PERTURBATIVE CHERN-SIMONS THEORY

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Abstract. We present the perturbation theory of the Chern-Simons gauge field theory and prove that to second order it indeed gives knot invariants. We identify these invariants and show that in fact we get a previously unknown integral formula for the Arf invariant of a knot, in complete agreement with earlier non-perturbative results of Witten. We outline our expectations for the behavior of the theory beyond two loops.

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1. Introduction

The aim of this paper is to prove, explain the motivation, and point at possible generalizations of the following theorem:

Theorem 1. *Let X be a parametrized oriented knot in $\mathbf{R}^3$. (that is to say, X is a smooth non-singular function from S^1 to $\mathbf{R}^3$ that has no self intersections). Then the integrals represented (as explained below) by the following two diagrams are convergent, and their sum $\tilde{\mathcal{W}}_2$ is an isotopy invariant of the knot X. This invariant can be identified to be $1/24$*

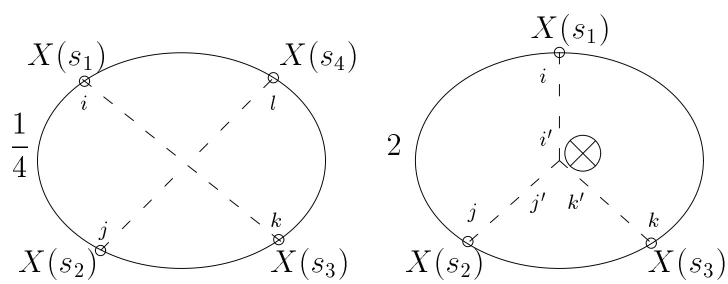

Figure 1. *The two contributing diagrams.*

plus the second non trivial coefficient in the Conway polynomial of X, whose reduction mod 2 is the well known Arf invariant of X.

The meaning of the two diagrams above still has to be explained. Each diagram represents an integral which can be read from its diagrammatic representation as follows:

(1) The ellipses represent the knot itself. It is parametrized as $X(s)$ and the points $X(s_1), \ldots, X(s_{3\text{ or }4})$ are points on the knot that always remain in the same cyclic

order as in the diagrams. We integrate over all such configurations of $X(s_1), \dots,$ $X(s_{3 \text{ or } 4})$:

$$\int_{\Delta_3} ds_{1,2,3} \qquad \text{or} \qquad \int_{\Delta_4} ds_{1,\dots,4},$$

where $\Delta_{3 \text{ or } 4}$ is the set of all cyclically ordered $s_{1,\dots,3 \text{ or } 4} \in S^1$.

(2) The dashed lines indicate the so called 'gauge propagator'. It is proportional to a particular choice of an inverse for the operator *curl*:

$$\underset{i}{\overset{X(s_1)}{-}} - - - - - \underset{k}{\overset{X(s_3)}{-}} = V_{ik}(X(s_1), X(s_3)) = \epsilon_{ikm} \frac{(X(s_1) - X(s_3))^m}{4\pi \left|X(s_1) - X(s_3)\right|^3} \tag{1}$$

Summation over repeated indices is of course understood, and ϵ_{ikm} denotes the totally antisymmetric tensor in three dimensions — $\epsilon_{ikm} = \text{sign }(ikm)$ if ikm is a permutation of $\{1, 2, 3\}$ and $\epsilon_{ikm} = 0$ otherwise.

(3) When a propagator (a dashed line) begins or ends on the knot, say at $X(s_1)$, its index that corresponds to the side of the propagator that is by $X(s_1)$ is contracted with $\dot{X}^i(s_1)$:

$$\dot{X}^i(s_1) \underset{i}{-} - - - -$$

(4) The symbol $\otimes$ near a vertex with three propagators carrying indices i', j', and k' emanating from it means that integration over the position of that vertex in $\mathbf{R}^3$ should be performed, and the three indices should be contracted with an $\epsilon^{i'j'k'}$:

$$-\frac{1}{6} \int_{\mathbf{R}^3} dz \, \epsilon^{i'j'k'}.$$

(5) The numerical factors in front of each diagram just indicates its weight.

Therefore, our invariant reads:

$$\begin{aligned} \tilde{\mathcal{W}}_2 = & \frac{1}{64\pi^2} \int_{\Delta_4} ds_{1,\dots,4} \dot{X}^i(s_1) \dot{X}^j(s_2) \dot{X}^k(s_3) \dot{X}^l(s_4) \epsilon_{ikm} \epsilon_{jln} \frac{(X(s_1) - X(s_3))^m}{\left|X(s_1) - X(s_3)\right|^3} \frac{(X(s_2) - X(s_4))^n}{\left|X(s_2) - X(s_4)\right|^3} \\ & - \frac{1}{192\pi^3} \int_{\Delta_3} ds_{1,2,3} \int_{\mathbf{R}^3} d^3z \, \dot{X}^i(s_1) \dot{X}^j(s_2) \dot{X}^k(s_3) \epsilon^{i'j'k'} \epsilon_{ii'i''} \epsilon_{jj'j''} \epsilon_{kk'k''} \\ & \qquad \frac{(X(s_1) - z)^{i''}}{\left|X(s_1) - z\right|^3} \frac{(X(s_2) - z)^{j''}}{\left|X(s_2) - z\right|^3} \frac{(X(s_3) - z)^{k''}}{\left|X(s_3) - z\right|^3} \end{aligned} \tag{2}$$

We shall explain presently the simple idea that lies behind these complicated-looking integrals, and see that using more or less the same building blocks as 1-5 above we can generate more combinations of diagrams that we expect to yield integral representations for higher coefficients of the Jones polynomial, and that using similar diagrammatic building blocks we can construct integral invariants of general three-manifolds and of knots embedded in them. These integrals appear to be divergent (just as the integrals above appear to be divergent on first sight), and more work needs to be done in order to show that it is possible to make sense out of these integrals anyway, and that they indeed converge (after some corrections) to invariants. None of this work has yet been done for the general case.

The organization of this paper is as follows. Section 2 will introduce the infinite dimensional integral whose asymptotic expansion should give link and three-manifold invariants, and briefly review the Feynman-diagram technique for obtaining this asymptotic expansion. Section 3 describes a formal invariance proof for the invariants introduced in section 2, and then sections 4-6 treat few of the simplest of those invariants — section 4 treats the linking number of two knots and the self linking number of a single knot from our point of view, while sections 5 and 6 contain the proof of the theorem 1, first proving the finiteness of $\tilde{\mathcal{W}}_2$ in section 5 and then its invariance in section 6. These proofs rely on some simple algebra that was carried out using a computer, and the relevant computer routine is included in the appendix. Section 7 compares our results with the earlier non-perturbative treatment of Witten, and section 8 discusses our expectations for the behavior of the theory beyond the few simple cases treated here. Sections 2,3 and 8 are non-rigorous and a bit speculative, while sections 4,5 and 6 use the ideas of sections 2 and 3 to produce some rigorous results.

This paper is a modified version of a preprint I first distributed almost 4 years ago, in April 1990. Section 9 describes in just a few words the new developments in this subject in the years 1990-94.

During the preparation of this paper we received a paper by E. Guadagnini, M. Martellini, and M. Mintchev, [19], in which they have conjectured the invariance of (2) and calculated it by explicit and numerical integration for several simple knots.

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2. The basic idea

The basic idea is simple and to make it even simpler we will ignore knots for a moment and explain it first for the case of a bare three manifold. Our invariants will be complex numbers. To get a complex number out of a bare three manifold, that has no additional structure on it, is hard. It is a lot easier to get numerical quantities when there is more structure to play with. So we look at an oriented three manifold with an additional piece of structure, generate a complex number using this additional structure, and then try to integrate our complex number over all possible choices of such an additional structure. The additional structure that we will pick will be a connection on a trivial pre-picked bundle on our three manifold M^3, and the complex number that we will generate, the integrand in our program, will essentially be the exponential of the 'Lagrangian' — the Chern-Simons number [14] associated with the connection A:

$$cs(A) = \frac{k}{4\pi} \int_{M^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

and so our invariant will be:

$$\mathcal{W}(M^3,k) = \int_{\mathcal{A}} \mathcal{D}A \; e^{\frac{ik}{4\pi}\int_{M^3} tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)}. \tag{3}$$

(k is an integer parameter whose importance for our purposes will be made clear shortly).

Luckily, the space of all connections $\mathcal{A}$ is an affine space and so there should be a canonical choice for a measure on it — the Lebesgue measure. Unfortunately, $\mathcal{A}$ is an infinite dimensional space and so that measure doesn't really exist. To go around this we will use perturbation theory techniques that were originally developed by physicists to be used in quantum field theory. Instead of attempting to calculate the integral (3) as it is, we will try to investigate its asymptotic behavior as $k/2\pi i \to \infty$. It will turn out that (assuming that infinite dimensional Lebesgue measures do exist) to determine this asymptotic behavior requires only evaluating finite dimensional integrals represented by so-called "Feynman diagrams", and therefore it is possible to *define* the asymptotic behavior of (3) to be given by those "Feynman diagrams", without ever giving meaning to the integral (3) itself. I will very briefly present these techniques here. For further information consult any quantum field theory textbook such as [29, 16, 22].

2.1. A finite dimensional analogue. To illustrate the technique of Feynman diagrams, let us first look at a simpler finite dimensional analogue — let us try to understand the $t \to \infty$ asymptotics of:

$$\mathcal{Z}_t = \int_{\mathbf{R}^n} d\vec{x} e^{it(\frac{1}{2}\lambda_{ij}x^ix^j+\lambda_{ijk}x^ix^jx^k)}. \tag{4}$$

(This case is in fact quite general — whenever an expression of the form $\int e^{itf}$ is encountered its $t \to \infty$ asymptotics is dominated by the contribution from small neighborhoods of the critical points of the real-valued function f, and at those points f can be replaced by a quadratic term plus a higher order correction.)

By a simple change of variables,

$$\vec{x} \to \vec{x}' = \sqrt{t}\vec{x}, \tag{5}$$

(suppressing primes)

$$\mathcal{Z}_t = t^{-n/2}\int_{\mathbf{R}^n} d\vec{x} e^{i\frac{1}{2}\lambda_{ij}x^ix^j} e^{\frac{i}{\sqrt{t}}\lambda_{ijk}x^ix^jx^k}$$

expanding the second exponential to a series, and suppressing the odd degree terms (which vanish upon integration on $\mathbf{R}^n$), we get

$$\mathcal{Z}_t = t^{-n/2}\int_{\mathbf{R}^n} d\vec{x} e^{i\frac{1}{2}\lambda_{ij}x^ix^j} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!t^m}(\lambda_{ijk}x^ix^jx^k)^{2m}. \tag{6}$$

And so the mth term in our asymptotic expansion will be given up to a multiplicative constant by:

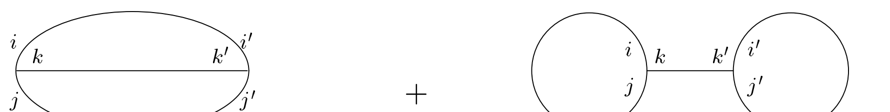
$$\int_{\mathbf{R}^n} d\vec{x} e^{i\frac{1}{2}\lambda_{ij}x^ix^j}(\lambda_{ijk}x^ix^jx^k)^{2m} =$$

this is a simple Gaussian integral, which we can evaluate using standard methods:

$$= \left[\left(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k} \right)^{2m} \int_{\mathbf{R}^n} d\vec{x} e^{i \frac{1}{2} \lambda_{ij} x^i x^j + i J_i x^i} \right]_{\vec{J}=0}$$

$$\propto \left[\left(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k} \right)^{2m} e^{-i \frac{1}{2} \lambda^{ij} J_i J_j} \right]_{\vec{J}=0}, \tag{7}$$

where λ^{ij} is the inverse of λ_{ij}: $\lambda_{ij}\lambda^{jk} = \delta_i{}^k$. Now this expression can clearly be expanded further, and a moment's reflection will convince that up to combinatorial factors and powers of i it is given by the sum of all "Feynman diagrams" that have exactly $2m$ vertices of order three. That is to say, to evaluate (7) we calculate a sum over all graphs with $2m$ vertices of order three where the contribution of each such graph is a product of λ_{ijk}'s for each vertex and λ^{ij}'s for each arc. So for example up to numerical factors the term with $m = 1$ will be:



$(num1)\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ii'}\lambda^{jj'}\lambda^{kk'}$ $\qquad$ $(num2)\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ij}\lambda^{kk'}\lambda^{i'j'}$

It is not hard to see that in general $2m$ is also equal to the number of independent loops in a diagram. Therefore we will also call the m'th order term in such an asymptotic expansion 'the $2m$-loop term'.

Looking back at our infinite dimensional situation we will by analogy *define* the $\mathcal{W}_j$'s to be those sums of diagrams, in which now every vertex will correspond to integration on M^3 and to a tri-linear form that comes from $A \wedge A \wedge A$, and every arc will correspond to a Green's function of the operator defining the quadratic part of $cs(A)$. (The cautious reader will notice that this quadratic part is not elliptic, and therefore does not have a Green's function. This problem will be dealt with later on).

2.2. The incorporation of knots. To incorporate a link $\mathcal{X} = \{X_\gamma\}_{\gamma=1}^{\Gamma}$ into the above picture, we have to supplement the integrand:

$$\mathcal{W}(M^3, \mathcal{X}, k) = \int_{\mathcal{A}} \mathcal{D}A \prod_{\gamma=1}^{\Gamma} \mathrm{Tr}_{R_\gamma} \mathcal{P}\mathrm{exp} \left(\int ds \dot{X}_\gamma^i(s) A_i(X_\gamma(s)) \right) e^{\frac{ik}{4\pi} \int_{M^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)} \tag{8}$$

Where

$$\mathrm{Tr}_R \mathcal{P}\mathrm{exp} \left(\int ds \dot{X}^i(s) A_i(X(s)) \right) = \dim R + \int ds \dot{X}^i(s) A_i^a(X(s)) R_{a\alpha}^{\alpha}$$

$$+ \int_{s_1 < s_2} ds_{1,2} \dot{X}^{i_1}(s_1) \dot{X}^{i_2}(s_2) A_{i_1}^{a_1}(X(s_1)) A_{i_2}^{a_2}(X(s_2)) R_{a_1\alpha_2}^{\alpha_1} R_{a_2\alpha_1}^{\alpha_2} + \cdots, \tag{9}$$

and R is a representation of the underlying Lie-algebra. (9) is, of course, just the trace of the holonomy of the connection A along X in the representation R, expanded in powers of the connection A which is assumed to be almost flat. The expansion (9) shows us that to understand the integral (8) we first have to understand integrals as in (4), only with an additional polynomial $P(\vec{x})$ multiplying the integrand. Moreover, after rescaling $\vec{x}$ as in (5) and carrying out exactly the same analysis as in (4) – (6) with an additional $P(\vec{x})$ multiplying each integrand we see that in the mth order term in our revised asymptotic expansion will be given by:

$$\sum_{m_1+m_2=2m} \int_{\mathbf{R}^n} d\vec{x} e^{i\frac{1}{2}\lambda_{ij}x^i x^j} P_{m_1}(\vec{x})(\lambda_{ijk}x^i x^j x^k)^{m_2}, \tag{10}$$

where $P_{m_1}(\vec{x})$ denotes the part of $P(\vec{x})$ which is homogeneous of degree m_1 in $\vec{x}$. Noticing that just as before we ended up having to calculate the expectation value of a polynomial $\left(P_{m_1}(\vec{x})(\lambda_{ijk}x^i x^j x^k)^{m_2}\right)$ with respect to a Gaussian measure, we can now use the same trick and replace the above integral by a sum of 'revised' Feynman diagrams that are also allowed to have a single exceptional vertex of some order m_1, weighted by the coefficients of $P_{m_1}(\vec{x})$.

Returning to the infinite-dimensional situation we see that the perturbative expansion of (8) will be given by Feynman diagrams that have 'propagators' (arcs) corresponding to Green's functions of *curl*, internal vertices corresponding to $A\wedge A\wedge A$, and additional vertices integrated on the link $\mathcal{X}$ with coefficients as in (9). Comparing again with (10), we see that the mth order term in the expansion of 8 is a sum of such diagrams having exactly $2m$ vertices, internal or not. This is exactly the form that $\tilde{\mathcal{W}}_2$ of the theorem has.

2.3. The ellipticity problem. Let us now return to a problem that was brushed aside temporarily. The quadratic part of $cs(A)$ is not elliptic, and as it stands it does not have a Green's function and therefore the Feynman diagrams technique is not available. The origin of the problem is a bit deeper - $cs(A)$ is invariant under the infinite-dimensional infinite-volume group of gauge transformations, and hence integrating $\mathcal{D}A$ we integrate over infinitely many superfluous variables and we cannot expect to get a finite integral. To resolve this complication we will once again look at our finite dimensional analogue, assume that the Lagrangian there, $\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k$, is invariant under the isometrical non-degenerate action of an l-dimensional Lie group G, and try to evaluate the integral (4) without redundant integration over the orbits of G.

We will visit each orbit of G just once by choosing a function $F : \mathbf{R}^n \to \mathbf{R}^l$ that has a unique zero on each G-orbit, and inserting a $\delta^l(F(\vec{x}))$ into the integral. If we want the result to be the same as the full integration and independent of F we need to add a correction term that corresponds to the volume of the G-orbit through $\vec{x}$ and as the action of G is by isometries this term can be calculated locally at a point $\vec{x}$ satisfying $F(\vec{x}) = 0$. It is given by the inverse ratio of the volume element of the Lie-algebra $\mathcal{G}$ of G and its image in $\mathbf{R}^l$ under the action of G composed with F. That is to say — we have to look at:

$$\mathcal{Z} = \int_{\mathbf{R}^n} d\vec{x} e^{it(\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k)} \delta^l(F(\vec{x})) \det\left(\frac{\partial F^a}{\partial \mathcal{G}_b}\right)(\vec{x}).$$

($\{\mathcal{G}_b\}_{b=1}^l$ is a set of generators for $\mathcal{G}$)

We will try to find a diagrammatic representation for the asymptotic expansion of $\mathcal{Z}$. The first additional term in the integral is easy — we can just replace it by its Fourier representation:

$$\delta^l(F(\vec{x})) = \frac{1}{(2\pi)^l} \int_{\mathbf{R}^l} d^l \phi e^{iF^a(\vec{x})\phi_a}$$

and then incorporate $F^a(\vec{x})\phi_a$ as a new term in the Lagrangian. The other new term, $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$, can be dealt with in two equivalent ways. The first way is to do the usual rescaling (5) and then to expand $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ in powers of $\frac{1}{\sqrt{t}}$ by first separating $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ into a constant part J_0 and a part $J_1(\vec{x})$ which is a series in $\frac{1}{\sqrt{t}}$, and then using

$$\det\left(J_0 + \frac{1}{\sqrt{t}} J_1(\vec{x})\right) = \det(J_0) \sum_m \left(\frac{1}{\sqrt{t}}\right)^m \operatorname{Tr}(\textstyle\bigwedge^m J_0^{-1})(\textstyle\bigwedge^m J_1(\vec{x})). \tag{11}$$

($\bigwedge^m J$ is the mth exterior power of the matrix J). Notice that J_0 is just a constant matrix, while $J_1(\vec{x})$ depends on $\vec{x}$. It will now be possible to regard (11) as a polynomial in $\vec{x}$ and get a Feynman diagram expansion. It is an easy exercise in elementary algebra to show that the polynomial (11) can itself be incorporated into the the Feynman diagrams by introducing a new type of propagator denoted by *directed* dotted lines that corresponds to J_0^{-1} and a collection of new types of vertices each connecting two dotted propagators with some dashed propagators — depending on the exact form of $J_1(\vec{x})$. (There will also be some alteration to the combinatorial rule of determining the numerical factor multiplying each diagram).

The other way of dealing with $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ is the one commonly used in the physics literature and the one that we will be using here. It involves the idea of anti-commutative integration. Non-commutative integration is treated in many places (see e.g. [9, 29, 16, 22]), and I will not explain it here in detail. Very briefly, 'anti-commuting' variables (called 'ghosts') $\{\bar{c}_a\}_{a=1}^l$ and $\{c^b\}_{b=1}^l$ are introduced together with a reasonable set of rules of integration with respect to them, and it is shown that for any matrix $J^a_{\ b}$

$$\int d^l \bar{c} d^l c e^{i\bar{c}_a J^a{}_b c^b} \propto \det(J).$$

(This is analogous and complementary to standard Gaussian integration — in which the resulting determinant is in the denominator).

Using this, $\mathcal{Z}$ can finally be written as

$$\mathcal{Z} \propto \int_{\mathbf{R}^n} d\vec{x} \frac{1}{(2\pi)^l} \int_{\mathbf{R}^l} d^l \phi \int d^l \bar{c} d^l c e^{i(t(\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k) + F^a(\vec{x})\phi_a + \bar{c}_a(\frac{\partial F^a}{\partial \mathcal{G}_b})c^b)}$$

and now we can use almost the same procedure as in (4) – (7) to get a diagrammatic expansion for the asymptotic behavior of $\mathcal{Z}$. Again it turns out that this involves introducing a new propagator and some new vertices.

As we will see below for the case of interest for us — the Chern-Simons Lagrangian — we will be able to choose F in a way so that the quadratic part of the supplemented Lagrangian will indeed be elliptic. This will be done in the next section.

3. The Chern-Simons Lagrangian and the BRST argument

Let M^3 be an oriented three manifold, G a compact semisimple Lie group with an invariant integral bilinear form tr on its Lie algebra $\mathcal{G}$ and $P \to M^3$ a principal G-bundle on M^3. Also A will always denote a connection on P, B a difference of two such connection - i.e. an $ad(P)$-valued 1-form on M^3, D^A and D_A covariant derivatives defined using A, and F^A the curvature of A.

The Chern-Simons Lagrangian $cs(A)$ is defined by (see [14]):

$$cs(A) = \frac{k}{4\pi}\int_{M^3} tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

where $tr(A_1 \wedge A_2 \wedge A_3) \stackrel{\text{def}}{=} \frac{1}{2}(trA_1 \wedge [A_2, A_3]) = \frac{1}{2}tr([A_1, A_2] \wedge A_3)$, and so relative to some choice of coordinates and a trivialization of P,

$$= \frac{k}{8\pi}\int_{M^3} \epsilon^{ijk} tr(A_i(\partial_j A_k - \partial_k A_j) + \frac{2}{3}A_i[A_j, A_k]).$$

It is invariant under infinitesimal gauge transformations in which $\delta A = -D^A c \stackrel{\text{def}}{=} -(dc + [A, c])$:

$$\begin{aligned}
\frac{4\pi}{k}\delta cs &= -\int_{M^3} tr\,((dc + [A, c]) \wedge dA + A \wedge d[A, c] \\
&\qquad\qquad +2(dc + [A, c]) \wedge A \wedge A) = \\
&= -\int_{M^3} tr([A, c] \wedge dA + A \wedge [dA, c] - A \wedge [A, dc] + 2dc \wedge A \wedge A) \\
&\quad -2\int_{M^3} tr[A, c] \wedge A \wedge A = \\
&= \int_{M^3} trc \wedge [A, [A, A]] = 0.
\end{aligned}$$

This implies that $cs(A)$ is invariant under gauge transformations that can be pathwise connected to the identity transformation. As it turns out (see [14]), $cs(A)$ is *not* invariant under general gauge transformations and, in fact, in our normalization it is defined only up to a multiple of 2π. This explains our choice of the normalization — we have chosen precisely that normalization for which the exponential in (3) *is* well defined.

The space $\mathcal{A}$ of all connections A on P is just an affine space and not a vector space in a natural way and the functional $cs(A)$ does not necessarily have the gauge orbit of zero as its only stationary gauge orbit on $\mathcal{A}$ — there is no well defined origin in $\mathcal{A}$ and $cs(A)$ might have more then just one stationary gauge orbit in $\mathcal{A}$. We will therefore repeat the analysis of the previous section separately for each of the stationary gauge orbits and then sum up the contributions from the various gauge orbits. Suppose now that A is an arbitrary stationary point for cs, i.e.: $\frac{\delta cs}{\delta A}=0$, which means $F^A = dA+\frac{1}{2}[A,A] = dA + A\wedge A = 0$, and for B an $ad(P)$ -valued 1-form on M^3 define $\mathcal{L}(B) = cs(A{+}B) - cs(A)$:

$$\begin{aligned}
\mathcal{L}(B) &= \frac{k}{4\pi}\int_{M^3} tr((A{+}B)\wedge d(A{+}B) + \frac{2}{3}(A{+}B)\wedge(A{+}B)\wedge(A{+}B)) - cs(A) \\
&= \frac{k}{4\pi}\int_{M^3} tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) + \\
&\quad + \text{ a vanishing linear term in } B \ + \\
&\quad + \frac{k}{4\pi}\int_{M^3} tr(B\wedge dB + \frac{2}{3}B\wedge B\wedge B + B\wedge [A,B]) - cs(A) \\
&= \frac{k}{4\pi}\int_{M^3} tr(B\wedge D^A B + \frac{2}{3}B\wedge B\wedge B).
\end{aligned}$$

Choose a trivialization of P, local coordinates $\{x^i\}$ and a metric g_{ij} on M^3 with $g \stackrel{\text{def}}{=} \det(g_{ij})$, and get

$$(D^A B)_{ij} = \partial_i B_j - \partial_j B_i + [A_i, B_j],$$

and

$$D^i_A \stackrel{\text{def}}{=} \sqrt{g}g^{ij}D^A_j = \sqrt{g}g^{ij}\partial_j + \sqrt{g}g^{ij}[A_j,\cdot].$$

Pick the gauge condition $\frac{k}{4\pi}D^A_i B^i = 0$, and get using the usual Faddeev-Popov procedure as described in the previous section:

$$\begin{aligned}
\mathcal{L}_{tot}(B,\phi,c,\bar{c}) &= \mathcal{L} + \mathcal{L}_{\textit{gauge-fixing}} + \mathcal{L}_{ghosts} = \\
&= \frac{k}{4\pi}\int_{M^3} tr(B\wedge D^A B + \frac{2}{3}B\wedge B\wedge B) + \\
&\quad + \frac{k}{4\pi}\int_{M^3} tr(\phi D^A_i B^i) + \\
&\quad + \frac{k}{4\pi}\int_{M^3} tr\bar{c}D^A_i(D^i_A + \operatorname{ad} B^i)c
\end{aligned}\tag{12}$$

ϕ, c, and $\bar{c}$ are Lie-algebra valued fields — $\phi = \phi^a\mathcal{G}_a$, $c = c^a\mathcal{G}_a$, and $\bar{c} = \bar{c}^a\mathcal{G}_a$ for a set of generators $\{\mathcal{G}_a\}$ of $\mathcal{G}$.

3.1. A "proof" of metric independence. To show that the Lagrangian that we obtained gives rise to a metric-independent theory in spite of the explicit appearance of a metric in it, we will have to introduce the 'BRST' operator Q [8] — the odd derivation acting on the space of all functionals of $B, \phi, \bar{c}, c$, defined by its action on the generators:

$$
\begin{aligned}
QB_i &= -(D_i^A + \operatorname{ad} B_i)c, && (13)\\
Q\phi &= 0, && (14)\\
Q\bar{c} &= \phi, && (15)\\
Qc &= \frac{1}{2}[c,c] = \frac{1}{2}\mathcal{G}_a f^a_{bc} c^b c^c. && (16)
\end{aligned}
$$

In (13) the expression "$\operatorname{ad} B_i$" stands for the operator defined by $(\operatorname{ad} B_i)c \stackrel{\text{def}}{=} [B_i, c]$, in (16) f^a_{bc} are the structure constants of $\mathcal{G}$, $[\mathcal{G}_b, \mathcal{G}_c] = f^a_{bc}\mathcal{G}_a$, and $[c,c]$ doesn't vanish because of the anti-commutativity of c.

Lemma 3.1. $Q\mathcal{L}_{tot}(B) = 0$.

Lemma 3.2. *There exists Λ (that depends on δg^{ij}) so that under $g^{ij} \to g^{ij} + \delta g^{ij}$*

$$\delta\mathcal{L}_{tot} = Q\Lambda$$

Lemma 3.3. *Q corresponds to a vector field of zero divergence.*

Let us first use the above three lemmas to prove that

$$\mathcal{W} = \int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\; e^{i\mathcal{L}_{tot}}$$

is formally metric independent [35]. In fact, more will be true: whenever $\mathcal{O}$ is a metric independent function of $B, \phi, c, \bar{c}$ that satisfies $Q\mathcal{O} = 0$,

$$\langle\mathcal{O}\rangle \stackrel{\text{def}}{=} \int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\; \mathcal{O}(B,\phi,c,\bar{c})e^{i\mathcal{L}_{tot}}$$

will be metric independent. (The case of $\mathcal{W}$ is when $\mathcal{O} \equiv 1$). Indeed, under $g^{ij} \to g^{ij} + \delta g^{ij}$

$$
\begin{aligned}
\delta\langle\mathcal{O}\rangle &= \delta\int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\; \mathcal{O}(B,\phi,c,\bar{c})e^{i\mathcal{L}_{tot}}\\
&= i\int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\; \mathcal{O}(B,\phi,c,\bar{c})e^{i\mathcal{L}_{tot}}\delta\mathcal{L}_{tot}\\
&= i\int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\; Q\Big(\mathcal{O}(B,\phi,c,\bar{c})e^{i\mathcal{L}_{tot}}\Lambda\Big). && (17)
\end{aligned}
$$

In the last equality we made use of the first two lemmas. Now we just use the third lemma and the well-known fact that the integral of a derivative taken using a divergence-free vector field is always zero to conclude our proof.

3.2. Proofs of lemmas 3.1-3.

Proof of lemma 3.1. The proof of $Q\mathcal{L} = 0$ is identical to the calculation showing gauge invariance.

$$\begin{aligned} Q\mathcal{L}_{\textit{gauge-fixing}} &= \frac{-k}{4\pi}\int_{M^3} tr\phi D_i^A(D_A^i + \operatorname{ad} B^i)c \\ Q\mathcal{L}_{\textit{ghosts}} &= \frac{k}{4\pi}\int_{M^3} tr\left(\phi D_i^A(D_A^i + \operatorname{ad} B^i)c + \bar{c}D_i^A[D_A^i c + [B^i, c], c]\right. \\ &\qquad \left. -\frac{1}{2}\bar{c}D_i^A(D_A^i + \operatorname{ad} B^i)[c,c]\right), \end{aligned}$$

and it is easy to see that $Q\mathcal{L}_{\textit{gauge-fixing}} + Q\mathcal{L}_{\textit{ghosts}} = 0$.

□

Proof of lemma 3.2. Suppose that $g^{ij} \to g^{ij} + \delta g^{ij}$. Then $\delta\mathcal{L} = 0$ while

$$\delta\mathcal{L}_{\textit{gauge-fixing}} = \frac{k}{4\pi}\int_{M^3} tr\left(\phi D_i^A(\sqrt{g}\delta g^{ij}B_j) - \frac{1}{2}\phi D_k^A(\sqrt{g}g_{ij}\delta g^{ij}g^{kl}B_l)\right)$$

and

$$\begin{aligned} \delta\mathcal{L}_{\textit{ghosts}} &= \frac{k}{4\pi}\int_{M^3} tr\left(\bar{c}D_i^A\left(\sqrt{g}\delta g^{ij}(D_j^A + \operatorname{ad} B_j)c\right)\right. \\ &\qquad \left. -\frac{1}{2}\bar{c}D_k^A\left(\sqrt{g}g_{ij}\delta g^{ij}g^{kl}(D_l^A + \operatorname{ad} B_l)c\right)\right) \end{aligned}$$

so that

$$\delta\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{M^3}\sqrt{g}\delta g^{ij}T_{ij}$$

with

$$T_{ij} = tr\left((D_i^A\phi)B_j + (D_i^A\bar{c})(D_j^A + \operatorname{ad} B_j)c - \frac{1}{2}g_{ij}\left((D_k^A\phi)g^{kl}B_l + (D_k^A\bar{c})g^{kl}(D_l^A + \operatorname{ad} B_l)c\right)\right)$$

and then $T_{ij} = Q\lambda_{ij}$ for

$$\lambda_{ij} = tr\left((D_i^A\bar{c})B_j - \frac{1}{2}g_{ij}(D_k^A\bar{c})g^{kl}B_l\right)$$

that is:

$$\delta\mathcal{L}_{tot} = Q\left(\frac{k}{4\pi}\int_{M^3}\sqrt{g}\delta g^{ij} tr\left((D_i^A\bar{c})B_j - \frac{1}{2}g_{ij}(D_k^A\bar{c})g^{kl}B_l\right)\right) \stackrel{\text{def}}{=} Q\Lambda.$$

□

Proof of lemma 3.3. To rigorously prove lemma 3.3 one first needs to understand what is meant by the divergence of a vector field on an infinite dimensional space. But as our metric independence “proof” is just a formal argument, it is sufficient to note that proving lemma 3.3 formally is completely trivial just by inspecting (13) – (16).

□

The value of the above metric independence "proof" is of course not in itself — so long as we do not give a proper definition for our infinite dimensional integrals it is far from being rigorous — but in the hints that it gives towards finding a *rigorous* proof that the Feynman diagrams expansion is metric independent. This independence appears to have been broken in (12), but the argument in (17) can quite straightforwardly be translated to a Feynman-diagrammatic argument just by expanding (17) in powers of $1/\sqrt{k}$ and reading the proofs of the above three lemmas as relations among the resulting diagrams[1]. Of course, the formal invariance proof thus obtained will have to be supplemented with analytic details concerning the convergence (or divergence) of the relevant diagrams, and with possible finite dimensional kernels of the differential operators that we need to invert. This will be done in detail for a simple case in this paper. Writing the formal proof in the general case is not very hard but I could not yet supplement it with the required analytic details.

4. The one-loop contribution

Having developed a general technique in the previous sections, let us now try to apply it in few particular cases, and let us start from the simplest case — the contribution of order $2\pi i/k$ to $\mathcal{W}(\text{flat }\mathbf{R}^3, \mathcal{X})$ where $\mathcal{X}$ is a two-component link in $\mathbf{R}^3$. There is just one flat connection on $\mathbf{R}^3$ — the trivial one — and we don't need to switch to the variable B. Furthermore, we will ignore the vacuum diagrams — those diagrams that have no vertices on the link. (As is well known, this corresponds to dividing by $\mathcal{W}(\text{flat }\mathbf{R}^3, \text{empty link})$). In this simple case the ghosts and the interaction term $A\wedge A\wedge A$ will not yet come into play, and of the infinitely many terms in the expansion of $\mathcal{P}\exp$ only terms up to the second order term will be relevant. That is to say, we just need to understand

$$
\begin{aligned}
\mathcal{W}' \;=\; & \int_{\mathcal{A}} \mathcal{D}A\mathcal{D}\phi\; e^{\frac{ik}{4\pi}\int_{\mathbf{R}^3} tr(\epsilon^{ijk}A_i\partial_j A_k + \phi\partial^i A_i)} \\
& \prod_{\gamma=1}^{2}\Bigg(\dim R_\gamma + \int ds \dot{X}^i_\gamma(s) A^a_i(X_g(s)) R^\alpha_{\gamma a\alpha} \\
& \qquad + \int_{s_1<s_2} ds_{1,2} \dot{X}^{i_1}_g(s_1)\dot{X}^{i_2}_\gamma(s_2) A^{a_1}_{i_1}(X_\gamma(s_1)) A^{a_2}_{i_2}(X_\gamma(s_1)) R^{\alpha_1}_{\gamma a_1\alpha_2} R^{\alpha_2}_{\gamma a_2\alpha_1}\Bigg)
\end{aligned}
$$

This is just a simple Gaussian integral. We can regard ϕ as a (Lie algebra valued) three-form on $\mathbf{R}^3$, A as a one-form, and write the quadratic form in our Gaussian integral as

$$
\frac{1}{2} tr\left(\epsilon^{ijk}A_i\partial_j A_k + \phi\partial^i A_i\right) = \frac{1}{2}\left\langle \begin{pmatrix} A \\ \phi \end{pmatrix}, L^- \begin{pmatrix} A \\ \phi \end{pmatrix} \right\rangle
$$

for $L^- \stackrel{\text{def}}{=} d\star + \star d$. Clearly $(L^-)^2 = \Delta$ and therefore V^-, the inverse of L^-, is given by $V^- = L^- \circ G$ where G is the Green's function of the (vector + scalar) Laplacian Δ. In the

[1] This was carried out in [6, sections 6 and 7]

Euclidean case this Green's function G is given by

$$G^{ab}(x,y) = \frac{t^{ab}}{4\pi|x-y|} \qquad (t^{ab} \text{ is the inverse of } t_{ab} \stackrel{\text{def}}{=} tr(\mathcal{G}_a\mathcal{G}_b))$$

for both the scalar and the vector cases, and so the A part of our propagator is given by

$$\underset{a,i}{x} - - - - - - \underset{b,j}{y} = V^{ab}_{ij}(x,y) = \langle A^a_i(x)A^b_j(y)\rangle = \epsilon_{ijk}\partial^k_x \frac{t^{ab}}{4\pi|x-y|} = \epsilon_{ijk}t^{ab}\frac{(x-y)^k}{4\pi|x-y|^3}$$

as anticipated in (1). The terms of order $2\pi i/k$ are given by the diagrams in figure 2.

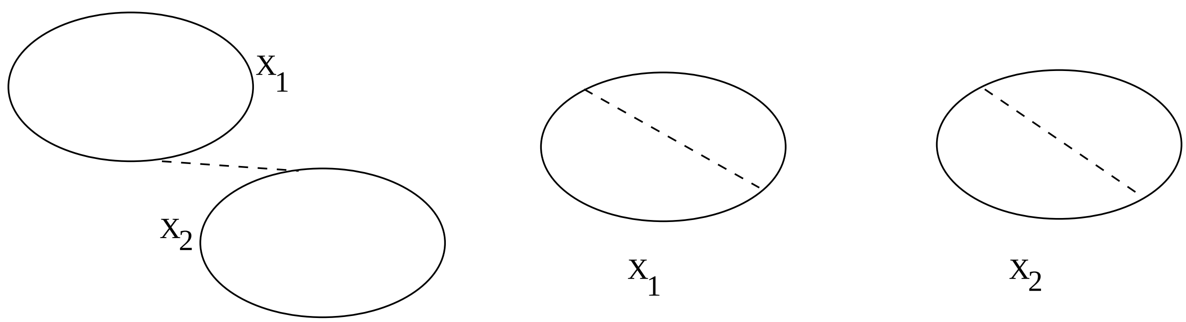


Figure 2. First order diagrams

4.1. The linking number of two knots. Let us first consider the left most diagram. Ignoring the constant numerical coefficient that the representations $R_{1,2}$ contribute it corresponds to the integral

$$\pounds(X_1,X_2) = \int ds_1 ds_2 V_{ij}(X_1(s_1),X_2(s_2))\dot{X}^i_1\dot{X}^j_2 \tag{18}$$

which is the well known Gauss integral representation for the linking number of two knots [30]. For the sake of completeness, and also as a preparation for the next section where we will use similar but more complicated considerations to deal with the two loop contribution, we will review here the proof of the invariance of (18) under isotopies and show that indeed it coincides with the linking number.

It is possible to view $V_{ij}(x,y)$ is as a $(1,1)$-form on $\mathbf{R}^3\times\mathbf{R}^3$ where $(x,y)\in\mathbf{R}^3\times\mathbf{R}^3$, i is the one form index for the variable x, and j is the one form index for the variable y. Viewed this way, (18) is just that form V evaluated on the cycle X_1 relative to its left variable and on the cycle X_2 relative to its right variable:

$$\pounds = \langle X_1|V|X_2\rangle \tag{19}$$

The key property required for the invariance proof is that there exists a $(2,0)$-form F (that is to say — a two variable form F which is a two form with respect to its left argument and a zero form with respect to its right argument) for which

$$d^LV = -d^RF \tag{20}$$

away from the diagonal, where d^L is the exterior derivative with respect to the left variable and d^R is the exterior derivative with respect to the right variable. Assuming such an F, under an infinitesimal deformation of X_1 we will have (using Stokes' theorem twice)

$$\delta \pounds = \delta\langle X_1|V|X_2\rangle = \langle{}^{\text{The surface } S \text{ spanned by the}}_{\text{infinitesimal deformation of } X_1}|d^L V|X_2\rangle = -\langle S|d^R F|X_2\rangle = 0. \tag{21}$$

As for the existence of F, notice that by our derivation of V, $V = \star d \circ G^{\text{vect}}$ and therefore $\star^L d^L V = \star d \star d \circ G$. By the commutativity of $\star d$ and G one gets $\star^L d^L V = G \circ \star d \star d$. Remembering that G is given by an integral kernel, one can integrate by parts $G \circ \star d \star d$ to get $\star^L d^L V = \star^R d^R \star^R d^R G = (\Delta^R - d^R \star^R d^R \star^R)G = \delta(x-y)I - d^R \star^R d^R \star^R G$. Multiplying from the left by $\star^L$ we obtain away from the diagonal

$$d^L V = -d^R \star^L \star^R d^R \star^R G \stackrel{\text{def}}{=} -d^R F.$$

The formula we just got for F can be expanded to give

$$F_{ij,-}(x,y) = \epsilon_{ijk} \frac{-(x-y)^k}{4\pi|x-y|^3},$$

and this can be used to verify (20) directly. Don't let yourself be mislead by the apparent equivalence of the formulae for V and for F! The indices are arranged a little differently and verifying (20) is a little more than just playing around with these indices — some differentiations do have to be carried and the verification *is* essentially the same calculation as the derivation in this paragraph.

Having shown that $\pounds$ is indeed an isotopy invariant we can now use it to show that it coincides with the linking number. Deform the knot so that it will be almost planar with only 'perpendicular crossings'. Now flip one of those crossings us shown in figure 3. Clearly,

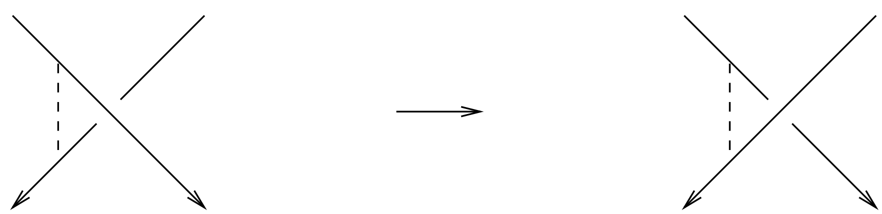


Figure 3. Flipping a crossing

when comparing the contribution to $\pounds$ from before and from after the flip we can integrate the propagator with its endpoints only nearby the crossing. If the crossed arcs are ϵ apart,

$$\pounds(\text{before}) - \pounds(\text{after}) = \frac{1}{2\pi} \int ds_{1,2} \frac{\epsilon}{(\epsilon^2 + s_1^2 + s_2^2)^{3/2}} = 1. \tag{22}$$

This is exactly the same relation is satisfied by the linking number, and together with $\pounds(\text{unlinked circles}) = 0$ (22) proves that $\pounds$ is indeed the linking number. To see that indeed $\pounds(\text{unlinked circles}) = 0$, use the already proven isotopy invariance to make sure that the two circles are very small relative to the separation between them and then the integral defining £ will tend to zero.

4.2. The self-linking of a single knot. The situation with the other diagrams in figure 2 is a bit more complicated. Let $£_s(X_1)$ be the 'self-linking' of X_1:

$$£_s(X_1) \stackrel{\mathrm{def}}{=} \frac{1}{2}\langle X_1|V|X_1\rangle = \frac{1}{2}\int ds_1 ds_2 V_{ij}(X_1(s_1), X_1(s_2))\dot{X}_1^i(s_1)\dot{X}_1^j(s_2). \tag{23}$$

(We have suppressed here the Lie-algebra coefficient which for R being the defining representation of $G = SU(N)$ in $\mathbf{C}^N$ and for tr being the usual matrix trace can easily be seen to equal $N^2 - 1$).

For three vectors A, B, C it will be convenient to denote $\epsilon_{ijk}A^iB^jC^k$, the volume of the parallelepiped spanned by $\vec{0}, A, B, C$ by $\det(A|B|C)$. Using this notation

$$£_s(X) = \frac{1}{8\pi}\int ds_1 ds_2 \frac{\det\left(X(s_1) - X(s_2)|\,\dot{X}(s_1)\,\middle|\dot{X}(s_2)\right)}{|X(s_1) - X(s_2)|^3}. \tag{24}$$

This integral appears at first sight to be divergent because of the cubic term in the denominator. Nevertheless when s_1 and s_2 are close, say ϵ apart, $X(s_1) - X(s_2) \sim \epsilon$ and the three vectors $X(s_1) - X(s_2)$, $\dot{X}(s_1)$, and $\dot{X}(s_2)$ are within a cone of opening $\sim \epsilon$. Therefore the volume of the parallelepiped spanned by these three vectors is $\sim \epsilon^3$ which is enough to suppress the singularity of the denominator. Unluckily, the argument in (21) doesn't go through — the key relation (20) holds only away from the diagonal, and in (23) our integration domain does intersect the diagonal.

This point has already been treated by Căalugăreanu [11, 12] (see also Pohl [27]) and later from a physical viewpoint by Polyakov [28] (see also Tze [31]). They found that indeed (23) is not an invariant, but yet it can be renormalized by the addition of a local term (the torsion of X) to give an invariant. It turns out that to properly define the torsion everywhere X needs to be 'framed', and therefore $£_s$ will just be an invariant of *framed* knots. We will arrive at the same results using a somewhat different regularization which makes the current calculation a bit less transparent but has a more straightforward generalization for the two-loop case to be treated in the next sections. Let us define $£_\epsilon$ by the integral (24) that defines $£_s$, only with the integration domain restricted to

$$\Delta_\epsilon \stackrel{\mathrm{def}}{=} [|s_1 - s_2| > \epsilon].$$

Assume that X undergoes an infinitesimal deformation $X \to X + \delta X \stackrel{\mathrm{def}}{=} X + \omega$. As in the invariance proof for the case of a link, (21), Stokes' theorem was used twice it will fail twice for this new case and $\delta £_\epsilon$ will pick up four non-zero contributions — one from each boundary term in each of the usages of Stokes' theorem. Denoting the evaluation of differential forms on Δ_ϵ by $\langle\ |\ |\ \rangle_\epsilon$ (compare with 19, and recall that the forms involved are forms on $\mathbf{R}^3 \times \mathbf{R}^3$ that can be evaluated on a *pair* of cycles in $\mathbf{R}^3$ — one on the "left" $\mathbf{R}^3$ and the other on the "right" $\mathbf{R}^3$) and on its two boundaries $[s_1 - s_2 = \pm\epsilon]$ by $\langle\ |\ |\ \rangle_\pm$ we will get: (S again is the surface spanned by the infinitesimal deformation of X)

$$\begin{aligned}\delta\pounds_\epsilon = \frac{1}{2}\delta\langle X|V|X\rangle_\epsilon &= \langle S|d^L V|X\rangle_\epsilon + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \\ &= -\langle S|d^R F|X\rangle_\epsilon + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \\ &= \langle S|F|-\rangle_+ + \langle S|F|-\rangle_- + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_-. \end{aligned} \tag{25}$$

We will try to understand the $\epsilon \to 0$ limit of $\delta\pounds_\epsilon$ by expanding (25) in powers of ϵ. For s a variable in S^1 let $X = X(s)$, $\dot{X} = \dot{X}(s)$, $\omega = \omega(s), \dots$,

$$\begin{aligned} X_{\pm\epsilon} = X(s\pm\epsilon) \quad &\sim \quad X \pm \epsilon\dot{X} + \frac{\epsilon^2}{2}\ddot{X} \pm \frac{\epsilon^3}{6}\dddot{X} \\ \dot{X}_{\pm\epsilon} = \dot{X}(s\pm\epsilon) \quad &\sim \quad \dot{X} \pm \epsilon\ddot{X} + \frac{\epsilon^2}{2}\dddot{X} \end{aligned}$$

Using these notations, with the dummy integration variable s picked to be at the point where ω is evaluated,

$$\langle\omega|V|X\rangle_\pm = \frac{1}{4\pi}\int ds \frac{\det\left(\dot{X}_{\pm\epsilon}|\omega|X_{\pm\epsilon} - X\right)}{|X - X_{\pm\epsilon}|^3}$$

$$\sim \frac{1}{4\pi}\int ds \frac{\det\left(\dot{X} \pm \epsilon\ddot{X} + \frac{\epsilon^2}{2}\dddot{X}|\omega| \pm \epsilon\dot{X} + \frac{\epsilon^2}{2}\ddot{X} \pm \frac{\epsilon^3}{6}\dddot{X}\right)}{|X - X_{\pm\epsilon}|^3}$$

$$\sim \frac{1}{4\pi}\int ds \frac{\epsilon^2 \det\left(\frac{1}{2}\ddot{X} \pm \frac{\epsilon}{3}\dddot{X}|\omega|\dot{X} \pm \frac{\epsilon}{2}\ddot{X}\right)}{|\epsilon|^{-3}|\dot{X}|^{-3}\left(1 \mp \epsilon\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\right)}.$$

Therefore (notice that the terms of order $\frac{1}{\epsilon}$ cancel!)

$$\langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \sim \frac{1}{4\pi}\int \frac{ds}{|\dot{X}|^3}\left(-\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\det(\ddot{X}|\omega|\dot{X}) + \frac{2}{3}\det(\dddot{X}|\omega|\dot{X})\right).$$

Similarly

$$\langle S|F|-\rangle_\pm = \frac{1}{4\pi}\int ds \frac{\det(\dot{X}|\omega|X_{\pm\epsilon} - X)}{|X_{\pm\epsilon} - X|^3}$$

$$\sim \frac{1}{4\pi}\int ds \frac{|\dot{X}|^3}{|\epsilon|}\left(1 \pm \epsilon\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\right)\det\left(\dot{X}\middle|\,\omega\,\middle|\frac{1}{2}\ddot{X} \mp \frac{\epsilon}{6}\dddot{X}\right)$$

and therefore (notice that again there is no term of order $\frac{1}{\epsilon}$)

$$-\langle S|F|-\rangle_+ + \langle S|F|-\rangle_- \sim \frac{1}{4\pi}\int \frac{ds}{|\dot{X}|^3}\left(-\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\det(\ddot{X}|\omega|\dot{X}) + \frac{1}{3}\det(\dddot{X}|\omega|\dot{X})\right).$$

This finally gives that the $\epsilon \to 0$ limit of $\delta\pounds_\epsilon$ is

$$\delta\pounds_s = \frac{1}{4\pi}\int \frac{ds}{|\dot{X}|^3}\left(-3\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^2}\det(\ddot{X}|\omega|\dot{X}) + \det(\dddot{X}|\omega|\dot{X})\right) \tag{26}$$

and we can see that indeed $\delta\pounds_s \neq 0$ and $\pounds_s$ is *not* a knot invariant.

4.3. The appearance of framings. Yet, some further investigation of $\delta\pounds_s$ shows that this can be corrected quite easily. Define τ to be $1/4\pi$ times the total torsion of the curve X — that is to say $1/4\pi$ times the integral with respect to arc length of the local torsion $\tau(s)$ (see [15, pp. 22]) of the curve, given by the standard formula

$$\tau(s) = -\frac{\det(\dot{X}(s)|\ddot{X}(s)|\dddot{X}(s))}{|\dot{X}(s)\times\ddot{X}(s)|^2} \tag{27}$$

whenever the denominator is non-zero. As I will comment below, under $X \to X+\omega$ one can show that $\delta\pounds_s$ and $-\delta\tau$ are given by exactly the same formula (26) so if one defines

$$\pounds_r = \pounds_s + \tau$$

then $\pounds_r$ *is* invariant under isotopies, so long as the denominator in (27) remains non-zero.

What if that denominator *is* equal to zero? On the normal bundle of X there is a canonically defined connection defined by the projection back to the normal bundle of the usual differentiation along the knot of vector functions normal to it. $1/4\pi$ times the total holonomy of that connection along the knot is some real number, well defined up to a half integer which depends on a choice of a trivialization for the normal bundle, and whenever τ is defined, it will be shown below to coincide with that number. Hence $\pounds_r$ is an invariant of *framed* knots — a framing is just a trivialization of the normal bundle which can be used to render τ and therefore $\pounds_s$ well-defined. This necessity of framing the knot X agrees with the results of Witten [32], but makes $\pounds_r$ quite useless for an unframed knot — it is a half integer which is well-defined only up to a half integer. For a framed knot it can be shown along the same lines as in (22) to be half the self-linking of a framed knot — half the linking number of that knot with its framing.

To complete the discussion we need to demonstrate the two differential geometric assertions made above. Very briefly, if $n(s)$ is *any* vector not tangent to the knot X then the holonomy discussed above can be calculated by measuring how much the projection of n to the normal bundle fails to be parallel. It is an elementary exercise to then find that relative to the framing given by n,

$$\tau = \frac{-1}{4\pi}\int ds|\dot{X}|\det\left(\frac{\dot{X}}{|\dot{X}|^2}\middle| n \middle| \frac{|\dot{X}|^2\dot{n} - (\dot{X}\cdot n)\ddot{X}}{|\dot{X}\times n|^2}\right). \tag{28}$$

Setting $n = \ddot{X}$ it is easy to see that (28) coincides with (27) and choosing n to be a constant vector that is not parallel to $\dot{X}(s)$ for any value of s simplifies it the most. One can then vary (28) under $X \to X+\omega$ and integrate by parts until all the derivatives of ω disappear. One is left with a huge and unfriendly expression that with a tremendous amount of labor and juggling with vector identities can be shown to equal (26). I could not verify this equality without the aid of a symbolic mathematics computer program [37].

5. The two-loop contribution

Let X be a parametrized knot in $\mathbf{R}^3$. In this section we will try to understand the two-loop contribution $\mathcal{W}_2$ to $\mathcal{W}(\text{flat } \mathbf{R}^3, \mathcal{X})$ — the contribution of order $-4\pi^2/k^2$. All the terms in the Lagrangian $\mathcal{L}_{tot}$ come in to play now, and on a flat $\mathbf{R}^3$ our $\mathcal{W}$ reads

$$\mathcal{W}(\text{flat } \mathbf{R}^3, \mathcal{X}) = \int \mathcal{D}B\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\ \mathrm{Tr}_R \mathcal{P}\exp\left(\int ds \dot{X}^i(s) A_i(X(s))\right) e^{i\mathcal{L}_{tot}}$$

where

$$\mathcal{L}_{tot} = \frac{k}{4\pi}\left(\int_{\mathbf{R}^3} tr(\epsilon^{ijk}A_i\partial_j A_k + \phi\partial^i A_i) + \frac{1}{3}\epsilon^{ijk} tr A_i[A_j, A_k] + \int_{\mathbf{R}^3} tr\ \bar{c}\partial_i(\partial^i c + [A^i, c])\right)$$

If R is a trace-free representation terms that have only one interaction point with X have a vanishing coefficient, and therefore the only potential contribution at two-loops come from the five diagrams in figure 4. In this figure dashed lines represent as before the

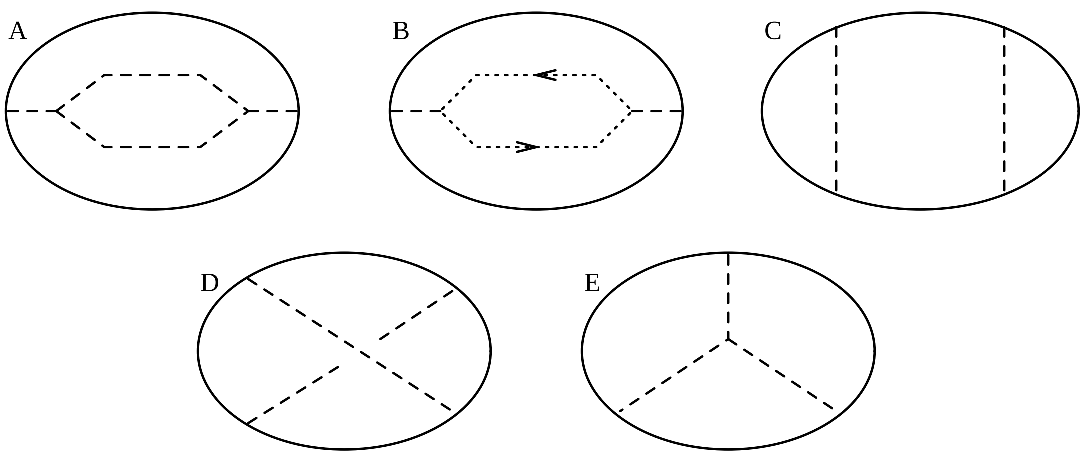


Figure 4. The five two-loop diagrams.

gauge-propagator V, the dotted lines represent the ghost $\bar{c}c$ propagator which is just the Green's function of the Laplacian $\partial_i\partial^i$, the symbol $\otimes$ represent the gauge-gauge-gauge interaction $-\frac{1}{6}\epsilon^{ijk} tr A_i[A_j, A_k]$ and the symbol $\oplus$ represents the ghost-gauge-ghost interaction $-\frac{1}{2} tr\bar{c}\partial_i[A^i, c]$.

The first two diagrams are divergent because of the integration over the location of the interaction vertices in $\mathbf{R}^3$. But as is readily verified and as was shown in [18] the integrands in these diagrams are exactly the opposites of each other so if we sum them before integrating we get zero. (We will accept at face value that A and B cancel and prove that $C + D + E$ is a topological invariant. It is very likely that the full story is a little more elaborate. In the context of a consistent regularization that could be used to all orders, A and B are likely to cancel only up to an imaginary multiple of the one loop contribution and thus what is calculated here is just the *real* part of the two-loop contribution. See[26, 1, 13]). Also, it is clear that if one ignores the Lie algebra coefficients of diagrams C and D then their sum is equal to the square of the one-loop one-knot contribution that was discussed in the previous section. It is therefore possible to subtract from $\mathcal{W}_2$ a multiple of $(\mathcal{W}_1)^2$ in such a way that diagram C will disappear. We will call the resulting quantity $\hat{\mathcal{W}}_2$. The coefficient of diagram D in $\hat{\mathcal{W}}_2$ will be the difference between the coefficients of diagrams C and D in $\mathcal{W}_2$, and these coefficients differ only because the Lie-Algebra indices are contracted in a slightly different way. So if $t_{ab} \stackrel{\text{def}}{=} tr(\mathcal{G}_a\mathcal{G}_b)$, t^{ab} is the inverse matrix of t_{ab} and we use t_{ab} and t^{ab} to raise and lower Lie-algebra indices, we get:

$$\binom{\text{Lie algebra con-}}{\text{tractions for } D} - \binom{\text{Lie algebra con-}}{\text{tractions for } C} = t^{bb'}t^{cc'}R^{\beta}_{b'\delta}R^{\delta}_{c'\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta} - t^{bb'}t^{cc'}R^{\beta}_{c'\delta}R^{\delta}_{b'\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta}$$

The fact that R is a representation is just the relation $f^{abc}R^{\beta}_{a\gamma} = t^{bb'}t^{cc'}(R^{\beta}_{b'\delta}R^{\delta}_{c'\gamma} - R^{\beta}_{c'\delta}R^{\delta}_{b'\gamma})$ and therefore

$$= f^{abc}R^{\beta}_{a\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta}.$$

These are exactly the Lie-algebra contractions for diagram E. Taking into account the different symmetry factors for these diagrams we finally get (after dividing by the Lie algebraic coefficient)

$$\tilde{\mathcal{W}}_2 = \frac{1}{4}D - 2E$$

as anticipated in Theorem 1.

In the case of $G = SU(N)$; $R = \mathbf{C}^N$ one can calculate that in $\mathcal{W}_2$ the Lie-algebraic coefficients of diagrams C, D, and E are $\frac{(N^2-1)^2}{N}$, $\frac{-N^2+1}{N}$, and $N(N^2-1)$ respectively, and therefore in this case

$$\tilde{W}_2 = \frac{1}{N(N^2-1)}\left(\mathcal{W}_2 - \frac{1}{2N}(\mathcal{W}_1)^2\right).$$

5.1. Finiteness of $\tilde{\mathcal{W}}_2$. It still isn't clear that the integrals represented by the diagrams D and E are finite. For diagram D there appears to be a singularity when three of the integration variables are close together but exactly the same analysis that has shown that the self-linking integral is finite shows that this integral is also finite. In diagram E there appears to be a problem when two or three of the knot integration variables are close together and are close to z — the variable of the vertex $\otimes$ integration. Up to a constant factor, diagram E represents the integral:

$$E = \int_{\Delta_3} ds_{1,2,3}\dot{X}^i(s_1)\dot{X}^j(s_2)\dot{X}^k(s_3)V_{ijk}(X(s_1), X(s_2), X(S_3)) \tag{29}$$

where

$$V_{ijk}(x_1,x_2,x_3) \stackrel{\text{def}}{=} \epsilon^{i'j'k'}\epsilon_{ii'i''}\epsilon_{jj'j''}\epsilon_{kk'k''}T^{i''j''k''}(x_1,x_2,x_3) \stackrel{\text{def}}{=} 6_{ijki''j''k''}T^{i''j''k''}(x_1,x_2,x_3) \tag{30}$$

and

$$T^{ijk}(x_1,x_2,x_3) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} d^3z \frac{(x_1-z)^i}{|x_1-z|^3}\frac{(x_2-z)^j}{|x_2-z|^3}\frac{(x_3-z)^k}{|x_3-z|^3}$$

The integral defining T is clearly finite for every choice of distinct x_{1-3} in $\mathbf{R}^3$, but it blows up rapidly when some of the x's coincide. To show that in spite of this the integral (29) is finite we need to understand the behavior of T as two or three of its arguments coincide.

Let us first rewrite T in a way that will make it easier to handle. Using

$$\frac{4}{3\sqrt{\pi}}\int_0^\infty e^{-\alpha^{2/3}N}d\alpha = \frac{1}{N^{3/2}}$$

we can rewrite T as

$$T^{ijk} = \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha \int_{\mathbf{R}^3} dz(x_1-z)^i(x_2-z)^j(x_3-z)^k e^{-\left(\alpha_1^{2/3}|x_1-z|^2+\alpha_2^{2/3}|x_2-z|^2+\alpha_3^{2/3}|x_3-z|^2\right)}.$$

Introducing the notation:

$$\begin{array}{rclcrcl} A &=& \sum\alpha_m^{2/3} & ; & \lambda_m &=& \frac{\alpha_m^{2/3}}{A} \\ t &=& \sum\lambda_m x_m & ; & s &=& \sum\lambda_m|x_m-t|^2 \end{array}$$

we get

$$\begin{aligned} T^{ijk}(x_1,x_2,x_3) &= \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha\int_{\mathbf{R}^3} dz(x_1-z)^i(x_2-z)^j(x_3-z)^k e^{-A(|z-t|^2+s)} \\ &= \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha e^{-As}\int_{\mathbf{R}^3} dz(x_1-t-z)^i(x_2-t-z)^j(x_3-t-z)^k e^{-A|z|^2}. \end{aligned}$$

This is just a Gaussian integral with respect to z, and it can be evaluated to give

$$\begin{aligned} T^{ijk} &= \frac{64}{27}\int_0^\infty d^3\alpha\frac{e^{-As}}{A^{3/2}}\left[\frac{1}{2A}\left((x_1-t)^i\delta^{jk}+(x_2-t)^j\delta^{ki}+(x_3-t)^k\delta^{ij}\right)\right. \\ &\qquad \left. +(x_1-t)^i(x_2-t)^j(x_3-t)^k\right]. \end{aligned}$$

Changing variables from $d^3\alpha$ to $d^2\lambda dA$ (there are just two integrations over the λ's because they are constrained to satisfy $\sum\lambda_m = 1$) we pick the Jacobian $\frac{27}{8}A^{7/2}\sqrt{\lambda_1\lambda_2\lambda_3}$ and get (after evaluating the A integral)

$$\begin{aligned} T^{ijk}(x_1,x_2,x_3) &= 4\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{(x_1-t)^i\delta^{jk}+(x_2-t)^j\delta^{ki}+(x_3-t)^k\delta^{ij}}{s^2}\right. \\ &\qquad \left. +4\frac{(x_1-t)^i(x_2-t)^j(x_3-t)^k}{s^3}\right]. \end{aligned} \tag{31}$$

Clearly the integral (29) is translation invariant, and invariant under reparametrizations of X of the form $s \to s+s_0$. So in the investigation of its possible divergencies we can assume that, say, 0 is the midpoint between s_2 and s_3, s_1 is farther away from s_2 or s_3 than the distance between these two:

$$s_1 = \tau \quad ; \quad s_2 = -\eta\tau \quad ; \quad s_3 = \eta\tau \quad ; \quad |\eta| < \frac{1}{3},$$

and that $X(0) = 0$. In this case we can write

$$T^{ijk}(X_\tau, X_{-\eta\tau}, X_{\eta\tau}) = 4 \int d^2\lambda \sqrt{\lambda_1\lambda_2\lambda_3} \left[\frac{S_1^{ijk}}{s^2} + 4\frac{S_2^{ijk}}{s^3} \right] \tag{32}$$

with

$$\begin{aligned} S_1^{ijk} &\stackrel{\text{def}}{=} (X_\tau - t)^i \delta^{jk} + (X_{-\eta\tau} - t)^j \delta^{ki} + (X_{\eta\tau} - t)^k \delta^{ij}, \\ S_2^{ijk} &\stackrel{\text{def}}{=} (X_\tau - t)^i (X_{-\eta\tau} - t)^j (X_{\eta\tau} - t)^k. \end{aligned}$$

The problematic regions are when η or τ are small, and we need to be able to estimate integrals like those in (32) for such values of η and τ.

Lemma 5.1. *Let A, B, and C be the three vertices of a triangle with sides $|A-B| \sim |A-C| \sim \tau$, and $|B-C| \sim \eta\tau$ with $\eta < 1/3$ (see figure 5). For positive λ's satisfying*

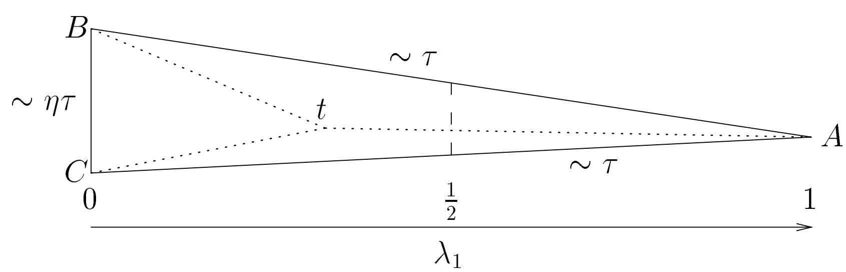


Figure 5. *The triangle ABC.*

$\lambda_1 + \lambda_2 + \lambda_3 = 1$ *define:*

$$\begin{aligned} t &= \lambda_1 A + \lambda_2 B + \lambda_3 C \\ s &= \lambda_1 |A - t|^2 + \lambda_2 |B - t|^2 + \lambda_3 |C - t|^2 \end{aligned}$$

Finally let Λ_A be one of $\{(1-\lambda_1), \lambda_2, \lambda_3\}$, Λ_B be one of $\{\lambda_1,\ (1-\lambda_2), \lambda_3\}$, and Λ_C be one of $\{\lambda_1, \lambda_2, (1-\lambda_3)\}$.

In this situation there exists constants c_{1-5} independent of η and τ for which:

$$\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{1}{s^2}\right] < \frac{c_1}{\eta\tau^4} \tag{33}$$

$$\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\lambda_1}{s^2}\right] < \frac{c_2}{\tau^4} \tag{34}$$

$$\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\Lambda_A\Lambda_B\Lambda_C}{s^3}\right] < \begin{cases} \frac{c_3}{\eta^3\tau^6} & \text{if neither of } \Lambda_B \text{ or } \Lambda_C \text{ is chosen to be } \lambda_1, \\ \frac{c_4}{\eta\tau^6} & \text{if exactly one of } \Lambda_B,\ \Lambda_C \text{ is chosen to be } \lambda_1, \\ \frac{c_5}{\tau^6} & \text{if both of } \Lambda_B \text{ and } \Lambda_C \text{ are chosen to be } \lambda_1. \end{cases} \tag{35}$$

5.2. Proof of the finiteness of diagram E. It is sufficient to show that (the symbol "6" was implicitly defined in (30))

$$6_{ijki'j'k'}T^{i'j'k'}(X_\tau, X_{-\eta\tau}, X_{\eta\tau}) < c/\tau. \tag{36}$$

Let us first deal with the contribution coming from S_1^{ijk}. Expanding S_1^{ijk} in powers of λ_1,

$$S_1^{ijk} = S_1^{0,ijk} + \lambda_1 S_1^{1,ijk} \tag{37}$$

we can use (33) and (34) and then all that is left to prove is:

$$6_{ijki'j'k'}\dot{X}^{i'}(\tau)\dot{X}^{j'}(-\eta\tau)\dot{X}^{k'}(\eta\tau)S_1^{p,ijk} = O(\eta^{1-p}\tau^3) \qquad ; \qquad p = 0, 1. \tag{38}$$

This can be done by expanding all the terms in the above expressions once in powers of η and once in powers of τ and showing that the low order coefficients in each of these expansions are zero. It is not hard to do it by hand, but as we are going to encounter some very similar but a bit harder expansions later on we will not do it here but postpone it to the appendix where it will be shown how all these expansions can be carried out in a uniform way using a computer.

The terms involving S_2^{ijk} are dealt with in a very similar way. Clearly, each of the factors of S_2^{ijk} is made of three summands, whose coefficients exactly correspond to the various possibilities for choosing Λ_A, Λ_B, and Λ_C in the lemma 5.1. Keeping $(X_\tau - t)^i$ unexpanded and expanding *only* the last two factors of S_2^{ijk} in powers of λ_1,

$$S_2^{ijk} = S_2^{0,ijk} + \lambda_1 S_2^{1,ijk} + \lambda_1^2 S_2^{2,ijk}, \tag{39}$$

and keeping in mind (35) what is left to prove is

$$6_{ijki'j'k'}\dot{X}^{i'}(\tau)\dot{X}^{j'}(-\eta\tau)\dot{X}^{k'}(\eta\tau)S_2^{p,ijk} = \begin{cases} O(\eta^3\tau^5) & \text{for } p=0, \\ O(\eta\tau^5) & \text{for } p=1, \\ O(\tau^5) & \text{for } p=2. \end{cases} \tag{40}$$

Again, the relevant expansions will be shown to vanish to the required order in the appendix using a computer.

5.3. Proof of lemma 5.1. We will write $\lambda_2 = (1-\lambda_1)\theta$ and $\lambda_3 = (1-\lambda_1)\bar{\theta}$ where $0 \leq \theta \leq 1$ and $\bar{\theta} = 1-\theta$. c will denote a positive constant that is allowed to change from line to line. It is easy to read from the geometry of figure 5 that when $\lambda_1 < 1/2$ (equivalently, when t is in the left portion of figure 5), $\lambda_1|A-t|^2 > c\lambda_1\tau^2$, that $\lambda_2|B-t|^2 > c\lambda_2\bar{\theta}^2|B-C| > c\theta\bar{\theta}^2\eta^2\tau^2$, that $\lambda_3|C-t|^2 > c\bar{\theta}\theta^2\eta^2\tau^2$, and thus that

$$s|_{\lambda_1<\frac{1}{2}} > c\left(\lambda_1\tau^2 + (\theta\bar{\theta}^2 + \bar{\theta}\theta^2)\eta^2\tau^2\right) = c\tau^2\left(\theta\bar{\theta}\eta^2 + \lambda_1\right). \tag{41}$$

In the region $\lambda_1 > 1/2$ the expressions which are integrated over $d^2\lambda$ in (33), (34), and (35) are bounded functions, and therefore (41) can be used to give upper bounds for the integrals we are considering.

Taking for example (35) with $\Lambda_A = (1-\lambda_1)$, $\Lambda_B = (1-\lambda_2)$, and $\Lambda_C = (1-\lambda_3)$ we get

$$\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\Lambda_A\Lambda_B\Lambda_C}{s^3}\right] < c\int_0^1 d\theta\int_0^{\frac{1}{2}} d\lambda_1\frac{\sqrt{\lambda_1\theta\bar{\theta}}(\lambda_1+\theta\bar{\theta})}{\tau^6\left(\theta\bar{\theta}\eta^2+\lambda_1\right)^3}. \tag{42}$$

The λ_1 integral can be explicitly evaluated. In fact, for a small α one has

$$\int_0^a d\lambda\,\frac{\sqrt{\lambda}}{(\alpha^2+\lambda)^3} = \frac{-\sqrt{a}}{2(a+\alpha^2)^2} + \frac{\sqrt{a}}{4\alpha^2\,(a+\alpha^2)} + \frac{\arctan(\frac{\sqrt{a}}{\alpha})}{4\alpha^3} < \frac{c}{\alpha^3}$$

and

$$\int_0^a d\lambda\,\frac{\sqrt{\lambda}\lambda}{(\alpha^2+\lambda)^3} = \frac{\sqrt{a}\alpha^2}{2(a+\alpha^2)^2} - \frac{5\sqrt{a}}{4\,(a+\alpha^2)} + \frac{3\arctan(\frac{\sqrt{a}}{\alpha})}{4\alpha} < \frac{c}{\alpha}$$

and plugging these two estimates into (42) gives the required result. The other assertions of the lemma are proved along the same lines.

$\square$

6. Proof of theorem 1

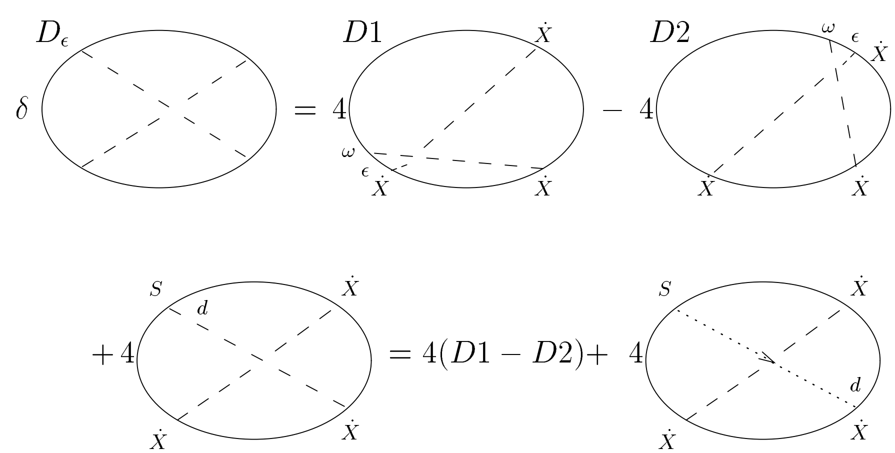
We will now show that $\tilde{\mathcal{W}}_2$ is indeed a knot invariant — that it is not changed under infinitesimal deformations. The proof presented here should be similar in spirit to invariance proofs (that are yet to be found) of higher terms in the perturbative expansion — we will first write a diagrammatic argument as expected from the results of section 3, (though our diagrammatic argument *is not* derived from the results there), and then supplement it with the required analytical details. As in the case of the analysis of the variation of the self linking number in the previous section, in analyzing the variation of $\tilde{\mathcal{W}}_2$ we will need take derivatives of V_{ijk} and of V_{ij} near the diagonal where there are singularities which will prevent a straight-forward invariance proof. To avoid these singular points define $\tilde{\mathcal{W}}_{2,\epsilon}$ to be given by the same integrals D and E as $\tilde{\mathcal{W}}_2$, only with the integration domain restricted by the condition that the s's would be at least ϵ apart — for $i \neq j$ we require

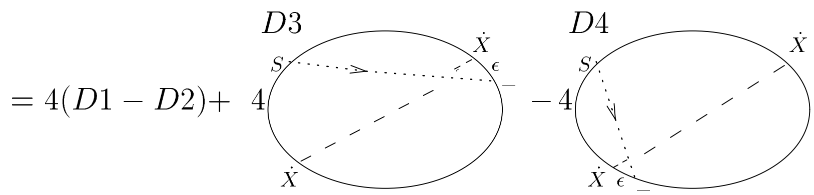
$$|s_i - s_j| > \epsilon. \tag{43}$$

We will denote these integrals by D_ϵ and E_ϵ, and the finiteness of $\tilde{\mathcal{W}}_2$ that was proven above just means

$$\tilde{\mathcal{W}}_{2,\epsilon} = \frac{1}{4}D_\epsilon - 2E_\epsilon \xrightarrow[\epsilon \to 0]{} \frac{1}{4}D - 2E = \tilde{\mathcal{W}}_2.$$

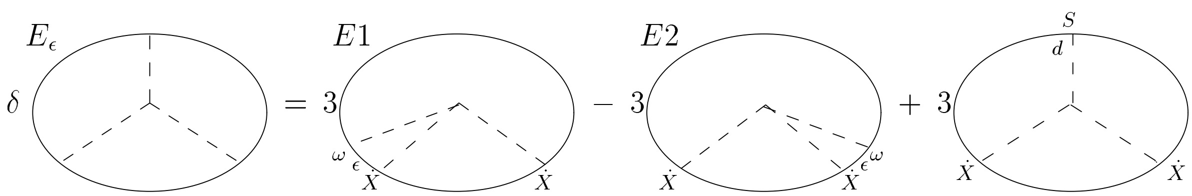
6.1. The variation of $\tilde{\mathcal{W}}_2$. We will now vary D_ϵ and E_ϵ under the infinitesimal deformation of X given by $X \to X + \omega$. It will be a lot more instructive to perform those calculations diagrammatically instead of working with the explicit formulae given for D and E in the introduction. First, let us vary diagram D_ϵ. When X moves to $X + \omega$ it sweeps an infinitesimal surface S, and our quantity of interest δD_ϵ is given by the evaluation of $d^L V$ on S which after using the key relation (20) reduces to diagrams $D3$ and $D4$ and by two boundary terms, diagrams $D1$ and $D2$:

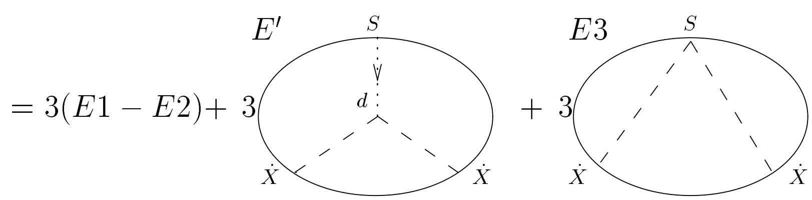




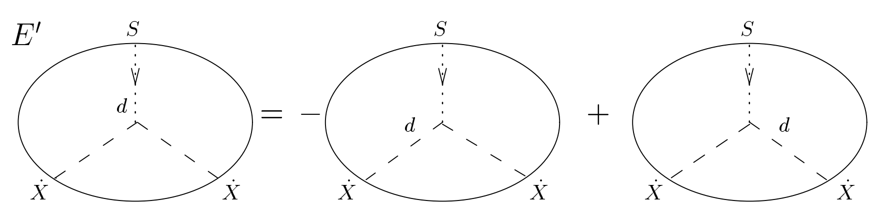
In these diagrams a dashed line represents as before the gauge propagator V_{ij} evaluated between the two vectors marked at its ends, a dotted represents the $(2,0)$-form F, a d symbol stands for exterior differentiation applied to the nearby end of the nearby propagator, and an ϵ between two interaction points on the knot means that these points are exactly ϵ apart.

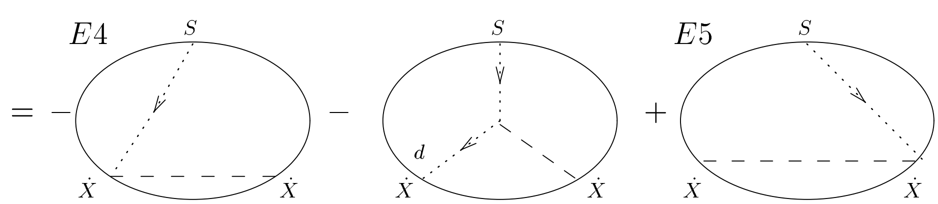
Similarly we can vary E_ϵ:

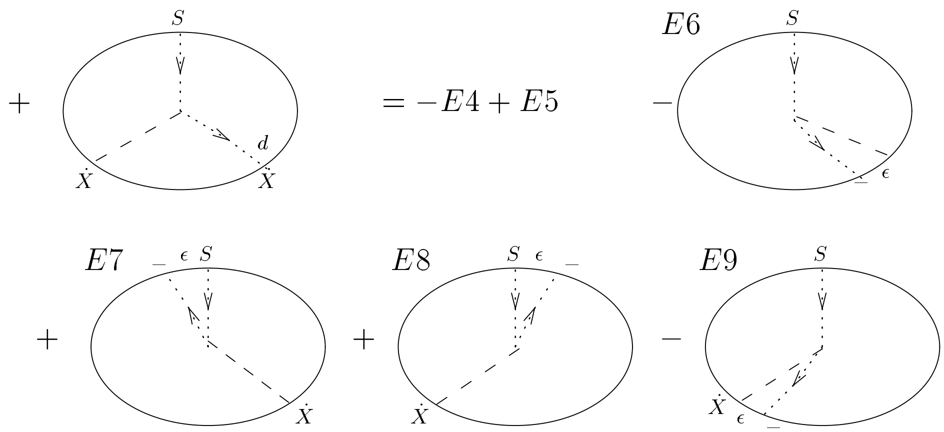




The diagram $E3$ appears because (20) is true only *off* diagonal. Actually $d^L V$ and $-d^R F$ differ by a $\star^L$ of a δ-function as was shown in the derivation of (20). Integrating by parts and using Leibnitz's rule we get:







6.2. Proof of Theorem 1. To show that $\tilde{\mathcal{W}}_2$ is indeed an invariant we just need to show that the limit as $\epsilon \to 0$ of $\delta(\frac{1}{4}D_\epsilon - 2E_\epsilon)$ vanishes. That is, we need to show that

$$\lim_{\epsilon\to 0} D1 - D2 - D3 + D4 + 6(E1 - E2 + E3 - E4 + E5 + E6 - E7 - E8 + E9) = 0.$$

In fact, we will show that

$$\lim_{\epsilon\to 0} D1 - D2 + 6E3 = 0, \tag{44}$$

$$\lim_{\epsilon\to 0} -D3 + D4 + 6(-E4 + E5) = 0, \tag{45}$$

and

$$\lim_{\epsilon\to 0} E1 - E2 + E6 - E7 - E8 + E9 = 0, \tag{46}$$

independently. For convenience, the symbol $\int_\epsilon$ will denote integration in which the integration variables are constrained to satisfy the restrictions (43), we will write X_ν for $X(s_\nu)$, and similarly for $\dot{X}_\nu$, $\ddot{X}_\nu$ and ω_ν.

Proof of (44). Diagram $D1$ represents the integral

$$D1 = \int_\epsilon ds_{1-3}\omega_3^i \dot{X}_4^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_4, X_2)\dot{X}_2^l \qquad ; \qquad s_4 = s_3 + \epsilon, \tag{47}$$

diagram $D2$ reads

$$-D2 = \int_\epsilon ds_{1-3} - \dot{X}_3^i \omega_4^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_4, X_2)\dot{X}_2^l \qquad ; \qquad s_4 = s_3 + \epsilon, \tag{48}$$

and diagram $E3$ is given by

$$E3 = \frac{1}{6}\int_\epsilon ds_{1-3}\dot{X}_3^p \omega_3^n \epsilon_{pnm}\epsilon^{mik} V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l. \tag{49}$$

Using

$$\epsilon_{pnm}\epsilon^{mik} = \delta_p^i\delta_n^k - \delta_p^k\delta_n^i$$

we can write $E3 = E3' + E3''$ with

$$E3' = \frac{1}{6}\int_\epsilon ds_{1-3}\dot{X}_3^i\omega_3^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l. \tag{50}$$

and

$$E3'' = -\frac{1}{6}\int_\epsilon ds_{1-3}\dot{X}_3^k\omega_3^i V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l.. \tag{51}$$

The nearness of s_3 and s_4 clearly implies that the integrand in (47) converges to the integrand of (51) and the integrand in (48) converges to the integrand of (50) as $\epsilon \to 0$. At the region where s_1 and s_2 are farther from $s_{3,4}$ than some fixed but small positive constant T, there is no problem with commuting integration with taking the $\epsilon \to 0$ limit. Concentrating first on comparing diagrams $D1$ and $E3''$ we see that nothing particularly harmful happens if *just* $|s_4 - s_2|$ is small — as it was shown in section 4 the integrand in this case remains finite. Otherwise, we are looking at one of the following exceptional cases (assuming for simplicity that $s_4 = 0$, $s_3 = -\epsilon$, and $X_4 = 0$):

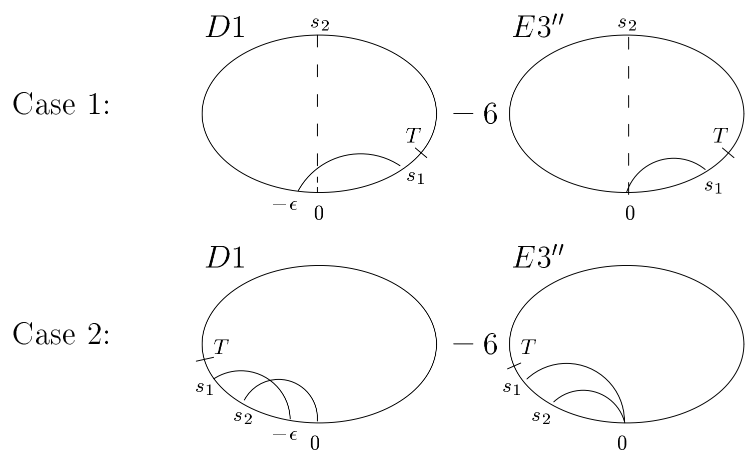


Figure 6. The two exceptional cases for $D1 \leftrightarrow E3''$.

Case 1: Disregarding the propagator connecting X_2 and $X_4 = 0$ the difference $D1 - 6E3''$ reads:

(52)

$$\int_\epsilon^T ds_1 \frac{\det\left(\omega(-\epsilon)\left|\dot{X}(s_1)\right| X(-\epsilon) - X(s_1)\right)}{|X(-\epsilon) - X(s_1)|^3} - \int_\epsilon^T ds_1 \frac{\det\left(\omega(0)\left|\dot{X}(s_1)\right| X(0) - X(s_1)\right)}{|X(0) - X(s_1)|^3}.$$

Expanding the integrands in (52) in powers of s_1 we can ignore all terms of order smaller than $1/s_1$ — evaluating the integrals in (52) for these terms would give a result bounded by a constant multiple of T in the $\epsilon \to 0$ limit, and as T was chosen small we can indeed ignore the contribution to (52) coming from these terms. There are no terms of order *higher* than $1/s_1$ in (52) and the term of order $1/s_1$ reads:

$$\int_\epsilon^T ds_1 \left(\frac{1}{2(s_1+\epsilon)} \frac{\det\left(\omega(-\epsilon)\left|\dot{X}(-\epsilon)\right|\ddot{X}(-\epsilon)\right)}{|\dot{X}(-\epsilon)|^3} - \frac{1}{2s_1}\frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)}{|\dot{X}(0)|^3}\right)$$

at the $\epsilon \to 0$ limit we get

(53)
$$\sim \frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)}{|\dot{X}(0)|^3}\int_\epsilon^T ds_1\left(\frac{1}{2(s_1+\epsilon)} - \frac{1}{2s_1}\right) \to \frac{-\log 2}{2}\frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)}{|\dot{X}(0)|^3}.$$

Reinstalling the propagator connecting X_2 and X_4 and the integration over s_2 we get the only non-vanishing contribution to $D1 - 6E3''$.

Case 2: Here the $\epsilon \to 0$ limit is in fact zero. To see that, one does analysis similar to the previous case, and notices that s_2 is integrated over an interval of length smaller than s_1 and thus remembering that the propagator connecting X_2 and X_4 is finite even near the diagonal the s_2 integral is $\sim s_1$, and this additional factor is sufficient to make the contribution coming from this case vanish.

A similar analysis to the above shows that the only non-vanishing contribution to $6E3'-D2$ comes from the case parallel to case 1 here, and that, in fact, these contributions exactly cancel. □

Proof of (45). Here are the integrals corresponding to the relevant diagrams:

$$-D3 = -\int_\epsilon ds_{1-3}\dot{X}_4^k V_{lk}(X_2,X_4)\dot{X}_2^l\dot{X}_1^i\omega_1^j F_{ij,-}(X_1,X_3), \quad ; \quad s_4 = s_3+\epsilon, \tag{54}$$

$$D4 = \int_\epsilon ds_{1-3}\dot{X}_3^k V_{kl}(X_3,X_1)\dot{X}_1^l\dot{X}_2^i\omega_2^j F_{ij,-}(X_2,X_4), \quad ; \quad s_4 = s_3+\epsilon, \tag{55}$$

$$-6E4 = \frac{1}{2} - \int_\epsilon ds_{1-3}\dot{X}_3^k\epsilon_{kmn}\epsilon^{mnp}V_{pl}(X_3,X_1)\dot{X}_1^l F_{ij,-}(X_2,X_3)\dot{X}_2^i\omega_2^j, \tag{56}$$

$$6E5 = \frac{1}{2}\int_\epsilon ds_{1-3}\dot{X}_3^k\epsilon_{kmn}\epsilon^{mnp}V_{pl}(X_3,X_2)\dot{X}_2^l F_{ij,-}(X_1,X_3)\dot{X}_1^i\omega_1^j. \tag{57}$$

Using

$$\epsilon_{kmn}\epsilon^{mnp} = 2\delta_k^{\ p}$$

and the nearness of s_3 and s_4 it is clear that so long as X_1 and X_2 are far away from X_3 the integrands of (54) and of (55) converge to the integrands of (57) and of (56) respectively, and that there is no problem with commuting integration with taking the $\epsilon \to 0$ limit. The cases when X_1 and X_2 are not far away from X_3 can be treated in the same way as in the previous proof. □

Proof of (46). It will be convenient here to replace ϵ by 2ϵ and then take the $\epsilon \to 0$ limit. In all of the relevant diagrams two of the s's are constrained to be exactly 2ϵ apart and the third to be farther then 2ϵ from any of the previous two. It is harmless to assume that $s_2 = -\epsilon$, $s_3 = \epsilon$, $X(0) = 0$, and $s_1 = \tau$ with $|\tau| > 3\epsilon$. We will denote the ratio ϵ/τ by η.

With these notations one can see that the integrands corresponding to our diagrams can be written in pairs as follows: (ignoring the overall coefficient $1/384\pi^3$)

$$\begin{aligned}
E1 - E2 &= \sum_{\beta=\pm} 6_{ijki'j'k'} \dot{X}^{i'}_{\tau} \omega^{j'}_{-\beta\eta\tau} \dot{X}^{k'}_{\beta\eta\tau} T^{ijk}(X_\tau, X_{-\beta\eta\tau}, X_{\beta\eta\tau}) \\
E6 + E9 &= \sum_{\beta=\pm} \epsilon_{mni}\epsilon_{ljk} \dot{X}^{m}_{\tau} \omega^{n}_{\tau} \dot{X}^{l}_{\beta\eta\tau} T^{ijk}(X_\tau, X_{-\beta\eta\tau}, X_{\beta\eta\tau}) \\
-E7 - E8 &= \sum_{\beta=\pm} \epsilon_{mnj}\epsilon_{lki} \dot{X}^{m}_{-\beta\eta\tau} \omega^{n}_{-\beta\eta\tau} \dot{X}^{l}_{\tau} T^{ijk}(X_\tau, X_{-\beta\eta\tau}, X_{\beta\eta\tau}).
\end{aligned}$$

Remembering (32), (37), (39), and lemma 5.1 we see that in considering the $\epsilon \to 0$ limit we just need to show that

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{|\tau|>T} \frac{d\tau}{\eta^a\tau^b} \sum_{\beta=\pm} \Big(6_{ijki'j'k'} \dot{X}^{i'}_{\tau} \omega^{j'}_{-\beta\eta\tau} \dot{X}^{k'}_{\beta\eta\tau} &+ \epsilon_{mni}\epsilon_{ljk} \dot{X}^{m}_{\tau} \omega^{n}_{\tau} \dot{X}^{l}_{\beta\eta\tau} \\
&+ \epsilon_{mnj}\epsilon_{lki} \dot{X}^{m}_{-\beta\eta\tau} \omega^{n}_{-\beta\eta\tau} \dot{X}^{l}_{\tau} \Big) S^{p,ijk}_{q} = 0
\end{aligned} \tag{58}$$

and that

$$\lim_{\epsilon\to 0} \int_{3\epsilon<|\tau|<T} \frac{(\text{same})}{\eta^a\tau^b} d\tau = O(T) \tag{59}$$

where T is some fixed small positive number and a and b are the exponents of η and τ as in equations (33), (34), and (35).

As in (58) τ is bounded from below we can use $\epsilon = \eta\tau$ to replace the limit there by an $\eta \to 0$ limit and then all that is required is to show that the summand there is $\sim \eta^{a+1}$. The relevant algebra will be carried out in the appendix using a computer.

The integration domain in (59) is symmetric and therefore we can replace the integration there with an integration over $3\epsilon < \tau < T$, replacing the integrand with

$$\sum_{\substack{\beta=\pm \\ \alpha=\pm}} \Big(6_{ijki'j'k'} \dot{X}^{i'}_{\alpha\tau} \omega^{j'}_{-\beta\eta\tau} \dot{X}^{k'}_{\beta\eta\tau} + \epsilon_{mni}\epsilon_{ljk} \dot{X}^{m}_{\alpha\tau} \omega^{n}_{\alpha\tau} \dot{X}^{l}_{\beta\eta\tau} + \epsilon_{mnj}\epsilon_{lki} \dot{X}^{m}_{-\beta\eta\tau} \omega^{n}_{-\beta\eta\tau} \dot{X}^{l}_{\alpha\tau} \Big) S^{ijk}_{1,2}\Big|_{\tau\to\alpha\tau}. \tag{60}$$

Simply integrating over τ now shows that to conclude the invariance proof we just need to show that (60) $= O(\eta^a\tau^b)$. Again, the relevant algebra will be carried out in the appendix using a computer. □

6.3. Identifying $\tilde{\mathcal{W}}_2$. The last assertion of theorem 1 is that the invariant $\tilde{\mathcal{W}}_2$ that we have produced is (up tp a constant shift) the second non-zero coefficient in the Conway polynomial of X. The Conway polynomial is defined by its behavior under flipping a crossing in a planar projection, so we will try to understand how $\tilde{\mathcal{W}}_2$ changes under such a flip.

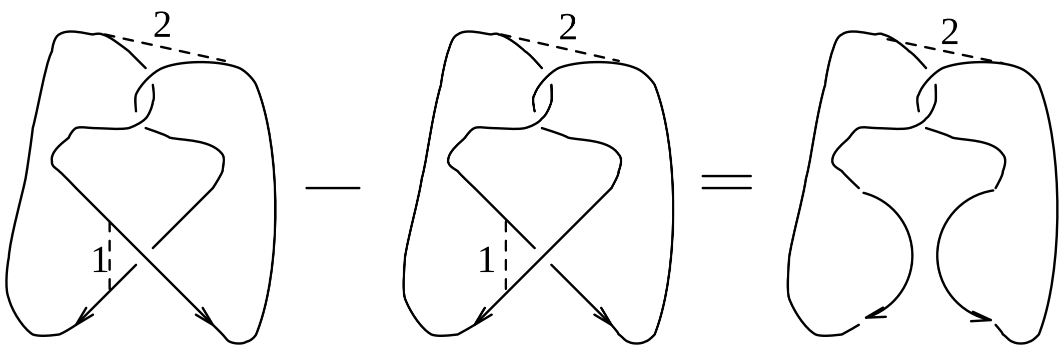


Figure 7. The change in $\tilde{\mathcal{W}}_2$ under a flip.

Very briefly, it is clear that the difference in the value of $\tilde{\mathcal{W}}_2$ before and after a flip comes from a singularity in either of V_{ijk} or V_{ij} at the point where the flip occurs. Using the invariance that we have just proven one can 'straighten' the knot near a crossing point before flipping, and then it is easy to check in this case V_{ijk} contracted with the tangents of the knot in fact vanishes near the crossing point except if one of its arguments is on the upper branch of the crossing and the other is on the lower. V_{ijk} is then inversely proportional to the distance between its two arguments, and the fact that $1/r$ is integrable on $\mathbf{R}^2$ shows that this singularity can be neglected. Similarly considering diagram D one finds that the only singularity that remains is the one that occurs when the two arguments of the *same* propagator are arranged as propagator 1 in figure 7, and the other propagator can then be assumed to be away from the crossing. Repeating (22) for propagator 1 and then integrating over the location of the other propagator, marked 2 in the figure, it is clear that effectively we are calculating the linking number of the two knots that are created if the original knot is cut at the crossing as in the figure. It is easy to check from the definitions (see [23]) that this is exactly the same relation as the one that is satisfied by the second non-zero coefficient in the Conway polynomial of X, and so they coincide up to a constant shift. This constant shift is given by $\tilde{\mathcal{W}}_2$(unknotted circle). By invariance we can just calculate $\tilde{\mathcal{W}}_2$(the unit circle in the XY plane) and an explicit calculation shows that (see [19])

$$\tilde{\mathcal{W}}_2(\text{the unit circle in the } XY \text{ plane}) = \frac{1}{24}.$$

This concludes the proof of theorem 1.

7. Comparison with Witten's non-perturbative treatment

In [32] Witten has used a very different approach resting on conformal field theory to give a non-perturbative definition for the infinite dimensional integral (8) defining $\mathcal{W}(M^3, \mathcal{X}, k)$. His definition is much more successful in that he can show how to use it to evaluate (8) precisely for *every* three manifold M^3 and link $\mathcal{X}$ in it, and not just calculate its leading large k asymptotics for $\mathbf{R}^3$, but it is less elementary and very particular to the Chern-Simons theory. There doesn't seem to be any direct relation between his way of calculating and the perturbative calculation shown here, and it is very interesting to compare the two view points. Let us start by reviewing his results for a link in $\mathbf{R}^3$, as presented in [33]. As is shown there, $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ considered as a function of k and the gauge group $G = SU(N)$ is in fact up to a simple change of variable the HOMFLY [17] polynomial of the link $\mathcal{X}$, which itself is a generalization of the Jones polynomial of $\mathcal{X}$.

Witten shows that to define $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ unambiguously one needs to consider *framed* links instead of just links. That is to say, each component X_γ of the link has to be accompanied with a prescribed 'framing' — a choice up to homotopy of a nowhere vanishing section F_γ of the normal bundle of X_γ in the language of section 4 or, more geometrically, a choice of a 'shadow' for each component as in the figure 8.

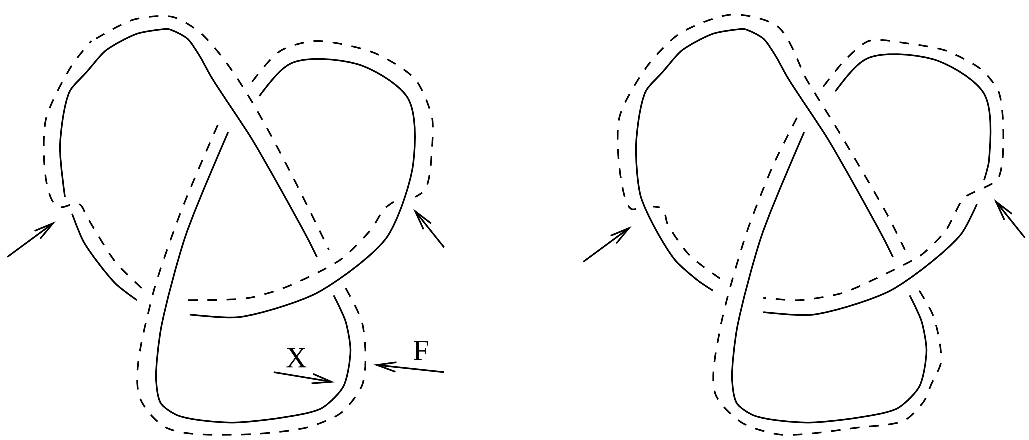


Figure 8. A knot with two of its possible framings. (The arrows indicate the differences between the two framings)

According to Witten, if the framing of link changes by a single twist, $\mathcal{W}$ get multiplied by $e^{2\pi i h}$, where h is a real number determined by k and the representation R_γ corresponding to the component of the link on which the twist was made. This is shown schematically in figure 9.

We will only be concerned with the case where the underlying group G is $SU(N)$ for some positive integer N, and all the representations R_γ are just the defining representation of

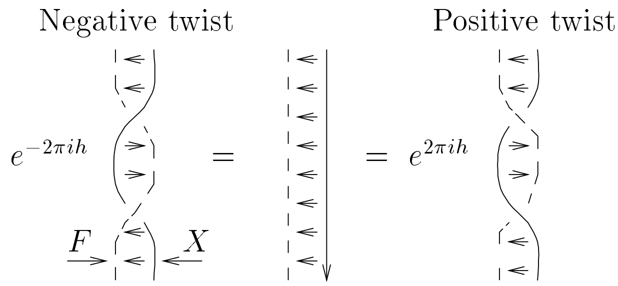


Figure 9. The change in $\mathcal{W}$ under a single twist.

$SU(N)$ in $\mathbf{C}^N$. In this case h is given by:

$$h = \frac{N^2-1}{2N(N+k)} \tag{61}$$

The difference between any two framings of a *single* knot is measured using a single integer — the number of signed twists required to change one framing to the other, and the above relation shows that for a link with several components we can in fact consider two framings to be equivalent if the total number of twists required to switch from one framing to the other is zero, counting all twists on all the components of the given link. With this identification for each link $\mathcal{X} = \{X_\gamma\}$ in $\mathbf{R}^3$ there is a unique preferred framing — the framing $\{F_\gamma\}$ for which the total linking number of $\mathcal{X}$ is 0:

$$\pounds(\mathcal{X}) \stackrel{\text{def}}{=} \sum_{\gamma_1,\gamma_2} \pounds(X_{\gamma_1}, F_{\gamma_2}) = 0$$

In this framing, Witten has shown that $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ has the following three properties which allows one to calculate it for any given link:

(1) For

$$q = e^{\frac{2\pi i}{N+k}} \tag{62}$$

one has

$$\mathcal{W}(\text{Unknotted circle in } \mathbf{R}^3, k) = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} \tag{63}$$

(In fact, this relation can be derived from the following two by using the third relation on the unknot whose planar projection is ∞)

(2) If the link $\mathcal{X}$ is the *unlinked* union of $\mathcal{X}_1$ and $\mathcal{X}_2$ then

$$\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k) = \mathcal{W}(\mathbf{R}^3, \mathcal{X}_1, k)\mathcal{W}(\mathbf{R}^3, \mathcal{X}_2, k) \tag{64}$$

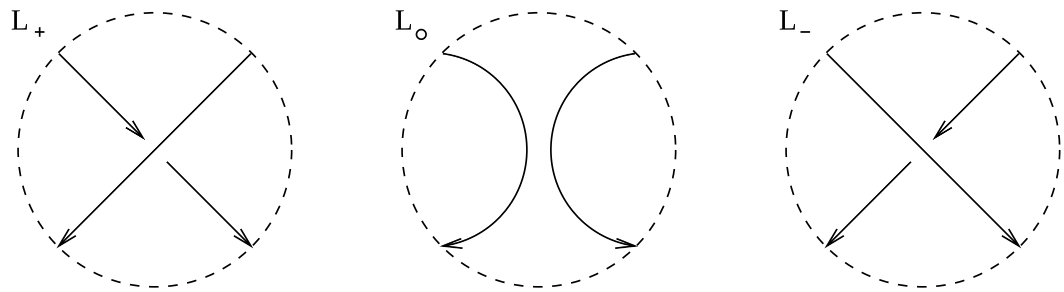


Figure 10. The links involved in the skein relation.

(3) Most important — the so called "skein relation" — if the three links L_0, L_+, and L_- differ only inside a small ball where they look as in figure 10, then the following relation holds:

$$-q^{N/2}L_+ + (q^{1/2} - q^{-1/2})L_0 + q^{-N/2}L_- = 0 \tag{65}$$

where for brevity we wrote $L.$ for $\mathcal{W}(\mathbf{R}^3, L., k)$.

To compare these results with ours we first need to expand them in powers of $2\pi i/k$, and thus we will write for a link $L.$

$$\mathcal{W}(\mathbf{R}^3, L., k) \sim N^{c.} + \frac{2\pi i a.}{k} - \frac{4\pi^2 b.}{k^2}.$$

From (63) and (64) it is clear that $c.$ is just the number of components of the link $L.$ if $L.$ is the unlinked union of unknotted circles. In addition, the zeroth order part of (65) reads $-N^{c_+} + 0 + N^{c_-} = 0$ and as L_+ and L_- always have the same number of components it means that the number of components of an arbitrary $L.$ is given by $c.$. The terms of orders $2\pi i/k$ and $-4\pi^2/k^2$ in (65) give the following two relations:

$$a_+ - a_- = NN^{c_\pm} - N^{c_0}, \tag{66}$$

$$b_+ - b_- = a_0 + \frac{1}{2\pi i}N(NN^{c_\pm} - N^{c_0}) - \frac{1}{2}N(a_+ + a_-). \tag{67}$$

If L^{tw} is the same one component link as L, only with its framing twisted positively once, expanding the relation in figure 9 in powers of $2\pi i/k$ gives two additional relations:

$$a = a^{tw} + \frac{1}{2}(N^2 - 1) \tag{68}$$

$$b = b^{tw} + a^{tw}\frac{N^2-1}{2N} + \frac{N^2-1}{4\pi N}\left(iN^2 + \frac{\pi}{2}(N^2-1)\right). \tag{69}$$

Theorem 2. *The following assertions hold for links in* $\mathbf{R}^3$*:*

(1) *For a two component link* L, $\frac{1}{N(N^2-1)}a$ *is the linking number of its two components.*

(2) *For a single component knot L not necessarily with its preferred framing,* $\frac{a}{(N^2-1)}$ *is half its self linking number.*
(3) *For a single component knot L not necessarily with its preferred framing,* $\tilde{b} \stackrel{def}{=} \frac{1}{N(N^2-1)} Re \left(b - \frac{a^2}{2N} \right)$ *is framing independent, and is in fact equal to our* $\tilde{\mathcal{W}}_2(L)$*.*

All of these assertions are easy consequences of (66)-(69). For example:

Proof of assertion (3). To get the framing independence of $\tilde{b}$ just use (68) and (69) to express it in terms of a^{tw} and b^{tw}, and then notice that the resulting expression differs from that of $\tilde{b}^{tw}$ only by the real part of an imaginary number. To show that $\tilde{b}$ is equal to $\tilde{\mathcal{W}}_2(L)$ we just need to show that they satisfy the same skein relation. But for knots $L_\pm$ with their preferred framings $a_\pm = 0$ by assertion (2), and therefore using (67) one gets

$$\tilde{b}_+ - \tilde{b}_- = \frac{1}{N(N^2-1)} \mathrm{Re}\, (b_+ - b_-) = \frac{1}{N(N^2-1)} a_0$$

which by assertion (1) equals to the linking number of the two knots obtained by cutting $L_\pm$ as in figure 7. It is easy to check that $\tilde{b}(\text{the unknot}) = 1/24$.

□

The above theorem is in complete agreement with the results of this paper.

8. Perturbation theory beyond two loops

Following Witten [36], I will sketch here how we expect the perturbation theory of the Chern-Simons gauge theory to behave on a general three manifold and to higher order in $1/k$.

In [32, 33] Witten used very different techniques than those presented here to find a complete non-perturbative definition of the Chern-Simons gauge theory. The part of his solution that is relevant for making a comparison with the results proven here was reviewed in the previous section, and that comparison showed a complete agreement between the two approaches. The solution involves three subtleties that are hard to predict by just observing the definition of the theory in equation (8):

(1) *Links have to be framed.* According to Witten's solution $\mathcal{W}(M^3, \mathcal{X}, k)$ *cannot* be defined as it is for a bare link $\mathcal{X}$, but one also has to choose a framing for each of the components of $\mathcal{X}$ and only then $\mathcal{W}(M^3, \mathcal{X}, k)$ can be defined. Its definition will then depend on the choice of the framing in a prescribed manner. This point was explained in some more detail in the previous section.
(2) *Three-manifolds have to be framed.* According to Witten's solution $\mathcal{W}(M^3, \mathcal{X}, k)$ *cannot* be defined as it is for a bare three-manifold M^3, but one also has to choose a framing for M^3 — a choice up to homotopy of a trivialization of the tangent bundle of M^3, and only then $\mathcal{W}(M^3, \mathcal{X}, k)$ can be defined [34, 2]. (Actually, something a little less than a framing of M^3 is enough [34, 2]–it is enough, roughly speaking, to have a framing modulo torsion.) Its definition will then depend on the choice of the framing

in a prescribed manner. As we were working on a flat $\mathbf{R}^3$ we have not encountered this subtlety in this paper. We can consider this subtlety and the previous one as cases of a broken symmetry — as framings do not at all appear in (8) it is trivialy invariant under a change of framing and this symmetry is broken in Witten's solution.

(3) *Analyticity near $k=\infty$ is lost.*[2] Naively one sees that as $k \to -k$ in (8), $\mathcal{W}(M^3, \mathcal{X}, k)$ transforms to its complex conjugate. This property of $\mathcal{W}$ together with analyticity near $k=\infty$ means that we expect the even powers in the $1/k$ asymptotics of $\mathcal{W}$ to be real and the odd ones to be imaginary. This property is lost in Witten's solution as can clearly be seen from equations (61), (62), (63) and (65). We have avoided this difficulty in a not very satisfactory way by claiming to have calculated only the real part of $\mathcal{W}_2$.

All of the above mentioned subtleties seem not to appear in a naive Feynman-diagrammatic expansion of $\mathcal{W}$, and the purpose of this section is to show how these points probably do appear in perturbation theory after all.

Formally writing down the sums of Feynman diagrams that we expect to yield higher three-manifold and link invariants and translating them into finite dimensional integrals is routine and easy. It is also not hard to produce a *formal* invariance proof for these integrals as explained at the end of section 3, ignoring the analytical difficulties arising from the divergence of those integrals. We will see below how resolving these analytical difficulties is likely to explain the three subtleties listed above.

The origin of the above mentioned analytical difficulties is the singularities Greens's functions have near the diagonal. These get milder for higher order differential operators. This suggests trying to regularize (8) by adding higher order terms to the Lagrangian preserving as much symmetries as possible so as not to spoil the metric independence argument in section 3. (Physicists call such a procedure Pauli-Villars regularization.) The main ingredient of this argument is BRST invariance (lemma 3.1), and if we wish to preserve it we can only add terms that preserve gauge invariance. The only such term of order two is the square of the norm of the curvature of the connection A and therefore we will make the replacement

$$\mathcal{L}_{tot} \to \mathcal{L}_{regularized} \stackrel{\text{def}}{=} \mathcal{L}_{tot} + \epsilon||F_A||^2.$$

(In fact, to preserve the ellipticity of the quadratic part of $\mathcal{L}_{regularized}$ one also has to change the gauge-fixing term of $\mathcal{L}_{tot}$ and this forces changing Q slightly. Making those changes is easy and does not affect the rest of our reasoning, so we will ignore them.)

Let as now *pretend* that $\mathcal{L}_{regularized}$ gives rise to a finite perturbation theory. (Actually, this is true except for the role of a few low order subdiagrams.) What will remain of the invariance argument (17)?

Lemma 3.1 and lemma 3.3 will still hold because we have preserved gauge invariance, but as the additional term in $\mathcal{L}_{regularized}$ is metric dependent, lemma 3.2 will not be true any

[2]Some authors [20, 21] dispute this point, which is usually referred to as "the shift in k". It is very likely that in the context of the regularization suggested below no changes need to be made to the assertions in this paper.

more. Instead, the variation of $\mathcal{L}_{regularized}$ under $g^{ij} \to g^{ij} + \delta g^{ij}$ will be given by

$$\delta \mathcal{L}_{regularized} = Q\Lambda + \epsilon\delta||F_A||^2$$

and therefore in the notations of (17) we will have

$$\delta\langle\mathcal{O}\rangle_\epsilon = \epsilon\langle\mathcal{O}\delta||F_A||^2\rangle_\epsilon \tag{70}$$

where the subscript ϵ in $\langle\ \cdot\ \rangle_\epsilon$ is meant to remind us that we are taking expectation values with respect to a Lagrangian that depends on ϵ. Of course, equation (70) (and equations (71)-(74) as well) should be understood as an equality of perturbative asymptotic expansions, and its proof will be based on (17) as explained in section 3. If $\langle\mathcal{O}\rangle_\epsilon$ had a limit as $\epsilon \to 0$ and $\langle\mathcal{O}\delta||F_A||^2\rangle_\epsilon$ was bounded as $\epsilon \to 0$ we could have taken this limit and it would have been metric independent. One cannot expect this to be true. However, the divergences in $\langle\mathcal{O}\delta||F_A||^2\rangle_\epsilon$ for $\epsilon \to 0$ originate from a very definite type of contribution to the Feynman diagrams, and by considering how such divergences can originate, one can obtain results that are nearly as good as the naive results that would have held if there were no divergences. In explaining this, we will consider the basic case $\mathcal{O} = 1$.

It is convenient to consider only the connected Feynman diagrams and as is well known [29, 16, 22] the sum of those is just $\log\langle 1\rangle_\epsilon$. Divergences in Feynman diagrammatic contributions to $\log\langle 1\rangle_\epsilon$ and to

$$\delta\left(\log\langle 1\rangle_\epsilon\right) = \frac{\epsilon\langle\delta||F_A||^2\rangle_\epsilon}{\langle 1\rangle_\epsilon} \tag{71}$$

come from a region of integration in which all integration points are separated by distances of order ϵ. This means that the divergences can be expanded in terms of local differential geometric invariants – the metric, the curvature tensor, and its covariant derivatives. This expansion is analogous to the short time expansion of the heat kernel. The most general divergent terms are of the form

$$\log\langle 1\rangle_\epsilon = \frac{c_1}{\epsilon^3}V + \frac{c_2}{\epsilon}R + \text{finite terms} \tag{72}$$

and

$$\frac{\langle\delta||F_A||^2\rangle_\epsilon}{\langle 1\rangle_\epsilon} = \frac{c_1}{\epsilon^4}\delta V + \frac{c_2}{\epsilon^2}\delta R + \frac{c_3}{\epsilon}\delta C + \text{finite terms.} \tag{73}$$

Here c_1, c_2, and c_3 are constants (or more exactly functions of k only, which must be computed order by order in perturbation theory, but do not depend on the particular three manifold or metric). Also, V is the volume of M^3, R is the integral over M^3 of its scalar curvature, C is the Chern-Simons number associated with the Levi-Civita connection and δV, δR, δC are the variations of these quantities with respect to $g^{ij} \to g^{ij} + \delta g^{ij}$. The expansion (73) is determined by the following principles. (i) The terms on the right hand side must be closed one forms on the space of metrics (since the left hand side of the equation has this property.) (ii) The coefficients of these closed one forms must be local functionals of the metric. What has been written on the right hand side of equation (73) is the most general expression with these properties. The general principles do not determine c_1, c_2, and c_3, which from this point of view must simply be computed order by order in perturbation theory.

Equation (73) means that $\langle 1\rangle_\epsilon$ does not converge as $\epsilon \to 0$ to a topological invariant. Indeed its variation (71) not only does not vanish as $\epsilon \to 0$; it diverges in this limit. If, however, we define[3]

$$\mathcal{W}_{renormalized} = \langle 1\rangle_{renormalized} \stackrel{\text{def}}{=} \exp\lim_{\epsilon\to 0}\left(\log\langle 1\rangle_\epsilon - \frac{c_1}{\epsilon^3}V - \frac{c_2}{\epsilon}R - c_3 C\right) \tag{74}$$

then (72) shows that $\mathcal{W}_{renormalized}$ is finite while (70) and (73) shows that it is an invariant. Here we see where the framing of M^3 comes in — to define C we must first pick a trivialization of the tangent bundle and so the invariants that we have just produced depend on a choice of such a trivialization.

Notice that δC, in equation (73) does not depend on the choice of a framing, but C does. What is entering here is clearly a sort of local cohomology of the space of metrics. The local, closed one forms δV, δR appearing in (73) can be written as variations (exterior derivatives) of local functionals of the metric. But δC, though itself a local functional and a closed one form, cannot be written as the variation of a local functional. (If δC were itself not local, it could not arise in the intrinsic local evaluation of Feynman diagrams that leads to equation (73).)

Similarly, in the case of a non-empty link $\mathcal{X}$ we do not expect that the higher order Feynman diagrams will converge to link invariants, but instead we expect them to converge to something whose variation with respect to a deformation of $\mathcal{X}$ will be equal to some constant multiple of the variation of the total torsion of $\mathcal{X}$. (The torsion will enter just as the Chern-Simons number C entered in the above discussion.) The total torsion can then be subtracted out yielding link *invariants* at the price of having to introduce a framing for $\mathcal{X}$ — the total torsion can be defined only given such a framing. This agrees with the results of Witten and with the results in section 4.

Unfortunately, we were just *pretending* that the theory defined by $\mathcal{L}_{regularized}$ is finite. In fact, it is not. One can figure out how badly divergent the theories defined by $\mathcal{L}_{tot}$ and $\mathcal{L}_{regularized}$ are by taking a diagram with a specified number of vertices and arcs, measuring the total degree of singularity of the arcs and vertices, and subtracting the number of integrations that the vertices induce. The result, the so-called "superficial degree of divergence" Δ of a

[3] This is consistent with what is usually called renormalization - it just corresponds to adding $-\frac{c_1}{\epsilon^3}V - \frac{c_2}{\epsilon}R - c_3C$ to the original Lagrangian as the limit $\epsilon \to 0$ is taken. In fact, the above paragraph can be summarized by saying that these three terms are the only possible local BRST invariant additions to the Lagrangian which are of the right dimension. Notice that all three terms depend on the metric alone and not on the fields, and therefore the n-point functions of the theory are not renormalized and thus no care needs to be taken of the renormalization of lower order diagrams when considering the renormalization of a fixed order in perturbation theory.

diagram with E_B external gauge lines, E_F external ghost lines and L internal loops is

$$\Delta(\mathcal{L}_{tot}) = 3 - E_B - \frac{1}{2}E_F \qquad ; \qquad \Delta(\mathcal{L}_{regularized}) = 4 - L - E_B - E_F. \tag{75}$$

Clearly, the regularized theory is less divergent than the original one, but (75) shows that even in the regularized theory the diagrams with a small number of loops and external lines will be divergent and as these diagrams appear as subdiagrams in diagrams with higher complexity we cannot just ignore them. One can check that even if higher terms than $\epsilon||F_A||^2$ are added to $\mathcal{L}_{tot}$ and even when considering the reduction in the divergence that comes from gauge invariance[4] one loop diagrams with one, two , or three external legs will remain divergent in the resulting theory. Yet, we believe that the following is true:

Conjecture 1. *(Witten,* [36]*) The analysis in (*72*), (*73*), and (*74*) can be justified, and the resulting invariants $\mathcal{W}_{renormalized}$ coincide with the expansion in powers of $1/k$ of the results in* [32, 33].

One-loop diagrams in the Chern-Simons theory have been regularized using ζ-function regularization in [32] and using Pauli-Villars regularization in [1]. Both of these regularizations give partial results consistent with the above conjecture, but presently I don't know how to complete these results and use them to prove the conjecture to all orders.

9. Epilogue

There has been major developments in this subject in the 4 years since this paper was first distributed in a preprint form[5].

In [3, 4] Axelrod and Singer found an additional symmetry obeyed by the Chern-Simons path integral (3), and used it to prove that perturbation theory on bare three manifolds (subject to some additional conditions) is indeed finite to all orders, and that the resulting integrals are 'almost' independent of the choice of a metric, with the residual metric dependence being proportional to δC, as predicted in (73). A similar construction was given (but never published in detail) by Kontsevich [24], who also states theorem 1 (without proof) in [25].

In [6] I have noticed that the "lie-algebraic" part of Chern-Simons perturbation theory can be "divorced" from the "integral" part, showing (modulo analytical difficulties) that there is a perturbative invariant corresponding to each "weight system", and in [7] I have shown that these "weight systems" underlying Chern-Simons perturbation theory are the same as the weight systems underlying the theory of Vassiliev invariants[6], thus establishing a relationship

[4] $Q\bar{c} = \phi$, and therefore $\langle\phi(x)\phi(y)\rangle = 0$. This together with the structure of the ϕB propagator proves that the amputated two-point function is given by $\star^L d^L$ of a $(1,1)$-form whose convergence properties are by one degree better. For a similar example, see e.g. [10, pp. 299-300].

[5] Though it seems that no one had yet published an alternative proof of theorem 1.

[6] The inclusion {Chern-Simons weight systems} $\subset$ {Vassiliev weight systems} was already proven in [5, 6].

between the two domains. This same relationship was later observed and vastly generalized by Kontsevich [24, 25].

Appendix A. Some algebra

We include here the short computer routine that verifies few assertions that were made in sections 5 and 6. First, the routine itself. It is written in *Mathematica*™ — a symbolic mathematics language. For more information about this language see [37].

```
X[mu_] := {X1[mu],X2[mu],X3[mu]} ; Xd[mu_] := D[X[nu],nu] /. nu -> mu
X1[0]=X2[0]=X3[0]=0 ; w[mu_] := {w1[mu], w2[mu], w3[mu]}
ser[expr_] := Series[#,{var,0,ord}]& /@ expr
Xdtau    = ser[Xd[a tau]]       ;  wtau    = ser[w[a tau]]
Xdeps    = ser[Xd[b eta tau]]   ;  weps    = ser[w[b eta tau]]
Xdnegeps = ser[Xd[-b eta tau]] ;  wnegeps = ser[w[-b eta tau]]

t = lambda1 X[a tau] + lambda2 X[-b eta tau] + lambda3 X[b eta tau]
z1 = X[a tau] - t ; z2 = X[-b eta tau] - t ; z3 = X[b eta tau] - t
delta = IdentityMatrix[3]
S=Table[ser[Which[
   var==eta,{(z1[[i]]delta[[j,k]]+z2[[j]]delta[[k,i]]+z3[[k]]delta[[i,j]])
         /. lambda1 -> c2 eta ,
      z1[[i]] (Expand[z2[[j]]z3[[k]]]
         /. {lambda1^2 -> c5 eta^3 , lambda1 -> c4 eta^2})/eta^2},
   var==tau,{(z1[[i]]delta[[j,k]]+z2[[j]]delta[[k,i]]+z3[[k]]delta[[i,j]])/tau,
      z1[[i]]z2[[j]]z3[[k]]/tau^3}]],
   {i,3},{j,3},{k,3}]

sign = (Signature /@ (perm = Permutations[{1,2,3}]))
eps[f_]:=Sum[sign[[p]]sign[[q]](f@@Join[perm[[p]],perm[[q]]]),{p,6},{q,6}]
six[f_]:=eps[f[#3,#1,#4,#6,#2,#5]&] + eps[f[#6,#1,#4,#2,#3,#5]&]

e[type_]  :=six[S[[#1,#2,#3,type]]Xdtau[[#4]]Xdnegeps[[#5]]Xdeps[[#6]]&] /. b->1
e12[type_]:=six[S[[#1,#2,#3,type]]Xdtau[[#4]]wnegeps[[#5]]Xdeps[[#6]]&]
e69[type_]:=eps[S[[#3,#5,#6,type]]wtau[[#1]]Xdtau[[#2]]Xdeps[[#4]]&]
e78[type_]:=eps[S[[#6,#3,#5,type]]Xdnegeps[[#1]]wnegeps[[#2]]Xdtau[[#4]]&]
de[type_] :=Sum[e12[type] + e69[type] + e78[type] , {b,-1,1,2}]
```

The first paragraph of the routine defines X, $\dot{X}$, ω, and their expansions with respect to the externally defined variable `var` to order `ord` at the points $\alpha\tau$, $-\epsilon = -\beta\eta\tau$, and $\epsilon = \beta\eta\tau$.

The second paragraph defines `S[[i,j,k,1 or 2]]` to be $S^{ijk}_{1\text{ or }2}$ expanded with respect to the relevant variable. `S` is defined *differently* for `var=eta` then for `var=tau` — if `var=eta` then (33) and (34) mean that in S_1 one can make the replacement `lambda1 -> c2 eta` while (35) means that in S_2 the replacement `{lambda1^2 -> c5 eta^3 , lambda1 -> c4 eta^2}` can be made. It is easy to see that after the latter replacement has been made the expansion for S_2 will begin at η^2, and this justifies dividing it by η^2 and expanding everything to an order two less than is mentioned in sections 5 and 6. If `var=tau` the expansions for `z1`, `z2`, and `z3` begin at τ, and thus the definitions `S[[i,j,k,1]]`$= S^{ijk}_1/\tau$ and `S[[i,j,k,2]]`$= S^{ijk}_2/\tau^3$. This allows us to expand `S[[i,j,k,1]]` (`S[[i,j,k,2]]`) to an order lower by one (three) than the order required for S_1^{ijk} (S_2^{ijk}).

The third paragraph contains the routines that do the $\epsilon_{...}$ and the $6_{......}$ contractions, and the last paragraph defines the relevant diagrams.

We now include a *Mathematica*™ session produced using the above routine, for which I have chosen the not very imaginative name "file".

```
Mathematica (sun4) 1.2 (November 6, 1989) [With pre-loaded data]
by S. Wolfram, D. Grayson, R. Maeder, H. Cejtin,
   S. Omohundro, D. Ballman and J. Keiper
with I. Rivin and D. Withoff
Copyright 1988,1989 Wolfram Research Inc.

In[1]:= var=eta; ord=1; << file

In[2]:= {e[1] , e[2]} /. {a->1 , eta->0}

Out[2]= {0, 0}

In[3]:= {de[1] , de[2]} /. a->1

               2        2
Out[3]= {O[eta] , O[eta] }

In[4]:= var=tau; ord=1; << file

In[5]:= {Sum[e[1],{a,-1,1,2}] , Sum[e[2],{a,-1,1,2}]}

               2        2
Out[5]= {O[tau] , O[tau] }

In[6]:= var=tau; ord=2; << file

In[7]:= {Sum[de[1],{a,-1,1,2}] , Sum[de[2],{a,-1,1,2}]}

               3        3
Out[7]= {O[tau] , O[tau] }
```

`Out[2]` and `Out[5]` prove equations (38) and (40), while `Out[3]` and `Out[5]` prove the assertions at the end of the invariance proof in section 6. This concludes the proof of the main theorem of this paper.

Comment: Obtaining these eight expansions takes few hours of CPU time on a 1989 workstation.

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