# MAT 1100, Algebra I, Fall 2019
## Homework 3, due on Tuesday, October 29
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1. (a) If $|G| = pq$ with $p < q$ both prime, show that $G$ is solvable.
   (b) If $|G| = pqr$ with $p < q < r$ all prime, show that $G$ is solvable. (Hint: if $G$ is simple, give a lower bound for $n_p$, $n_q$, $n_r$ and hence for the number of elements of order $p$, $q$, $r$. Show that their sum is greater than $|G|$.)

2. Suppose that $G$ is a finite solvable group. Let $M \lhd G$ be a *minimal* normal subgroup (i.e. $M \neq 1$, and if $N \lhd G$, $N \leq M$, then $N = 1$ or $N = M$). Show that $M$ is abelian and that there exists a prime number $p$ such that every non-identity element of $M$ has order $p$. (In fact, we will see later that this implies $M \cong (\mathbb{Z}/p)^r$ for some $r$.)

   (Hint: use characteristic subgroups of $M$ to deduce first that $M$ is abelian, then that $M$ is a $p$-group, finally the claim.)

3. Suppose that $H$, $K$ are subgroups of a group $G$. Let $[H, K]$ denote the subgroup generated by all commutators $[h, k] = hkh^{-1}k^{-1}$ for $h \in H$, $k \in K$.

   (a) Show that $[H, K] \lhd \langle H, K\rangle$, where the right-hand side denotes the subgroup generated by $H$ and $K$. (Hint: try to express $h_0[h, k]h_0^{-1}$ in terms of commutators...)
   (b) Show that $[G, H]H$ is a normal subgroup of $G$.
   (c) Show that $[G, H]H$ is the smallest normal subgroup of $G$ that contains $H$.

4. Suppose that $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$ with $m < n$. The purpose of this exercise is to show that $G$ is infinite. (Note that this can fail if $m = n$.)

   (a) Show that it suffices to prove the existence of a *non-zero* homomorphism $G \to \mathbb{Q}$.
   (b) As a warmup, use the universal property of $\langle S \mid R\rangle$ to construct a non-zero homomorphism $\langle x, y \mid x^{-1}y^2x^3y\rangle \to \mathbb{Q}$. In fact, find all possible such homomorphisms.

(c) Now show that in general there is a non-zero homomorphism $G \to \mathbb{Q}$. It may help to use facts from linear algebra...

5. The goal of this exercise is to prove that

$$S_n \cong G_n := \langle s_1, \ldots, s_{n-1} \mid (s_i s_j)^{m(i,j)} = 1 \rangle,$$

where $m(i,j) = \begin{cases} 1 & i = j \\ 3 & |i-j| = 1 \\ 2 & |i-j| > 1 \end{cases}$. Said in another way, $s_i^2 = 1$; $s_i$ and $s_j$ commute when $|i-j| > 1$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

(a) Show that there is a surjective homomorphism $G_n \to S_n$.

(b) Prove using induction that $|G_n| \leq n!$, using the following hints. Assume this is true for $n$. Let $H$ be the subgroup of $G_{n+1}$ generated by $s_1, \ldots, s_{n-1}$. Show by induction that $|H| \leq n!$.

(c) Continuing, now start showing that $|G_{n+1}/H| \leq n+1$: let $H_i := s_{i+1} s_{i+2} \ldots s_n H$ $(0 \leq i \leq n)$. Show that it suffices to show that left multiplication by any $s_i$ permutes the set $\{H_0, H_1, \ldots, H_n\}$.

(d) Continuing, show that $s_i H_i = H_{i-1}$ and $s_i H_{i-1} = H_i$.

(e) Continuing, show that $s_i H_j = H_j$ for $j > i$ or $j < i-1$. (If you get stuck on the second case, try $s_n H_{n-2}$ first.)

(f) Deduce that $S_n \cong G_n$.