MAT 1100, Algebra I, Fall 2019
Homework 4, due on Friday November 19

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All rings in these problems are commutative.

1. (a) Suppose that $p = 0$ in the ring R, where p is a prime number. Show that $\varphi : R \to R$, $x \mapsto x^p$ is a ring homomorphism.

   (b) Suppose R is any ring. Show that if $x \in R$ is *nilpotent* (i.e., $x^n = 0$ for some $n > 0$), then $1 + x \in R^\times$.

   (c) Suppose that R is as in part (a). Show that if x is nilpotent, then $1 + x$ is *unipotent* (i.e., $(1 + x)^n = 1$ for some $n > 0$).

2. The goal of this exercise is to show that the intersection J of all prime ideals of R equals the set of all nilpotent elements of R.

   (a) Show that if $x \in R$ is nilpotent, then x is contained in J.

   (b) Suppose now that $x \in R$ isn't nilpotent. Let $X := \{1, x, x^2, \dots\}$. Use Zorn's lemma to show that there is an ideal P that is maximal among all ideals that are disjoint from X (i.e. $P \cap X = \varnothing$).

   (c) Continuing with (b), show that P is a prime ideal. (Hint: if $a, b \notin P$ but $ab \in P$, consider the ideals $P + (a)$ and $P + (b)\dots$)

   (d) Deduce that J consists precisely of all nilpotent elements of R.

3. The ring $R = \mathbb{Z}[\sqrt{-2}]$ is a UFD (you can assume this as a fact). Consider the norm function $N : R \to \mathbb{Z}_{\geq 0}$, $z \mapsto |z|^2$ as in class.

   (a) Determine the units $R^\times$.

   (b) Factor the following elements into primes in the ring R: 2, 5, 19.

   (c) Show that a prime number p is prime in R iff $x^2 + 2y^2 = p$ has no solutions with $x, y \in \mathbb{Z}$.

   (d) Find the gcd of $-2 + \sqrt{-2}$ and 3 in R.

4. Consider the ring $R = K[x^2, x^3]$, where K is any field. (This is the smallest subring of $K[x]$ containing x^2, x^3. Concretely it's given by all polynomials $\sum a_i x^i$ with $a_1 = 0$.)

(a) Show that x^2, x^3 are both irreducible in R.

(b) Find two different factorisations of x^6 into irreducibles, and deduce that neither of x^2, x^3 are prime in R.

(c) Is R a UFD?

(d) Show that the ideal (x^2, x^3) is not principal.

5. (a) Suppose that p is a prime and that $i \in \mathbb{C}$ satisfies $i^2 = -1$. Use the first isomorphism theorem to write the rings $\mathbb{F}_p[x]$ and $\mathbb{Z}[i]$ as quotient rings of $\mathbb{Z}[x]$.

(b) Use part (a) and the third isomorphism theorem to show that $\mathbb{F}_p[x]/(x^2+1) \cong \mathbb{Z}[i]/(p)$. (Hint: express both rings as quotient rings of $\mathbb{Z}[x]$.)

(c) Find prime numbers p such that $\mathbb{F}_p[x]/(x^2+1)$ is a field (resp. contains a nonzero nilpotent element, resp. neither of two previous conditions hold).

(d) Suppose p is a prime number such that $\mathbb{F}_p[x]/(x^2+1)$ is a field. Show that the rings $\mathbb{F}_p[x]/(x^2+1)$ and $\mathbb{Z}/n\mathbb{Z}$ are not isomorphic, where $n := |\mathbb{F}_p[x]/(x^2+1)|$.

6. (a) If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_n a_0 \neq 0$ has a root $\alpha \in \mathbb{Q}$, show that $\alpha = \frac{r}{s}$ with integers r, s such that $r|a_0$ and $s|a_n$. (Hint: use a version of Gauss' lemma to deduce that f has a linear factor in $\mathbb{Z}[x]$.)

(b) Show that $x^4 + 5x^3 + 3x^2 - x - 1$ is irreducible in $\mathbb{Q}[x]$. (Hint: use Gauss' lemma. To rule out quadratic factors, write it as a product of unknown quadratic polynomials in $\mathbb{Z}[x]\ldots$)

(c) Show that $x^n + y^n + 1$ is irreducible in $\mathbb{C}[x, y]$ for any $n \geq 1$. (Hint: use Eisenstein's criterion with $R = \mathbb{C}[x]$.)

(d) Show that $x^p + y^p + 1$ is reducible in $\mathbb{F}_p[x, y]$ for any prime p.