)

6. (Optional) If $M$, $N$, $P$ are $R$-modules, construct a (nice) isomorphism $\theta_{M,N,P} : \mathrm{Hom}_R(M \otimes_R N, P) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ of $R$-modules, where $\mathrm{Hom}_R(N, P)$ denotes the $R$-module of $R$-linear maps $N \to P$

(under pointwise operations). In fact, show that your isomorphism is "natural" in the sense that if $M \to M'$ is an $R$-linear map you get a commutative square of $R$-modules with top row $\theta_{M',N,P}$ and bottom row $\theta_{M,N,P}$ (say explicitly what the vertical maps are, they should be the "obvious" ones). Similarly if $N \to N'$ or $P \to P'$.

7. (Optional and fun!) The goal is to construct an explicit non-zero ring $R$ such that $R \cong R \oplus R$ as left $R$-modules. We saw in class that this is impossible when $R$ is commutative.

   Let $K$ be a field and $V$ be a countable-dimensional vector space with basis $v_i$ ($i \geq 1$). Let $R := \mathrm{End}_K(V)$. Define $\phi, \psi \in R$ as follows: $\phi(v_i) = v_{i/2}$ if $i$ even and $\phi(v_i) = 0$ if $i$ odd; on the other hand, $\psi(v_i) = v_{(i+1)/2}$ if $i$ odd and $\psi(v_i) = 0$ if $i$ even. Show that $R$ is an internal direct sum $R = R\phi \oplus R\psi$. Finally show that $R\phi \cong R \cong R\psi$ as left $R$-modules.