NON-ASSOCIATIVE TANGLES

DROR BAR-NATAN

To appear in the Georgia International Topology Conference proceedings

This edition: June 7, 1995; First edition: Dec. 20, 1993.

Abstract. Following Drinfel'd, Kontsevich and Piunikhin, we study the iterated integral expression for the holonomy of the formal Knizhnik-Zamolodchikov connection, finding that by introducing *non-associative tangles*, tangles whose strands are grouped in some particular way, the computation of these integrals can be reduced to the computation of just two holonomies R and Φ. A category **NAT** of non-associative tangles is thus introduced, and the 'kernel' of the natural forgetful functor $\mathbf{NAT} \to \{\text{tangles}\}$ is analyzed and is shown to be generated by relations reminiscent of the relations defining a quasitriangular quasi-Hopf algebra. It follows that any solution of these relations can be used to produce tangle (and link) invariants. An iterative combinatorial/algebraic procedure for finding such solutions is described, and thus we have a first completely combinatorial construction of a universal Vassiliev invariant.

Contents

This work was supported by NSF grant DMS-92-03382.

This paper and the computer programs used in it are available via anonymous file transfer from `ftp.math.harvard.edu`, user name `ftp`, subdirectory `dror`. Read the file `README` first. For easier access, point your WWW browser at

`http://www.math.harvard.edu/HTML/Individuals/Dror_Bar-Natan.html`

1. Introduction

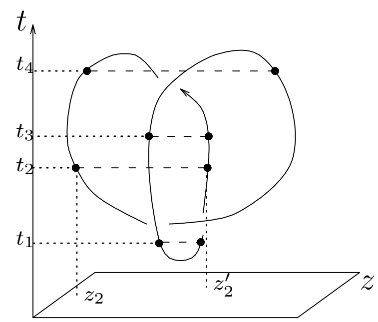
1.1. Vassiliev invariants and the Kontsevich integrals. A Vassiliev invariant of type $m \in \mathbf{N}$ (see [27, 28, 6]) can be viewed [2] as a knot (or link) invariant whose $m+1$st derivative (in some reasonable sense) vanishes, and thus whose mth derivative is (in some sense) a constant. Upon further thought, this constant turns out to actually be a system of constants indexed by some kind of diagrams, subject to some relations (the "Birman-Lin relations"; the $4T$ and framing independence relations). Such a system of constants is called a *weight system*, and for some time after the introduction of these notions it was not clear whether or not these $4T$ and framing independence relations are the *only* relations satisfied by an mth order derivative of a type m invariant. Then in the winter of 1991/2 Kontsevich [14] found that if one introduces a certain formal generalization Ω of the Knizhnik-Zamolodchikov connection, writes a certain integral formula for the holonomy of Ω, generalizes a bit and makes a small correction, one finds an integral formula ("the Kontsevich integral") for a type m Vassiliev invariant in terms of a general weight system, thus resolving the above mentioned difficulty.

Somewhat more specifically, using the notation and definitions of section 4 of [2], the Kontsevich integral formula[1] for a universal $\mathcal{A}^r$-valued Vassiliev invariant $\tilde{Z}(K)$ has the following form[2]:

$$(1)\ Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{\min}<t_1<\ldots<t_m<t_{\max}} \sum_{\substack{\text{applicable}\\ \text{pairings}\\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \in \mathcal{A}^r,$$

$$\tilde{Z}(K) = Z(K) \Big/ (Z(\infty))^{\frac{c}{2}}\ .$$

The standard mnemonic for (1) is the figure

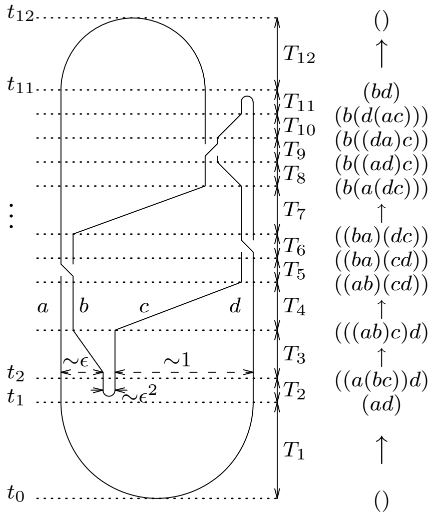


- Sum over all possible choices for the horizontal chords,
- pull the form $d\log(z - z')$ back to $\mathbf{R}_t$ using the ends of each chord,
- wedge together the resulting 1-forms,
- and integrate over $t_1, \ldots, t_m$,
- taking the chord diagram D_P defined by the horizontal chords and the knot as a coefficient.

1.2. A better knot projection. The problem with this lovely integral expression is that, as is, nobody seems to know how to compute it for any knot K. However, as it is known that (1) is independent of the particular embedding chosen for K, it might be that (1) becomes more manageable when the embedding of K is chosen wisely. For example, if ϵ is some very small number, a wiser choice of embedding for the knot displayed in the previous figure is:

[1] A detailed understanding of the Kontsevich formula is not necessary for understanding this paper. We advise readers unfamiliar with the formula to read the rest of the introduction without too much worry about the details of the formula; these will not be used later in the paper.

[2] Our normalization convention is different than in [2]; with c the number of critical points in a specific embedding of K, we replace $\frac{c}{2} - 1$ by $\frac{c}{2}$. See [2, problem 4.9].



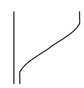
- In all marked time slices, $t_0, \dots, t_{12}$, all distances between various strands of the knot are approximately equal to some power of ϵ. (At time t_1, say, the distance between the two strands a and d is $\sim 1 + \epsilon + \epsilon^2 \sim 1$).
- Furthermore, pretending that strands are elements in some non-commutative non-associative algebra, in each of the marked time slices the order and distance between the strands gives rise to a complete choice of how to multiply the strands. At time t_2, say, the corresponding 'product' is $((a(bc))d)$, as marked in the right most column of the figure.
- In each of the time intervals $T_1, \dots, T_{12}$ only one change occurs to the 'product' corresponding to the strands, and only three types of changes occur:
  - (i) *Pair creation (annihilation)*, in which a pair of *neighboring strands* is created (or annihilated). *Neighboring strands* are strands for which the distance between them is smaller than the distance between them and any other strand. (intervals T_1, T_2, T_{11}, and T_{12}).
  - (ii) *Braiding morphism*, in which two neighboring strands are braided. (intervals T_5, T_6, and T_9).
  - (iii) *Associativity morphism*, in which the associative law is applied once. (intervals T_3, T_4, T_7, T_8, and T_{10}).

1.3. Why is this a better choice? Because the Kontsevich-KZ integrals are multiplicative in a sense explained in [2], and thus it is sufficient to understand the Kontsevich-KZ integrals associated with any of the time intervals $T_1, \dots, T_{12}$. Each of these is easy (or at least *easier*), depending on the interval's type:

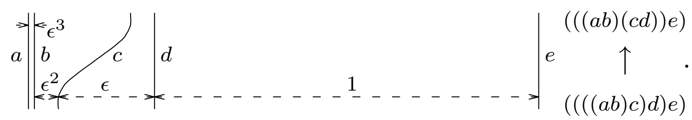
(i) *Pair creation (annihilation)*: In this case diagrams that have a chord connecting two strands other than the two newly created do not contribute because on parallel strands $(dz - dz')/(z - z')$ vanishes. The contribution from diagrams in which there is a chord connecting a newly created strand with an 'old' strand is negligible ($\sim O(\epsilon)$) because the domain of integration here would be very small relative to the size of the integrand, and diagrams that only have chords connecting the two strands in the newly created pair vanish because of the framing independence relation. So the only contribution from a pair creation (or annihilation) is the *skeleton* diagram, the diagram having no chords at all.

(ii) *Braiding morphism*: Here we may ignore all strands other than the two being braided, as they are all parallel and too far to interfere with the important two. And the contribution R from diagrams in which all chords connect the two strands being braided is easily computable — it is simply the holonomy of the 2 variable KZ connection, which is an Abelian connection.

(iii) *Associativity morphism*: Here something slightly more complicated happens. The basic holonomy to compute here is that of

(2)  .

Compute this holonomy once and for all and call the result Φ. (the computation might be hard — the connection here is not Abelian. But we need to do it only once[3]). More typically, we will meet an associativity morphism like

(3) 

.

Luckily, the above holonomy is easily computable from Φ — clearly, as all other strands are parallel, the only non-vanishing contributions are from diagrams all of whose chords begin (or end) on the strand labeled c. Clearly, chords whose other end is on e give a negligible contribution, as e is too far to matter. Also, *in as much as c is concerned*, strands a and b are equivalent — c is always too far from them to tell them apart. Therefore, chords ending on a give (almost) the same contribution as chords ending on b, and (almost) the same contribution as in (2). Thus the integral corresponding to (3) is computable from Φ.

With the above considerations in mind, it seems natural to replace tangles by *non-associative tangles*, tangles that come equipped with further 'associativity' information, as in our 'wiser' representation of the trefoil knot. The rest of this paper is organized as follows: In section 2 we realize this idea by introducing the category **NAT** of non-associative tangles and by listing the relations generating the 'kernel' of the obvious forgetful functor **For** : **NAT** $\to$ **PT**, where **PT** is a minor variation of the usual category of tangles. In section 3 we use an arbitrary solution (R, Φ) to the relations generating the 'kernel' of **For** to construct a functor on **NAT** with values in some category of chord diagrams, and use that functor to construct a universal Vassiliev tangle invariant. The following two sections are devoted to the construction of such a solution, with section 4 giving some necessary cohomological preliminaries

[3] The magnanimous reader will be forgiving to the fact that this holonomy diverges as $\epsilon \to 0$. There is an easy way to fix it but it would take us too far aside to describe it. Drinfel'd's Φ, as described in [8], is precisely the result of this fix.

and section 5 describing the actual solution method. Section 6 contains some remarks, and in section 7 we show how to use mathematica to compute R and Φ and how to compute the corresponding knot invariant using the methods of the previous sections.

The idea for writing this paper came from reading Piunikhin's [24], in which a universal Vassiliev link invariant is constructed in terms of an appropriate pair (R, Φ), constructed using the Knizhnik-Zamolodchikov equation. Our additions are that we make the construction somewhat clearer and more general (as it applies to tangles and not just to links), and that we give an explicit combinatorial construction of the necessary pair (R, Φ).

During the preparation of this paper and after the research for it was completed, I received other papers dealing with closely related subjects: T. Q. T. Le and J. Murakami wrote four papers [15, 16, 17, 18] in which they cover essentially the same grounds as sections 2, 3, and 4 of this paper, and then proceed to a different direction and discuss the rationality of the Kontsevich integral and its relation with the HOMFLY and Kauffman polynomials and with the theory of multiple ζ-numbers. S. Shnider and S. Sternberg [26] wrote a book with an extensive discussion of quasi-Hopf algebras, mostly as deformations of universal enveloping algebras of Lie algebras. P. Cartier [7] formalized the same ideas as in our sections 2 and 3 in the language of monoidal categories. C. Kassel wrote an account (with many additional results) of Cartier's paper in a chapter of his forthcoming book [12]. N. Bergeron [5] wrote a short paper announcing our joint work on section 4 here. S. Piunikhin [25] made some low order computations for some specific knots.

One comment has to be made, though. Most of the results in this paper, as well as in most other papers on this subject, are contained, either explicitly or implicitly, in the two seminal papers by Drinfel'd [8, 9].

1.4. Acknowledgement. I would like to thank J. Birman, R. Bott, S. Garoufalidis, D. Kazhdan, S. Piunikhin, B. Sanderson, J. D. Stasheff, S. Sternberg, D. P. Thurston, S. Willerton and H. L. Wolfgang for their many useful comments. Especially I would like to thank V. G. Drinfel'd for making a crucial remark in his short visit to Boston in September 1993, N. Bergeron for for jointly working with me on the results of section 4, and A. Stoimenow for finding a gap in an earlier version of this paper. This paper is essentially an expanded version of talks I gave in in conferences in Georgia and in Warwick in the summer of 1993. I would like to thank the organizers of these conferences for inviting me to attend and giving me the possibility to speak.

2. The category of non-associative tangles

2.1. The category.

Definition 2.1. A *parenthesized string* is a string (of letters in some alphabet) together with a *balanced* and *complete* choice pairs of parentheses enclosing parts of it.

The word *balanced* means that the number of 'open parenthesis' symbols "(" is always bigger than or equal to the number of 'close parenthesis' symbols ")" in any initial segment of the string, with equality if and only if the initial segment is the whole string. The word *complete* means that every pair of parentheses encloses either a pair of letters, or a letter and a substring enclosed in parentheses, or a pair of substrings enclosed in parentheses, and that the full string is enclosed in a pair of parentheses. As an exception, we will also consider the empty and single-character strings enclosed in a pair of parentheses as a parenthesized strings, and so the following is a complete list of all possible parenthesized strings of length at most 3 on an alphabet with a single letter $\uparrow$: $\{(), (\uparrow), (\uparrow\uparrow), ((\uparrow\uparrow)\uparrow), (\uparrow(\uparrow\uparrow))\}$.

Definition 2.2. If A and B are parenthesized strings and $\star$ is a letter appearing exactly once in A, there is an obvious *substitution operation* which we will denote by $A/\{\star \to B\}$. For example, $((ab)\star)/\{\star \to ((cd)(ef))\} = ((ab)((cd)(ef)))$. Similarly, $(ab)/\{a \to (cd), b \to (ef)\} = ((cd)(ef))$.

Definition 2.3. If A and B are parenthesized strings, the substitution $(ab)/\{a \to A, b \to B\}$ is called *the product* of A and B, and is simply denoted by (AB).

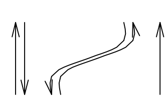
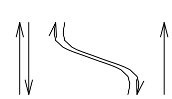
Definition 2.4. A *parenthesized $\updownarrow$-string* is a parenthesized string on the alphabet $\{\uparrow, \downarrow\}$. Often we will represent such a string by a collection of vertical arrows along a horizontal line, whose distances from each other reflect the parenthesization in the obvious way:

$$((\uparrow\downarrow)(\downarrow\uparrow)) \quad \longleftrightarrow \quad \uparrow\downarrow \quad \downarrow\uparrow$$

A *parenthesized $\updownarrow\star$-string* is a parenthesized string on the alphabet $\{\uparrow, \downarrow, \star\}$, in which the letter $\star$ appears exactly once.

Definition 2.5. (See also [1, 7, 18, 12]) Let **NAT** (**N**on-**A**ssociative **T**angles) be the category whose objects are parenthesized $\updownarrow$-strings, and whose morphisms are freely generated by the following morphisms: (for a precise definition of "the free category generated by a graph", consult e.g. Mac Lane [21])

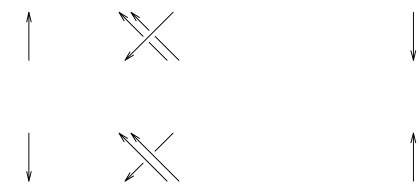
(G1) *Associativity morphisms:* A morphism $S^{|\swarrow|}_{A,B,C}$ and a morphism $S^{|\nwarrow|}_{A,B,C}$ for each parenthesized $\updownarrow\star$-string S, and each triple (A, B, C) of parenthesized $\updownarrow$-strings. The domains of these morphisms are $S/\{\star \to ((AB)C)\}$ and $S/\{\star \to ((A(BC))\}$ and their targets are $S/\{\star \to ((A(BC))\}$ and $S/\{\star \to ((AB)C)\}$ respectively. Graphically, we represent these morphisms as follows:

$(\star)^{|\swarrow|}_{(\uparrow\downarrow),(\downarrow\uparrow),(\uparrow)} \quad \longleftrightarrow$  ; $(\star)^{|\nwarrow|}_{(\uparrow\downarrow),(\uparrow\downarrow),(\uparrow)} \quad \longleftrightarrow$ 

(G2) *Braiding morphisms:* A morphism $S^{\times}_{A,B}$ and a morphism $S^{\times}_{A,B}$ for each parenthesized $\updownarrow\star$-string S, and each pair A, B of parenthesized $\updownarrow$-strings. The domain of these morphisms is $S/\{\star \to (AB)\}$, and the target is $S/\{\star \to (BA)\}$. Graphically, we represent these morphisms as follows:

$$((\uparrow\star)\downarrow)^{\times}_{(\downarrow),(\uparrow\uparrow)} \longleftrightarrow$$

$$((\downarrow\star)\uparrow)^{\times}_{(\downarrow),(\uparrow\uparrow)} \longleftrightarrow$$

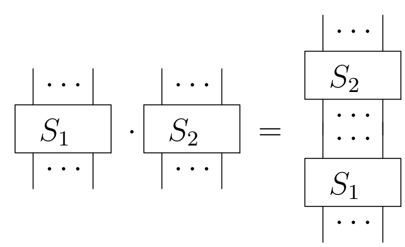


(G3) *Pair creations (annihilations):* A morphism $S^{\cup}$ for each parenthesized $\updownarrow\star$-string S, whose domain is $S/\{\star \to ()\}$ and whose target is $S/\{\star \to (\downarrow\uparrow)\}$. Graphically we will represent such a morphism by a diagram as follows:

$$((\uparrow\downarrow)\star)^{\cup} \longleftrightarrow$$

Similarly, we define morphisms $S^{\cup}$, $S^{\cap}$, and $S^{\cap}$, and regard them as generators in **NAT**. $S^{\cap}$, for example, is a morphism whose domain is $S/\{\star \to (\uparrow\downarrow)\}$ and whose target is $S/\{\star \to ()\}$.

Graphically, we will represent the composition $S_1 \cdot S_2$ of any two composable morphisms S_1 and S_2 by stacking the graphical representation of S_2 on top of that of S_1, as shown on the right.

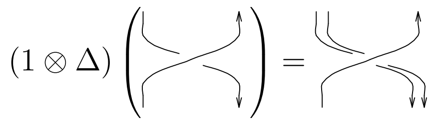


Definition 2.6. Let the categories **NAP** (**N**on-**A**ssociative **P**roducts) and **NAB** (**N**on-**A**ssociative **B**raids) have the same objects as **NAT**. The morphisms of **NAP** are freely generated by the associativity morphisms (G1). The morphisms of **NAB** are freely generated by the associativity morphisms (G1) and the braiding morphisms (G2).

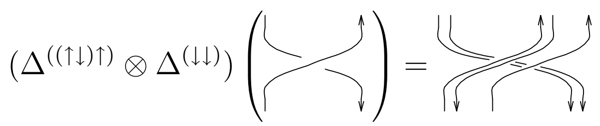
There are a few operations that take an arbitrary morphism M in **NAB** (or **NAP**) into other morphisms in **NAB** (or **NAP**). Let us mention two of these:

(1) Any of the strands of M can be *doubled* — get replaced by a pair of neighboring parallel strands. If M has n strands, we will denote the operation of doubling

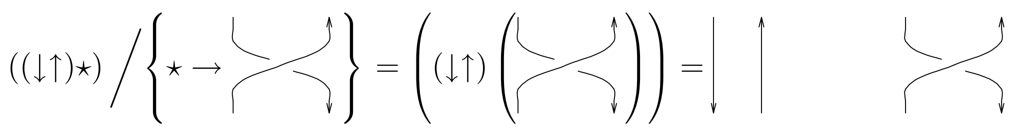
the kth (counting at the bottom) strand by $1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1$, where the Δ is in the kth slot:



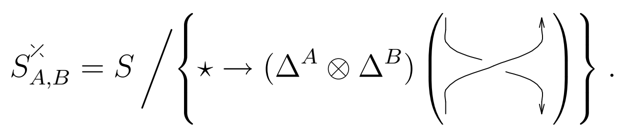
There are also some more general *cabling* operations in which more than one of the strands is affected, and the affected strands are replaced by an arbitrary number of ("parenthesized") strands rather than just doubled. For example, if M has two strands and A and B are parenthesized $\updownarrow$-strings, $(\Delta^A \otimes \Delta^B)(M)$ is obtained from M by replacing its first strand by a "bundle" of strands specified by A and replacing its second strand by the bundle specified by B:



(2) Vertical strands can be added to the left and to the right of M. In general, if S is a parenthesized $\updownarrow\star$-string, $S/\{\star \to M\}$ ("M *extended* by S") will be the morphism in **NAB** obtained by replacing the symbol $\star$ in S by M:



Notice that the generating morphisms of **NAB** are all obtained from the four 'basic' morphisms , ,  and  by the two operations above. For example,



2.2. The forgetful functor NAT → PT. Recall that the category **T** of tangles (see e.g. [23]) is the category whose objects are (non-parenthesized) strings of ↑ and ↓ signs, and whose morphisms are *tangles* — disjoint unions of knotted oriented strands (regarded up to ambient isotopy) whose "ends" are the domain and target strings. For a typical example, see figure 1. Similarly, **FT** is the category of *framed tangles* — disjoint unions of knotted oriented strands regarded only up to regular isotopy [13]. The categories **PT** and **PFT** are defined in exactly the same way, only with parenthesized $\updownarrow$-strings as their objects.

There are obvious forgetful functors **For** : **NAT** → **PFT** and $\widetilde{\textbf{For}}$: **NAT** → **PT** whose action on a parenthesized string is simply to do nothing, and whose action on morphisms of **NAT** is evident from their graphical representations in definition 2.5.

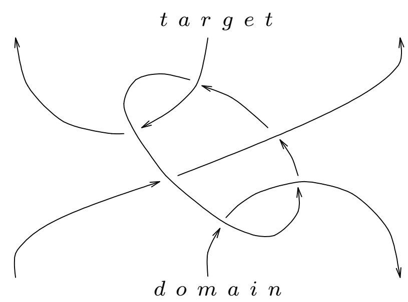
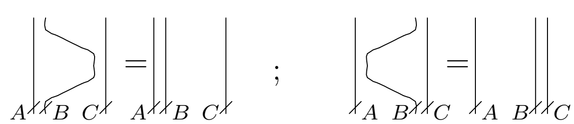


Figure 1. A tangle whose domain is $(\uparrow\uparrow\downarrow)$ and whose target is $(\uparrow\downarrow\uparrow)$.

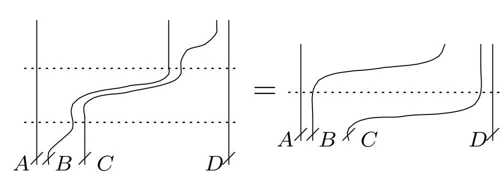
We will shortly prove that the "kernel" of **For** is generated by the relations (R1)–(R10) below, and that the kernel of $\widetilde{\mathbf{For}}$ is generated by the same relations, with the addition of a single relation (R11).

- *Associativity relations:*
  - (R1) $S^{|\nearrow|}_{A,B,C}$ is equal to an identity morphism if any of A, B, or C is the empty string. Similarly for $S^{|\searrow|}_{A,B,C}$.
  - (R2) $S^{|\searrow|}_{A,B,C}$ is a two sided inverse of $S^{|\nearrow|}_{A,B,C}$:

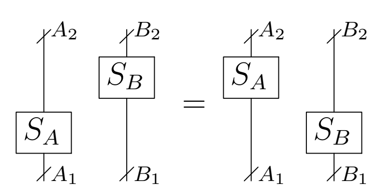


Notice that we've drawn only the "active" part of each relation; strands in the above figure are actually bundles of strands (determined by A, B, or C), and we don't bother to indicate the other "far away" strands in $S^{|\searrow|}$ and $S^{|\nearrow|}$.

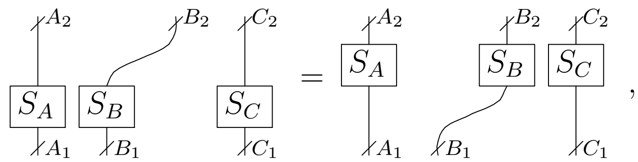
  - (R3) *The pentagon:* (named after the pentagon relation of category theory)

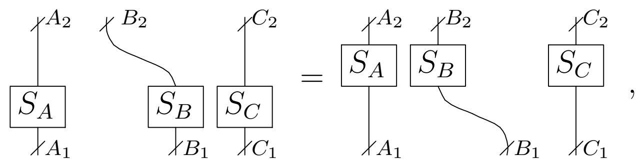


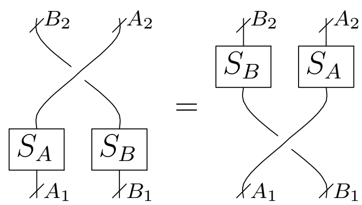
- *Locality relations:*
  - (R4) *Locality in space:* (for any pair S_A and S_B of morphisms)



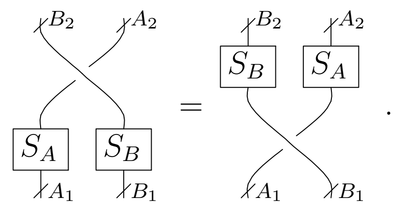
(R5) *Locality in scale:* for any morphisms S_A, S_B, and S_C,







and

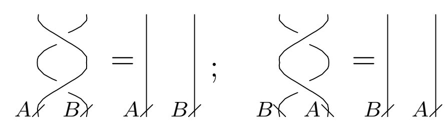


A locality relation (of either kind) will be called *elementary* if one of S_A and S_B (and S_C, when relevant) is one of the generators of **NAB**, while the other one (or two) are the identity morphisms.

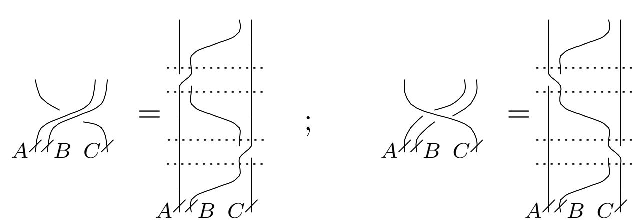
- *Braiding relations:*

(R6) $S^{\times}_{A,B}$ is equal to an identity morphism if A or B is the empty string. Similarly for $S^{\times}_{A,B}$.

(R7) $S^{\times}_{B,A}$ is a two sided inverse of $S^{\times}_{A,B}$:

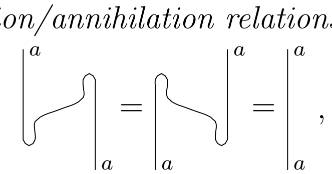


(R8) *The hexagons:* (named after the hexagon relations of category theory)

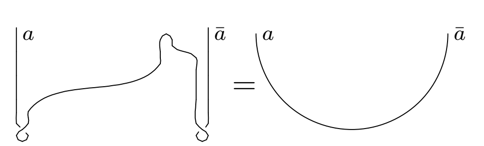


- *Creation/annihilation relations:*

(R9)

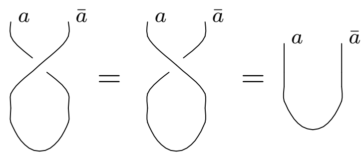


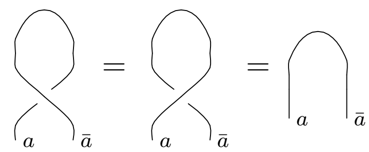
(R10)



where $a \in \{\uparrow, \downarrow\}$ is an arrow and $\bar{a}$ is its opposite.

(R11)

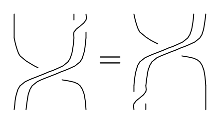


and


.

Theorem 1. *(See also* [1, 7, 18, 12]*) The kernel of* ***For*** *is indeed generated by relations (R1)–(R10), while the kernel of* $\widetilde{\textbf{\textit{For}}}$ *is generated by (R1)–(R11).*

Proof. The proof is basically an amalgamation of the Mac Lane coherence theorem [21] and the standard facts about "Reidemeister" moves for tangles as in [23]. Roughly speaking, the Mac Lane coherence theorem allows us to ignore 'associativity information' in non-associative tangles and in the relations (R1)–(R11), and when associativity information is ignored in (R1)–(R11), what remains is the usual "Reidemeister" moves for tangles. For more details see e.g. [1]. Let us just comment that the third Reidemeister move is a simple consequence of our locality-in-scale axiom:


. □

Let **PB** (**P**arenthesized **B**raids) be the image via **For** of **NAB**. An additional easy result is that the kernel of **For** : **NAB** → **PB** is generated by (R1)–(R8).

3. A universal Vassiliev invariant

Theorem 1 suggests the following method for constructing invariants of parenthesized framed tangles:

- Construct a functor $\mathbf{Z^{PFT}} : \mathbf{NAT} \to \mathbf{C}$, where $\mathbf{C}$ is some arbitrary 'target category'. This can be done by simply stating the values of $\mathbf{Z^{PFT}}$ on the generators of **NAT**.
- Check that $\mathbf{Z^{PFT}}$ respects relations (R1)–(R10).
- If it does, it descends to a functor (denoted by the same symbol) $\mathbf{Z^{PFT}} : \mathbf{PFT} \to \mathbf{C}$, which is the required invariant of parenthesized framed tangles.

Checking in addition the relation (R11), we get a parenthesized tangle invariant $\mathbf{Z^{PT}} : \mathbf{PT} \to \mathbf{C}$.

In this section, we will carry out this procedure in the case where $\mathbf{C} = \mathbf{AT}$ is a certain category of trivalent graphs, and the idea for the construction of $\mathbf{Z^{PFT}}$ is borrowed from the considerations of section 1.3.

Recall that in [2, section 1] we've defined $\mathcal{D}^a$ to be the collection of all diagrams made of directed solid lines ("strands"), undirected dashed lines, trivalent vertices in which a dashed line ends on a strand, oriented trivalent vertices in which three dashed lines end, and univalent vertices in which a line (dashed or solid) begins or ends. After picking some characteristic 0 ground field $\mathbf{F}$ we've then set

$$\mathcal{A}^a = \operatorname{span}(\mathcal{D}^a) \,/\, \{\text{anti-symmetric vertices, } STU \text{ and } IHX \text{ relations}\}\,,$$

where the anti-symmetry, STU, and IHX relations were also discussed in [2]. Notice that these three relations do not involve univalent vertices at all, and so $\mathcal{A}^a$ is the direct sum of its "end-homogeneous" components $\mathcal{A}^a_{ij} = \operatorname{span}\left(\mathcal{D}^a_{ij}\right)/\{\text{same relations}\}$, where $\mathcal{D}^a_{ij}$ is the subset of $\mathcal{D}^a$ containing all diagrams with i univalent vertices in which a strand (solid line) begins (and thus i univalent vertices in which a strand ends) and j univalent vertices in which a dashed line begins (or ends). Notice also that when $j = 0$, the number of trivalent vertices in a diagram has to be even, and so $\mathcal{A}^a_{i0}$ can be graded by half the number of trivalent vertices in a diagram. Let $\bar{\mathcal{A}}^a_{i0}$ be the graded completion of $\mathcal{A}^a_{i0}$. Finally, let the *skeleton* of a diagram $D \in \mathcal{D}^a_{*0}$ be the diagram obtained from D by removing all the dashed lines in it. It is always a diagram of degree 0. We will say that a diagram is a skeleton if it is equal to its own skeleton.

Definition 3.1. Let $\mathbf{AT}$ be the graded category[4] whose objects are (non parenthesized) strings of $\uparrow$ and $\downarrow$ arrows (same as in $\mathbf{T}$). The morphisms of $\mathbf{AT}$ are pairs (S, L) where S is a skeleton diagram whose univalent vertices are in a bijective direction-preserving correspondence with the $\uparrow$ and $\downarrow$ arrows in the domain and target objects, and L is an element of $\bar{\mathcal{A}}^a_{*0}$ that can be presented as a series of diagrams whose skeleton is S. For a typical example, see figure 2.

Let $\mathbf{AB}$ be the category whose objects are the same objects as those of $\mathbf{AT}$ and whose morphisms are those morphisms of $\mathbf{AT}$ in which all the strands connect an arrow ($\uparrow$ or $\downarrow$) in the domain object to an arrow in the target (pointing to the same direction).

On $\mathbf{AB}$ it is possible to define operations similar to those defined on $\mathbf{NAB}$. Let D be some diagram representing a morphism of $\mathbf{AB}$.

(1) Any of the strands of D can be doubled. If l is the kth such line and it is connected to n dashed lines, $(1 \otimes \cdots \otimes \overset{k}{\Delta} \otimes \cdots \otimes 1)(D)$ is the sum of diagrams obtained by doubling the kth strand in the skeleton of D and summing over the 2^n possible ways of 'lifting' D to the new skeleton thus obtained. (Compare with the definition of the Adams operations ψ^q in [2]). In general, it is clear how to define the cabling operation $\Delta^A \otimes \Delta^B \otimes \cdots$ on skeletons. If D_0 is the skeleton of D, let $(\Delta^A \otimes \Delta^B \otimes \cdots)(D)$ be the sum of all possible liftings of D

[4] A *graded category* is a category for which $\operatorname{mor}(A, B)$ is a graded Abelian group for every objects A and B, and for which the composition maps $\operatorname{mor}(A, B) \times \operatorname{mor}(B, C) \to \operatorname{mor}(A, C)$ are graded bilinear.

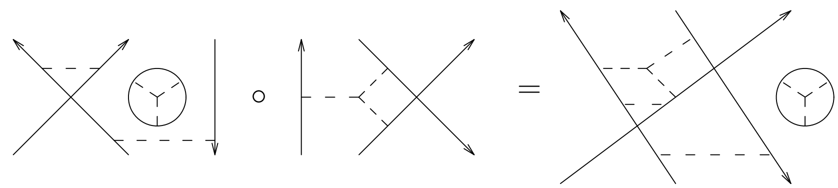
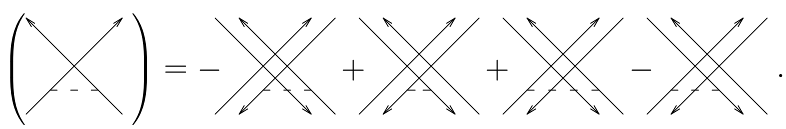
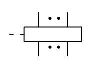
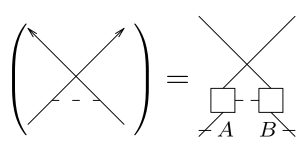


Figure 2. The composition of a degree 4 morphism in $\mathrm{mor}(\uparrow\uparrow\downarrow,\uparrow\uparrow\downarrow)$ with a degree 2 morphism in $\mathrm{mor}(\uparrow\uparrow\downarrow,\uparrow\downarrow\uparrow)$ is a degree 6 morphism in $\mathrm{mor}(\uparrow\uparrow\downarrow,\uparrow\downarrow\uparrow)$. The general composition law of **AT** is obtained from the one indicated here by bilinear extension. Notice that we compose from the bottom to the top, and that in these diagrams apparent quadrivalent vertices are to be ignored, and trivalent vertices are always oriented counterclockwise.

to $(\Delta^A\otimes\Delta^B\cdots)(D_0)$, multiplying each lifting by -1 raised to the number of vertices that change their orientation when lifted:

$(\Delta^{\uparrow\downarrow}\otimes\Delta^{\downarrow\uparrow})$ 

We will use the notation  as shorthand for summations as in the above equation. With this notation, we have:

$(\Delta^A\otimes\Delta^B)$ 

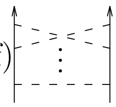
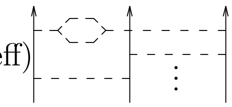
(2) There is no problem with defining the extension operation $S/\{\star\to D\}$ in the same way as in the case of **NAB**.
(3) ‘Strand removal’ operations are defined as follows: If there are no dashed lines ending on the ith strand of D, $s_i(D)$ is just D with its ith strand removed. Otherwise $s_i(D)=0$.

Exercise 3.2. Let **AP** be the subcategory of **AB** in which all strands are vertical. Show that the above three operations restrict to **AP**, and their restrictions are self-functors **AP** $\to$ **AP**. Formulate and prove the corresponding multiplicativity property of these operations regarded as self-maps (not functors!) **AB** $\to$ **AB**.

The following definition will be used in the next section:

Definition 3.3. A morphism D in **AP** is called a *perturbation of the identity* if its degree 0 piece is the identity 1. We say that D is *non-degenerate* if $D_{>0}$ is in the kernels of all the s_i's, where $D_{>0}$ is D with its degree 0 piece removed. That is, D is non-degenerate if every strand in $D_{>0}$ touches some dashed line in D.

3.1. An ansatz for $\mathbf{Z}^{\mathbf{PB}}$. Let us now construct a functor $\mathbf{Z}^{\mathbf{PB}} : \mathbf{NAB} \to \mathbf{AB}$. Pick an invertible non-degenerate automorphism R of the object $(\uparrow\uparrow)$ of $\mathbf{AP}$ and an invertible non-degenerate automorphism Φ of the object $(\uparrow\uparrow\uparrow)$, and assume that both R and Φ are perturbations of the identity:

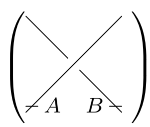
$R = \uparrow\uparrow + \sum(\text{coeff})$  , $\quad \Phi = \uparrow\uparrow\uparrow + \sum(\text{coeff})$  .

On objects $\mathbf{Z}^{\mathbf{PB}}$ acts by forgetting all pairs of parenthesis. On morphisms define

$$\mathbf{Z}^{\mathbf{PB}}(\overcrossing) = R \cdot \times \overset{\text{def}}{=} \tilde{R}, \quad \mathbf{Z}^{\mathbf{PB}}(\undercrossing) = \times \cdot R^{-1} = \tilde{R}^{-1},$$

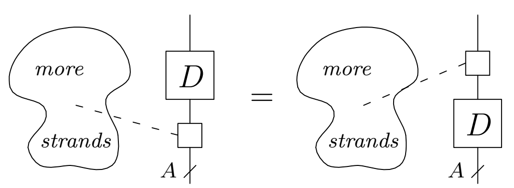
$$\mathbf{Z}^{\mathbf{PB}}(\text{↑↗↑}) = \Phi, \quad \mathbf{Z}^{\mathbf{PB}}(\text{↑↖↑}) = \Phi^{-1},$$

and extend this definition in the only way compatible with the cabling and extension operations of $\mathbf{NAB}$ and $\mathbf{AB}$. For example, set

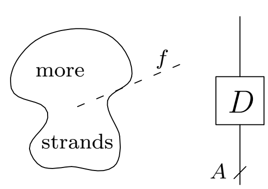
$\mathbf{Z}^{\mathbf{PB}}$  $= (\Delta^A \otimes \Delta^B)(\tilde{R}).$

We wish now to check whether $\mathbf{Z}^{\mathbf{PB}}$ respects the relations (R1)–(R8) generating the kernel of the forgetful functor $\mathbf{NAB} \to \mathbf{PB}$. From the definition of $\mathbf{Z}^{\mathbf{PB}}$ and from exercise 3.2 it readily follows that $\mathbf{Z}^{\mathbf{PB}}$ respects (R2) and (R7) (triviality of ↑↗↑ · ↑↖↑ and of $\overcrossing \cdot \undercrossing$). (R1) and (R6) follow from the non-degeneracy of R and Φ and the fact that they are perturbations of the identity. Locality in space (R4) is obvious, and locality in scale (R5) easily follows from the following lemma:

Lemma 3.4. *When comparable, cablings and extensions commute. In figures, if D is a diagram representing an automorphism of some string A in $\boldsymbol{AB}$, then*

(4) 

Proof. The proof is essentially the same as the proof of lemma 3.1 in [2]. Namely, add three little hooks near each trivalent vertex of the 'D' part of the diagram



,

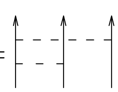
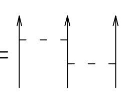
and consider the (properly signed) sum of all the possible ways of connecting f to these hooks. Grouping this sum by vertices we get zero by the IHX and STU relations, and regrouping by arcs we get the difference between the left-hand-side and right-hand-side of (4). $\square$

We see that the only conditions imposed on $\tilde{R}$ and Φ by the relations (R1)–(R8) are the pentagon (R3) and the hexagons (R8). In our notation, these relations read:

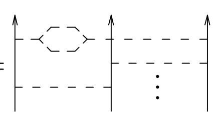
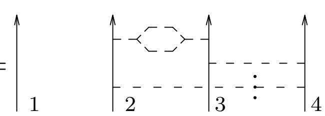
$$(\Phi\uparrow)\cdot(1\otimes\Delta\otimes 1)(\Phi)\cdot(\uparrow\Phi)=(\Delta\otimes 1\otimes 1)(\Phi)\cdot(1\otimes 1\otimes\Delta)(\Phi),$$
$$(\Delta\otimes 1)(\tilde{R})=\Phi\cdot(\uparrow\tilde{R})\cdot\Phi^{-1}\cdot(\tilde{R}\uparrow)\cdot\Phi,$$
$$(\Delta\otimes 1)(\tilde{R}^{-1})=\Phi\cdot(\uparrow\tilde{R}^{-1})\cdot\Phi^{-1}\cdot(\tilde{R}^{-1}\uparrow)\cdot\Phi.$$

Let us rewrite these three relations in terms of R, rather than $\tilde{R}$. First, we need a definition:

Definition 3.5. For a natural number n, let $\mathbf{AP}_n$ be the algebra of all automorphisms (in $\mathbf{AP}$) of the object $\underbrace{\uparrow\uparrow\cdots\uparrow}_{n}$. The unit of $\mathbf{AP}_n$ is the morphism $\uparrow\uparrow\cdots\uparrow$ (no dashed lines), and we will simply denote it by 1. On $\mathbf{AP}_n$ there is a natural action of the symmetric group on n letters $\mathcal{S}_n$, acting by "permuting the strands":

$\Psi =$  , $\Psi^{231} =$  .

Furthermore, for each n there are $n!$ injections of $\mathbf{AP}_{n-1}$ into $\mathbf{AP}_n$, each determined by an ordered subset of size $n-1$ of the set $\{1,\ldots,n\}$:

$\Phi =$  , $\Phi^{243} =$ 

.

(The minor ambiguity in the above notation can be resolved by identifying $\mathbf{AP}_{n-1}$ with its image in $\mathbf{AP}_n$ by the map that adds a "free standing" strand on the right, and will never cause any problem.)

The pentagon and the hexagons can now be written as equations in $\mathbf{AP}_4$ and $\mathbf{AP}_3$ respectively; setting $R^+=R$ and $R^-=(R^{21})^{-1}$ they read:

$$(\pentagon)\qquad \Phi^{123}\cdot(1\otimes\Delta\otimes 1)(\Phi)\cdot\Phi^{234}=(\Delta\otimes 1\otimes 1)(\Phi)\cdot(1\otimes 1\otimes\Delta)(\Phi),$$

$$(\hexagon_\pm)\qquad (\Delta\otimes 1)(R^\pm)=\Phi^{123}\cdot(R^\pm)^{23}\cdot(\Phi^{-1})^{132}\cdot(R^\pm)^{13}\cdot\Phi^{312}.$$

Notice that (apart from our somewhat different conventions) these are precisely the pentagon and hexagons of Drinfel'd's [8, 9]. In section 5 we will present the details of a perturbative approach, hinted at in [9], to solving these equations.

3.2. The norm of Φ. The following proposition, also due to Drinfel'd [8, 9], shows that under certain circumstances it is enough to consider just one of the two hexagon equations $\hexagon_\pm$:

Definition 3.6. For $\Phi \in \mathbf{AP}_n$ define the *transpose* of Φ to be $\Phi^T = \Phi^{n\dots 21}$. For $\Phi \in \mathbf{AP}_3$ define the *norm squared* of Φ to be $||\Phi||^2 = \Phi \cdot \Phi^T = \Phi \cdot \Phi^{321} \in \mathbf{AP}_3$. We say that Φ is *normalized* if it is of unit norm; i.e., if $||\Phi||^2 = 1$.

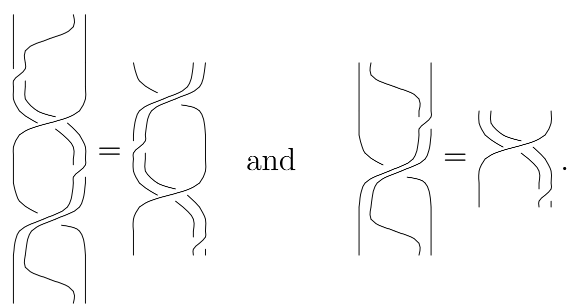
Proposition 3.7. *If $R \in \boldsymbol{AP}_2$ and $\Phi \in \boldsymbol{AP}_3$ are perturbations of the identity and R is symmetric (namely, $R = R^{21}$), then any two of the equations $\hexagon_+$, $\hexagon_-$, and $||\Phi||^2 = 1$ implies the third.*

Proof. Assume $\hexagon_-$ and that Φ is normalized, i.e. that $\Phi^{-1} = \Phi^{321}$. Using the latter equality, $\hexagon_-$ can be rewritten as

$$(\Delta \otimes 1)(R^{-1}) = (\Phi^{-1})^{321} \cdot (R^{-1})^{23} \cdot \Phi^{231} \cdot (R^{-1})^{13} \cdot (\Phi^{-1})^{213}.$$

Inverting both sides of this equation and applying the permutation 213 to the result, we get $\hexagon_+$. A similar computation shows that $\hexagon_+$ and $\Phi^{-1} = \Phi^{321}$ imply $\hexagon_-$.

Now assume $\hexagon_\pm$. Defining $\mathbf{Z}^{\mathbf{PB}}$ as above and noticing that no pentagons can be formed with just a 3-strand non-associative braid, we see that $\mathbf{Z}^{\mathbf{PB}}$ descends to a well defined invariant of parenthesized 3-strand braids. In particular, consider the following two braid equalities:



Applying $\mathbf{Z}^{\mathbf{PB}}$, we get the equalities (see also [8, proposition 3.5])

$$\Phi^{-1} \cdot \Big((\Delta \otimes 1)(R) \cdot R^{12}\Big)^2 \cdot \Phi = \Big(R^{23} \cdot (1 \otimes \Delta)(R)\Big)^2,$$

$$\Phi^{-1} \cdot (\Delta \otimes 1)(R) \cdot R^{12} \cdot (\Phi^{-1})^{321} = R^{23} \cdot (1 \otimes \Delta)(R).$$

Taking the square root of the first equality (possible using power series because both sides are perturbations of the identity) and comparing with the second, we get $\Phi = (\Phi^{-1})^{321}$, as required. $\square$

3.3. An ansatz for $\mathbf{Z^{PFT}}$. Assume that a solution (R, Φ) to the pentagon and hexagons is given, that $R = R^{21}$ is a non-degenerate perturbation of the identity, and that Φ is a normalized non-degenerate perturbation of the identity. Define an element $Z(\infty) \in \mathbf{AP}_1$ by reversing the middle strand of Φ, regarding the result $(1 \otimes \Delta^{\downarrow} \otimes 1)\Phi$ as a morphism in $\mathbf{AT}$, pre-composing it with the degree 0 morphism $\curvearrowleft\uparrow$, composing the result with the degree 0 morphism $\uparrow\curvearrowleft$, and finally regarding the result as an element of $\mathbf{AP}_1$, as in figure 3.

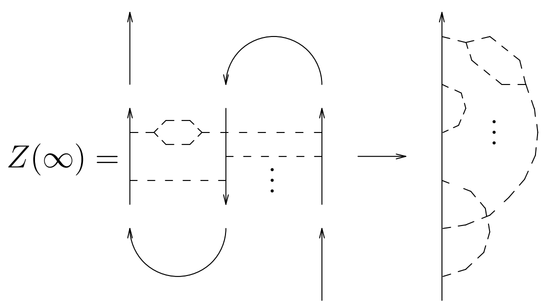
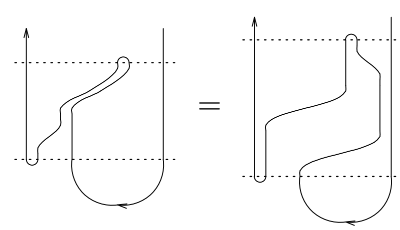


Figure 3. The definition of $Z(\infty)$.

Exercise 3.8. $Z(\infty)$ is the linear combination of chord diagrams corresponding to the non-associative tangle $\uparrow\!\diagup\!|$ by some obvious extension $\mathbf{Z}^{\text{naive}}$ of $\mathbf{Z^{PB}}$. Show that $Z(\infty)$ is also the image of $|\diagdown\!\uparrow$, $|\diagdown\!\downarrow$, and $|\diagup\!\downarrow$ by $\mathbf{Z}^{\text{naive}}$.

Hint 3.9. $\mathbf{Z}^{\text{naive}}\left(|\diagup\!|\right) = \mathbf{Z}^{\text{naive}}\left(|\diagdown\!|\right)$ follows from $\Phi^{-1} = \Phi^{T}$, while $\mathbf{Z}^{\text{naive}}\left(\uparrow\!\diagup\!|\right) = \mathbf{Z}^{\text{naive}}\left(\uparrow\!\diagdown\!|\right)$ follows by 'closing' the pentagon as on the right, and then using the fact that Φ is a non-degenerate perturbation of the identity.



Notice that $\mathbf{AP}_1$ is a commutative algebra; indeed it is simply the algebra $\mathcal{A}$ of [2]. Furthermore, $Z(\infty)$ is also a perturbation of the identity and thus its square root can be computed using a power series expansion. Now extend $\mathbf{Z^{PB}}$ to a functor $\mathbf{Z^{PFT}} : \mathbf{NAT} \to \mathbf{AT}$ by defining[5]

$$\mathbf{Z^{PFT}}(\cup\!\uparrow) = \mathbf{Z^{PFT}}(\cap\!\downarrow) = \mathbf{Z^{PFT}}(\uparrow\!\cup) = \mathbf{Z^{PFT}}(\downarrow\!\cap) = Z(\infty)^{-1/2},$$

and extending $\mathbf{Z^{PFT}}$ to more general creation/annihilation morphisms in the natural way.

Theorem 2. *(See also* [7, 18, 12]*)* $\mathbf{Z^{PFT}}$ *respects (R1)–(R10), and thus it is descends to an* $\boldsymbol{AT}$*-valued parenthesized framed tangle invariant.*

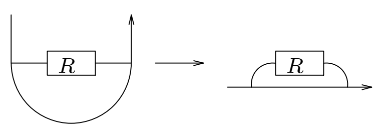
---

[5]In this equality $Z(\infty)^{-1/2}$ is interpreted as a member of five different spaces!

Proof. (R1), (R2) and (R4)–(R7) hold because they held for $\mathbf{Z^{PB}}$ (there is no problem with generalizing lemma 3.4 to the current situation). (R3) and (R8) hold by our choice of R and Φ, and the definition of $\mathbf{Z^{PFT}}$ on creation/annihilation morphisms was clearly cooked up so that (R9) would hold (I wish to thank D. P. Thurston for encouraging me to include exercise 3.8). The remaining relation, (R10), follows by sliding the two factors $\tilde{R}$ and $\tilde{R}^{-1}$ on the left hand side of (R10) until they are adjacent to each other (this is possible using the commutativity of $\mathcal{A}$ proven in [2] and essentially re-proven here as lemma 3.4). When they are adjacent, they can be cancelled using $R = R^{21}$, and the remaining factors can be dealt with using the previous relations. □

3.4. Invariants of unframed tangles. There is a standard way to "renormalize" a regular isotopy invariant Z satisfying $Z\left(\,\circlearrowright\,\right) = cZ(\uparrow)$ for some scalar c, to become a complete isotopy invariant satisfying $Z\left(\,\circlearrowright\,\right) = Z(\uparrow)$. Simply replace Z by $Zc^{-writhe}$. We can apply a variant of the same procedure to our $\mathbf{Z^{PFT}}$, getting an invariant $\mathbf{Z^{PT}}$ of unframed parenthesized tangles:

Definition 3.10. Let C be the result of composing $\cup$ with $(\Delta^{\downarrow} \otimes 1)(R^{-1})$, and regarding the result as an element of $\mathcal{A}$:

(7) $C =$ 

.

Definition 3.11. The *writhe* of a component of a tangle is half the number of positive crossings it goes through, minus half the number of negative crossings it goes through (counting self-crossings of the same component twice).

Finally, to compute $\mathbf{Z^{PT}}(M)$ for a given $M \in \mathbf{NAB}$, attach $C^{-writhe}$ to each of the components (i.e., strands) in $\mathbf{Z^{PFT}}(M)$. The usual commutativity property of $\mathcal{A}$ shows that it doesn't matter where the attachement is made. Remembering that the writhes are regular isotopy invariants, the commutativity property also shows that $M \mapsto \mathbf{Z^{PT}}(M)$ is a functor. Clearly, relations (R1)–(R10) are not affected, and meanwhile, we've fixed (R11) to hold for $\mathbf{Z^{PT}}$.

Theorem 3. *(See also* [7, 18, 12]*)* $\mathbf{Z^{PT}}$ *respects (R1)–(R11), and thus it descends to an* ***AT****-valued parenthesized framed tangle invariant.* □

3.5. Invariants of knots and links. The restriction $\mathbf{Z^{K}}$ of $\mathbf{Z^{PT}}$ to knots has values in the algebra $\mathcal{A}$ of chord diagrams [2]. Assume now that $R = 1 + \uparrow\!\!-\!\!\uparrow/2 + \text{higher degree}$ terms.

Theorem 4. *(See also* [7, 18, 12]*)* $\mathbf{Z^{K}}$ *is a universal Vassiliev invariant.*

In other words, if D is a chord diagram and the singular knot K_D is an embedding (as defined in [2, section 2.2]) of D in $\mathbf{R}^3$, then

$$\mathbf{Z}^{\mathbf{K}}(K_D) = D + \text{higher degree terms}.$$

(This of course means that if W is a degree m weight system, then the numerical knot invariant $W \circ \mathbf{Z}^{\mathbf{K}}$ is Vassiliev of type m and its underlying weight system is W, meaning that we can reconstruct *any* Vassiliev invariant out of $\mathbf{Z}^{\mathbf{K}}$.)

Proof. The proof is essentially identical to the proofs that all standard knot polynomials are series of Vassiliev invariants, as in [2]. The main point is that $R - R^{-1} = \uparrow\!\!-\!\!\uparrow$ +higher degree terms, and so every double point in K_D forces a chord in $\mathbf{Z}^{\mathbf{K}}(K_D)$, in just the right place. □

Similarly, one can use $\mathbf{Z}^{\mathbf{PT}}$ to define a link invariant $\mathbf{Z}^{\mathbf{L}}$, and $\mathbf{Z}^{\mathbf{L}}$ will have a parallel universality property.

Remark 3.12. Following Drinfel'd's analysis of the non-uniqueness of (R, Φ) Le and Murakami [18] had proven that if R is central, the resulting link invariant $\mathbf{Z}^{\mathbf{L}}$ is independent of the choice of Φ and if in addition $R = \exp \uparrow\!\!-\!\!\uparrow /2$, then $\mathbf{Z}^{\mathbf{K}}$ is equal to the Kontsevich integral $\tilde{Z}$ as in (1).

4. Cohomological preliminaries

Before we can present our algorithm for computing a pair (R, Φ) which solves the equations of the previous section, we need to discuss some cohomological preliminaries. Most of the homological algebra that we will use can be found in [20, section 1.6] (though we need the cohomological versions of the homological results there).

4.1. Defining the complexes.

Definition 4.1. Let $C^n = C^n(\mathbf{AP}) = \mathbf{AP}_n$ and define coface maps $d_i^n : C^n \to C^{n+1}$ $(0 \le i \le n+1)$ by

$$d_i^n(\xi) = \begin{cases} \xi^{234\ldots(n+1)} & i = 0 \\ (1 \otimes \cdots \otimes \underset{i}{\Delta} \otimes \cdots \otimes 1)(\xi) & 1 \le i \le n \\ \xi^{123\ldots n} & i = n+1 \end{cases}$$

(Namely, d_0^n adds one strand on the left, d_{n+1}^n adds one strand on the right, and otherwise d_i^n doubles the ith strand.) Define a differential $d^n : C^n \to C^{n+1}$ by

$$d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n,$$

and let the *nth formal Hochschild cohomology of* $\boldsymbol{AP}$, $H^n(\mathbf{AP})$, be the cohomology of the resulting complex; namely, set $Z^n = \ker d^n$, $B^n = \operatorname{im} d^{n-1}$, and $H^n = Z^n / B^n$.

Similarly, the *twisted formal Hochschild cohomology of* $\boldsymbol{AP}$, $H^n(\mathbf{AP}_\star \otimes \mathbf{AP}_1)$, will be defined by "leaving the rightmost strand alone":

Definition 4.2. Let $C^n_{\star\otimes 1} = C^n(\mathbf{AP}_\star \otimes \mathbf{AP}_1) = \mathbf{AP}_{n+1}$, define $d^n_i \otimes 1 = d^{n+1}_i$ for $i \leq n$, and set

$$(d^n_{n+1} \otimes 1)(\xi) = \xi^{12\dots n(n+2)}.$$

Let the differential $d^n \otimes 1$ be the alternating sum of the $d^n_i \otimes 1$'s, and set $Z^n_{\star\otimes 1} = \ker d^n \otimes 1$, $B^n_{\star\otimes 1} = \operatorname{im} d^{n-1} \otimes 1$, and $H^n_{\star\otimes 1} = Z^n_{\star\otimes 1}/B^n_{\star\otimes 1}$. In the context of $C_{\star\otimes 1}$, we will denote the transpose operation (parallel to the transpose of definition 3.6) by $T \otimes 1$. In other words, for $\xi \in C^n_{\star\otimes 1}$ define $\xi^{T\otimes 1} = \xi^{n\dots 21(n+1)}$. Similar definitions can be made for $C^n_{1\otimes\star}$, $1 \otimes d$, etc.

Definition 4.3. Let the *symmetric subcomplex* of $C^\star$ be given by

$$C^n_{sym} = C^n_{sym}(\mathbf{AP}) = \left\{ \xi \in C^n : \xi + (-1)^{n(n+1)/2}\xi^T = 0 \right\}.$$

It is easy to check that $C^\star_{sym}$ is indeed a subcomplex of $C^\star$, and so one can define Z^n_{sym}, B^n_{sym}, and H^n_{sym}. Define the groups $C^n_{sym\otimes 1}$, $Z^n_{sym\otimes 1}$, $B^n_{sym\otimes 1}$, and $H^n_{sym\otimes 1}$ by replacing C and T by $C_{\star\otimes 1}$ and $T \otimes 1$ in the previous sentence.

Proposition 4.4. *(Proof on page 24, see also* [8, proposition 3.11]*) If n is even, $H^n_{sym}(\boldsymbol{AP}) = 0$.*

Proposition 4.5. *(Proof on page 25) If n is even, $H^n_{sym\otimes 1}(\boldsymbol{AP}) = 0$.*

One further complex is of interest for us. To define it, we first need the notion of a *shuffle:*

Definition 4.6. Let p and q be non-negative integers. A permutation σ of $\{1, \dots, p+q\}$ is called a *(pq)-shuffle* if σ^{-1} preserves the order of $\{1, \dots, p\}$ and the order of $\{p+1, \dots, p+q\}$. Define $\omega_{pq} : \mathbf{AP}_{p+q} \to \mathbf{AP}_{p+q}$ by

$$\omega_{pq}(\xi) = \sum_{\text{all } (pq)\text{-shuffles } \sigma} (-1)^\sigma \xi^\sigma.$$

Definition 4.7. The Harrison subcomplex [10] of $C^\star$ is defined by

$$C^n_{Harr} = C^n_{Harr}(\mathbf{AP}) = \{\xi \in C^n : \omega_{pq}(\xi) = 0 \text{ whenever } p + q = n \text{ and } p, q \geq 1\}.$$

Make the usual definitions for Z^n_{Harr}, B^n_{Harr}, and H^n_{Harr}.

Exercise 4.8. Show that $C^\star_{Harr}$ is indeed a subcomplex of $C^\star$.

Theorem 5. *(Proof on page 24; the case $n = 3$ is in* [18]*) If $n \geq 2$, $H^n_{Harr}(\boldsymbol{AP}) = 0$.*

Remark 4.9. The cofaces d_i^n together with the strand removal operations s_i of page 14 make all of the complexes discussed here into cosimplicial sets. Thus by the normalization theorem for simplicial cohomology the non-degenerate subcomplex of each of these complexes, defined by

$$\tilde{C}^n_\bullet = \bigcap_i \ker s_i|_{C^n_\bullet},$$

has the same cohomology as the original complex.

Finally, we will briefly need the following definition and proposition:

Definition 4.10. Similarly to definition 4.2, define $d^n \otimes 1 \otimes 1$ by "leaving the two rightmost strands alone", and $1 \otimes 1 \otimes d^n$ by "leaving the two leftmost strands alone". Finally define $d^p \otimes d^q : \mathbf{AP}_{p+q} \to \mathbf{AP}_{p+q+2}$ by having d^p act on the first p strands and d^q act on the last q strands. For example, $d^1 \otimes d^1 = (d \otimes 1 \otimes 1)(1 \otimes d) = (1 \otimes 1 \otimes d)(d \otimes 1)$.

Proposition 4.11. *(Proof on page 25) The kernel Z^{11} of the map $d \otimes d : \boldsymbol{AP}_2 \to \boldsymbol{AP}_4$ is the (non direct) sum of the kernels of $d \otimes 1 : \boldsymbol{AP}_2 \to \boldsymbol{AP}_3$ and $1 \otimes d : \boldsymbol{AP}_2 \to \boldsymbol{AP}_3$. Furthermore, any antisymmetric element of Z^{11} is the antisymmetrization of an element of $\ker(d \otimes 1)$. I.e., if $\rho \in Z^{11}$ satisfies $\rho + \rho^T = 0$, then $\rho = \bar{\rho} - \bar{\rho}^T$ for some non-degenerate $\bar{\rho} \in \ker(d \otimes 1)$.*

4.2. Computing some of their cohomologies. Let us start by computing the cohomology of a much simpler related complex.

Definition 4.12. Let $C^n = C^n(\mathbf{F})$ be $\mathbf{F}^n$, the vector space of dimension n over the field $\mathbf{F}$. Let $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbf{F}^n$. Define coface maps $d_i^n : C^n \to C^{n+1}$ $(0 \leq i \leq n+1)$ by

$$d_i^n(e_j) = \begin{cases} e_{j+1} & i < j \\ e_j + e_{j+1} & i = j \\ e_j & i > j \end{cases},$$

and set $d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n$. Let $H^n(\mathbf{F})$ be the cohomology of the resulting complex.

Remark 4.13. Notice that if we draw the basis vector e_j of $\mathbf{F}^n$ as a sequence of n vertical strands with the j'th one marked by a bead ($\bullet$),

$e_4 =$  $\in \mathbf{F}^5,$

then the d_i^n of the above definition correspond to the d_i^n of definition 4.1; d_i^n can be viewed as adding one strand on the left, d_{n+1}^n as adding a strand on the right, and otherwise d_i^n can be viewed as doubling the i'th strand, summing over the possible ways of marking the resulting two children strands in case the parent strand was marked.

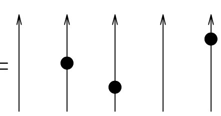
Lemma 4.14. $\dim H^1(\mathbf{F}) = 1$, *while* $\dim H^n(\mathbf{F}) = 0$ *for* $n > 1$.

Proof. First notice that $C^\star$ is a cosimplicial vector space — one may define codegeneracies $s_i^n : C^n \to C^{n-1}$ $(1 \le i \le n)$ by

$$s_i^n(e_j) = \begin{cases} e_{j-1} & i < j \\ 0 & i = j \\ e_j & i > j \end{cases} .$$

Therefore by the normalization theorem for simplicial cohomology the complex $C^\star$ has the same cohomology as the complex $\tilde{C}^\star$ defined by $\tilde{C}^n = \bigcap_i \ker s_i^n$. But it is clear that $\tilde{C}^n = 0$ unless $n = 1$, and that $\tilde{C}^1 = \mathbf{F}$. □

Definition 4.15. For any natural number k, set $C^n(\mathbf{F}, k) = (\mathbf{F}^n)^{\otimes k}$. The elements of $C^n(\mathbf{F}, k)$ can be pictured as linear combinations of diagrams made of n vertical strands marked by k beads set on k different time levels:

$e_3 \otimes e_2 \otimes e_5 =$  $\in C^5(\mathbf{F}, 3)$.

$C^\star(\mathbf{F}, k)$ inherits the structure of a cosimplicial vector space from $C^n(\mathbf{F})$, and again, the coface maps d_i^n have pictorial interpretations as strand addition and strand doubling as in definition 4.1. From the Eilenberg-Zilber theorem and the Künneth formula it follows that the only non-vanishing cohomology of $C^\star(\mathbf{F}, k)$ is in degree k, and that $\dim H^k(\mathbf{F}, k) = 1$. Finally, a careful study of the proof of the Eilenberg-Zilber theorem and the Künneth formula reveals that $H^k(\mathbf{F}, k)$ is generated by the class of the "determinant"

$$\delta_k = \sum_{\sigma \in S_k} (-1)^\sigma e_{\sigma 1} \otimes e_{\sigma 2} \otimes \cdots \otimes e_{\sigma k}.$$

(Or else, it is easy to verify directly that $d\delta_k = 0$ and that δ_k is orthogonal to the image of d relative to the Euclidean scalar product on $C^k(\mathbf{F}, k)$.)

Notice that S_n acts on $C^n(\mathbf{F}, k)$ by permuting the e_j's, and so we can define $H^n_{sym}(\mathbf{F}, k)$ and $H^n_{Harr}(\mathbf{F}, k)$ as in the previous section. The total anti-symmetry of δ_k implies that $\delta_k^T = (-1)^{n(n-1)/2}\delta_k$, and therefore $H^n_{sym}(\mathbf{F}, k) = 0$ for any even n.

Definition 4.16. Let k be a natural number and let R be a representation of S_k. Let $C^n(k, R) = (R \otimes (\mathbf{F}^n)^{\otimes k})^{S_k}$ be the S_k-invariant part of $R \otimes (\mathbf{F}^n)^{\otimes k}$, using the given action of S_k on R and the obvious action[6] of S_k on $(\mathbf{F}^n)^{\otimes k}$. $C^\star(k, R)$ inherits a structure of a cosimplicial vector space from $C^\star(\mathbf{F}, k)$, and for each n, $C^n(k, R)$ inherits an S_n action from $C^n(\mathbf{F}, k)$. Thus we can define the cohomologies $H^n(k, R)$, $H^n_{sym}(k, R)$, and $H^n_{Harr}(k, R)$.

[6] This action commutes with the action of S_n discussed in the previous paragraph.

Notice that the assignment $R \mapsto H^n(k,R)$ is additive: $H^n(k,R_1 \oplus R_2) = H^n(k,R_1) \oplus H^n(k,R_2)$ for any two representations $R_{1,2}$ of S_k.

Lemma 4.17. *$H^n(k,R)$ vanishes unless $n = k$, and the dimension of $H^k(k,R)$ is equal to the number of times the alternating representation* Alt *appears as a summand in the decomposition of R into irreducible representations.*

Proof. Let us first check the case of $R = \mathbf{F}S_k$, the regular representation of S_k. In this case $(R \otimes (\mathbf{F}^n)^{\otimes k})^{S_k}$ is isomorphic to $(\mathbf{F}^n)^{\otimes k}$, and the lemma follows from the discussion following definition 4.15. Furthermore, it is easy to verify that the above isomorphism carries δ_k to $(\sum_\sigma (-1)^\sigma \sigma) \otimes \delta_k$, meaning that already $\dim H^k(\mathrm{Alt}) = 1$. The general case of the lemma now follows from the additivity of $R \to H^n(k,R)$ and from the fact that the regular representation contains every irreducible representation of S_k. $\square$

Corollary 4.18. *$H^n_{sym}(k,R)$ vanishes if n is even.* $\square$

Corollary 4.19. *$H^n_{Harr}(k,R)$ vanishes if $n \geq 2$.*

Proof. Using the Eulerian idempotents $e^n_{(l)}$ one gets a decomposition (see [20, pp. 144] and [19, pp. 222])

$$H^n(k,R) = H^n_{(1)}(k,R) \oplus \cdots \oplus H^n_{(n)}(k,R),$$

with $H^n_{(1)}(k,R) = H^n_{Harr}(k,R)$ and $H^n_{(n)}(k,R)$ being the cohomology of the complex obtained by taking the total anti-symmetrization of $C^n(k,R)$ for each n. Lemma 4.17 implies that all of the cohomology $H^n(k,R)$ lies in $H^n_{(n)}(k,R)$, and so there is nothing left for $H^n_{(1)}(k,R) = H^n_{Harr}(k,R)$. $\square$

Proof of proposition 4.4 *and of theorem* 5. Given corollaries 4.18 and 4.19, all we have to do is to show that the complex $C^\star(\mathbf{AP})$ is isomorphic to the complex $C^\star(k,R)$ for some R, or to a direct sum of such complexes, via an isomorphism that respects the S_n actions on $C^n(\mathbf{AP})$ and on $C^n(k,R)$. This is easy. Indeed, let R_k be the space spanned by all Chinese characters[7] having exactly k univalent vertices labeled by the numbers 1 to k, modulo the usual IHX and AS relations. By permuting the labels of the univalent ends, R_k becomes a representation of S_k. $R_k \otimes (\mathbf{F}^n)^{\otimes k}$ can be viewed as the space spanned by Chinese characters whose k univalent ends are ordered and are colored by integers between 1 and n, corresponding to the n basis vectors $\{e_j\}$ of $\mathbf{F}^n$. The S_k invariant subspace $(R_k \otimes (\mathbf{F}^n)^{\otimes k})^{S_k}$ is the space spanned by Chinese characters whose k univalent ends are *unordered* and are colored by integers between 1 and n. Taking the direct sum over k, we get exactly the space $\mathcal{B}^{sl}$ of [3] (with

[7] as defined in [2].

$\Upsilon = \{1, \dots, n\}$) which is isomorphic to the space $\mathcal{A}^{sl}$ of the same paper. $\mathcal{A}^{sl}$ with $\Upsilon = \{1, \dots, n\}$ is just a different name for our $\mathbf{AP}_n$. Summarizing, we have that

$$C^n(\mathbf{AP}) = \bigoplus_k C^n(k, R_k). \tag{8}$$

Rereading remark 4.13, we see that the above isomorphisms respect the coface maps, the differentials, and the S_n actions on the two sides of (8). □ □

Proof of proposition 4.5. Simply repeat the constructions and considerations leading to the proof of proposition 4.4, only always ignoring the $(n+1)$st strand/basis vector/color. □

Proof of proposition 4.11. Use the same techniques as in the proof of (8) to reduce the problem to a problem about the kernel of $d \otimes d$ acting on $(\mathbf{F}^2)^{\otimes k}$. The vector space $(\mathbf{F}^2)^{\otimes k}$ is naturally isomorphic to the space $\langle x, y\rangle_k$ of degree k polynomials in two non-commuting variables x and y. Introduce new non-commuting variables $x_{1,2}$ and $y_{1,2}$, and for any polynomial p in the variables x, $x_{1,2}$, y, and $y_{1,2}$ define

$$\begin{aligned} d_x p &= p\big|_{x\to x_2} - p\big|_{x\to x_1+x_2} + p\big|_{x\to x_1}, \\ d_y p &= p\big|_{y\to y_2} - p\big|_{y\to y_1+y_2} + p\big|_{y\to y_1}, \end{aligned} \qquad \text{and} \quad p^T = p\big|_{x\leftrightarrow y}.$$

It is easy to check that using this language, the operators $d \otimes 1$, $1 \otimes d$, $d \otimes d$ and $\rho \mapsto \rho^T$ become d_x, d_y, $d_x \circ d_y$, and $p \mapsto p^T$ respectively, all restricted to $\langle x, y\rangle_k$, and that the condition "ρ is non-degenerate" becomes "$p \in \langle x, y\rangle_k$ vanishes if either x or y is set to 0". To finish the proof of the proposition, simply notice that $\ker d_x$ ($\ker d_y$) is the linear span of the monomials of degree exactly 1 in x (y), and that $\ker d_x \circ d_y$ is the linear span of the monomials of degree exactly 1 in either x or y. □

5. Solving the pentagon and the hexagons

In this section we will present our inductive prescription for solving the pentagon and the hexagons. A similar (and about as complicated) prescription can be read from Drinfel'd's [9, proposition 3.1]. Our prescription (suggested by [9, remark 2 following proposition 5.8]) uses the vanishing of a 'smaller' cohomology, H^4_{Harr}, whereas Drinfel'd's uses the vanishing of the bigger H^4_{sym}. In our case, both groups indeed vanish as proven in the previous section. But it may be that in some similar but different circumstances, (see e.g. section 6.1), the inclusion $H^4_{Harr} \subset H^4_{sym}$ is proper and only the smaller of the two vanishes.

The algebras in which the pentagon ⬠ and the hexagons ⬡$_\pm$ are written are graded, and thus we may hope to solve ⬠ and ⬡$_\pm$ degree by degree. Let us start with a definition:

Definition 5.1. Let $\mathbf{M}$ be a graded module. Denote by $\mathcal{G}_m\mathbf{M}$ the degree m piece of $\mathbf{M}$ and set $\mathcal{F}_m\mathbf{M} = \bigoplus_{i=0}^m \mathcal{G}_i\mathbf{M}$ and $\mathcal{R}_m\mathbf{M} = \bigoplus_{i>m} \mathcal{G}_i\mathbf{M}$.

Start the induction with $R_1 = 1 + \uparrow\!\!\!-\!\!\!\uparrow/2$ and $\Phi_1 = 1$. Then for an arbitrary $m > 1$ assume that a non-degenerate $\Phi_{m-1} \in \mathcal{F}_m\mathbf{AP}_3$ and a symmetric non-degenerate $R_{m-1} \in \mathcal{F}_m\mathbf{AP}_2$ solve ⬠ and $⬡_\pm$ up to and including degree $m-1$, that $R_{m-1} = 1 + \uparrow\!\!\!-\!\!\!\uparrow/2 \mod \mathcal{R}_m\mathbf{AP}_2$, and that

$$||\Phi_{m-1}||^2 = 1 \qquad \mod \mathcal{R}_m\mathbf{AP}_3. \tag{9}$$

Set $R_{m,0} = R_{m-1}$ and $\Phi_{m,0} = \Phi_{m-1}$. We will attempt to 'improve' $(R_{m,0}, \Phi_{m,0})$ via a five-step procedure, each time replacing $(R_{m,i-1}, \Phi_{m,i-1})$ $(i = 1, \dots, 5)$ by $(R_{m,i}, \Phi_{m,i})$, with each step bringing us closer to a solution of ⬠ and $⬡_\pm$ in all degrees up to and including degree m. The result of the last step, $(R_{m,5}, \Phi_{m,5})$, will actually solve ⬠ and $⬡_\pm$ to the required degree, and will serve as the seed (R_m, Φ_m) for the solution in the following degree.

Let $\mu_i \in \mathbf{AP}_4$ and $\psi_{\pm,i} \in \mathbf{AP}_3$ be the degree m 'mistakes' in ⬠ and $⬡_\pm$ when using $R_{m,i}$ and $\Phi_{m,i}$:

$$1 + \mu_i = \Phi_{m,i}^{123} \cdot (1 \otimes \Delta \otimes 1)(\Phi_{m,i}) \cdot \Phi_{m,i}^{234} \cdot (1 \otimes 1 \otimes \Delta)(\Phi_{m,i}^{-1}) \cdot (\Delta \otimes 1 \otimes 1)(\Phi_{m,i}^{-1}) \tag{10}$$

$$1 + \psi_{\pm,i} = (\Delta \otimes 1)(R_{m,i}^{\mp 1}) \cdot \Phi_{m,i}^{123} \cdot (R_{m,i}^{\pm 1})^{23} \cdot (\Phi_{m,i}^{-1})^{132} \cdot (R_{m,i}^{\pm 1})^{13} \cdot \Phi_{m,i}^{312} \tag{11}$$

(both equations taken modulo $\mathcal{R}_m\mathbf{AP}$).

Setting $R_{m,i} = R_{m,i-1} + r_i$ and $\Phi_{m,i} = \Phi_{m,i-1} + \varphi_i$, we will search for $r_i \in \mathcal{G}_m\mathbf{AP}_2$ and for $\varphi_i \in \mathcal{G}_m\mathbf{AP}_3$ that will make the mistakes μ_i and $\psi_{\pm,i}$ be 'simpler' than their predecessors μ_{i-1} and $\psi_{\pm,i-1}$. From (10) and (11) it is easy to read the following formulas for the 'new' mistakes in terms of the 'old' mistakes and the correction terms φ_i and r_i:

$$\begin{aligned}\mu_i = \mu_{i-1} + \varphi_i^{234} - (\Delta \otimes 1 \otimes 1)(\varphi_i) + (1 \otimes \Delta \otimes 1)(\varphi_i) - (1 \otimes 1 \otimes \Delta)(\varphi_i) + \varphi_i^{123} \\ = \mu_{i-1} + d\varphi_i,\end{aligned} \tag{12}$$

$$\psi_{\pm,i} = \psi_{\pm,i-1} + \left[\varphi_i^{123} - \varphi_i^{132} + \varphi_i^{312} \pm (r_i^{23} - (\Delta \otimes 1)(r_i) + r_i^{13})\right]. \tag{13}$$

Setting $\psi_i = (\psi_{+,i} + \psi_{-,i})/2$ and $\psi_{\mathit{diff},i} = (\psi_{+,i} - \psi_{-,i})/2$, we see that (13) is equivalent to the following two equations:

$$\psi_i = \psi_{i-1} + \text{ш}_{21}(\varphi_i); \qquad \psi_{\mathit{diff},i} = \psi_{\mathit{diff},i-1} + (d \otimes 1)(r_i). \tag{14}$$

We aim to find φ_i and r_i that will make the left hand sides of the above equations as simple as possible, hopefully even vanishing. The first step towards doing so is to find relations between the determined parts of the right hand sides of the above equations, namely between μ_{i-1} and $\psi_{\pm,i-1}$.

5.1. Finding relations between μ_i and $\psi_{\pm,i}$. Rearranging (10) and then cabling and extending it in some arbitrary way, (and similarly for (11)), we see that it can be regarded as a way of writing the identity of $\mathbf{AP}_n$ (for some n) as a product of five 'variants' of Φ, at the cost of some error proportional to μ. Each variant of Φ, say

,

appears in many different pentagons (and hexagons), and thus the product we just obtained can be expanded further at the cost of some more error terms. Continuing in this manner, if at some point our product becomes a product of pentagons and hexagons, we can simply cross them all out, again at the cost of some more error terms proportional to μ and ψ. What we have left at the end is an expression for the identity in terms of the identity and some error terms — in other words, we have a relation between those error terms, which is precisely what we now seek.

As a first example, start from the inner most pentagon in figure 4. Reading its edges counterclockwise beginning from the vertex labeled $((ab)c)d$, we get equation (10), and we've indicated the error term of that equation in the center of the pentagon. Now each of the R and Φ terms in (10) can be expanded further, by either using a locality relation (a square) or by using a hexagon (or the image of a hexagon equation by some cabling or permutation operation). Locality relations always hold and cost nothing, but each hexagon contributes some error term, as indicated in its center. Continuing to expand in this manner until we reach the outermost hexagon and then replacing that hexagon by its corresponding error term, we get the equation[8]

$$\mu_i^{1234} - \mu_i^{1243} + \mu_i^{1423} - \mu_i^{4123} = \psi_{+,i}^{234} - (\Delta\otimes 1\otimes 1)\psi_{+,i} + (1\otimes\Delta\otimes 1)\psi_{+,i} - \psi_{+,i}^{124}, \tag{15}$$

or

$$\text{ш}_{31}(\mu_i) = (d\otimes 1)(\psi_{+,i}). \tag{16}$$

Similarly, figure 5 proves equation (17), figure 6 proves equation (18), and figure 7 proves equation (19). For a similar proof of (19), see [8, pp. 1449] and [22].

$$\text{ш}_{22}(\mu_i) = (1\otimes 1\otimes d)\psi_{+,i} + (d\otimes 1\otimes 1)\psi_{-,i}^{231} \tag{17}$$

$$\psi_{+,i}^{123} - \psi_{+,i}^{213} + \psi_{-,i}^{321} - \psi_{-,i}^{231} = 0 \tag{18}$$

$$d\mu_i = 0 \tag{19}$$

---

[8]Actually, the above process doesn't quite give (15), but rather it gives a similar equation in which each of the terms is multiplied by some further factors proportional to $R_{m,i}$ and $\Phi_{m,i}$. But remembering that $R_{m,i}$ and $\Phi_{m,i}$ are perturbations of the identity, that μ_i and $\psi_{\pm,i}$ are of degree m, and taking only the degree m piece, we get equation (15).

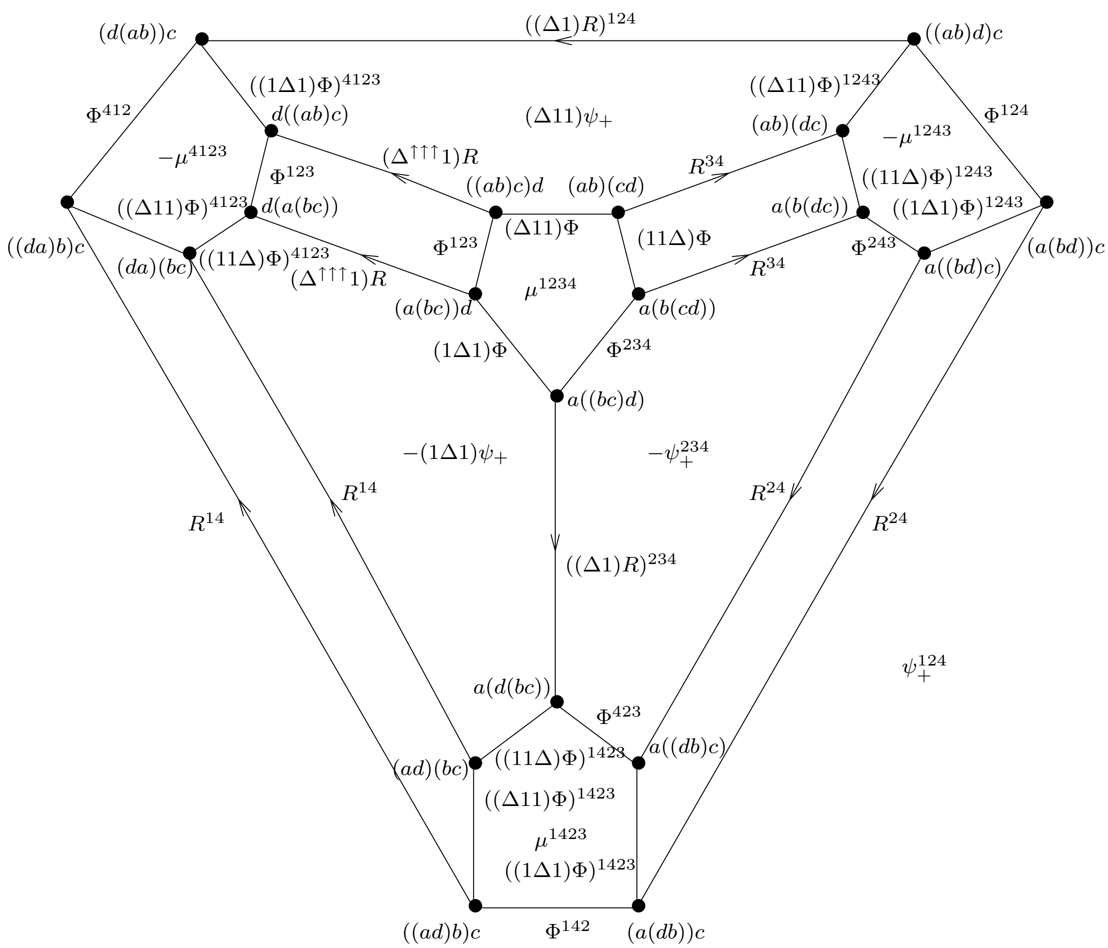


Figure 4. Proof of equation (16). To save space, we've suppressed all the $\otimes$ symbols, and the subscripts m and i. Notice that the orientations of the edges labeled by a variant of Φ can be read from the parenthesized strings labeling their ends, and so we also suppressed these orientations.

Proposition 5.2. *If* $R_{m,i} = R^{21}_{m,i}$ *and* $||\Phi_{m,i}||^2 = 1 \mod \mathcal{R}_m \boldsymbol{AP}_3$*, then* $\psi_{+,i} = -\psi^{213}_{-,i}$ $(= -\psi^{T\otimes 1}_{-,i})$ *and* $\mu_i = -\mu_i^{4321}$ $(= -\mu_i^T)$*.*

Proof. The first assertion follows from a trivial modification of the first few lines of the proof of proposition 3.7. To prove the second, invert both sides of (10), take the transpose and use $\Phi^{-1}_{m,i} = \Phi^T_{m,i}$ whenever possible, getting the equation

$$1-\mu_i^T = (1\otimes 1\otimes\Delta)\Phi^{-1}_{m,i}\cdot(\Delta\otimes 1\otimes 1)\Phi^{-1}_{m,i}\cdot\Phi^{123}_{m,i}\cdot(1\otimes\Delta\otimes 1)\Phi_{m,i}\cdot\Phi^{234}_{m,i} \qquad \mod \mathcal{R}_m\mathbf{AP}_3.$$

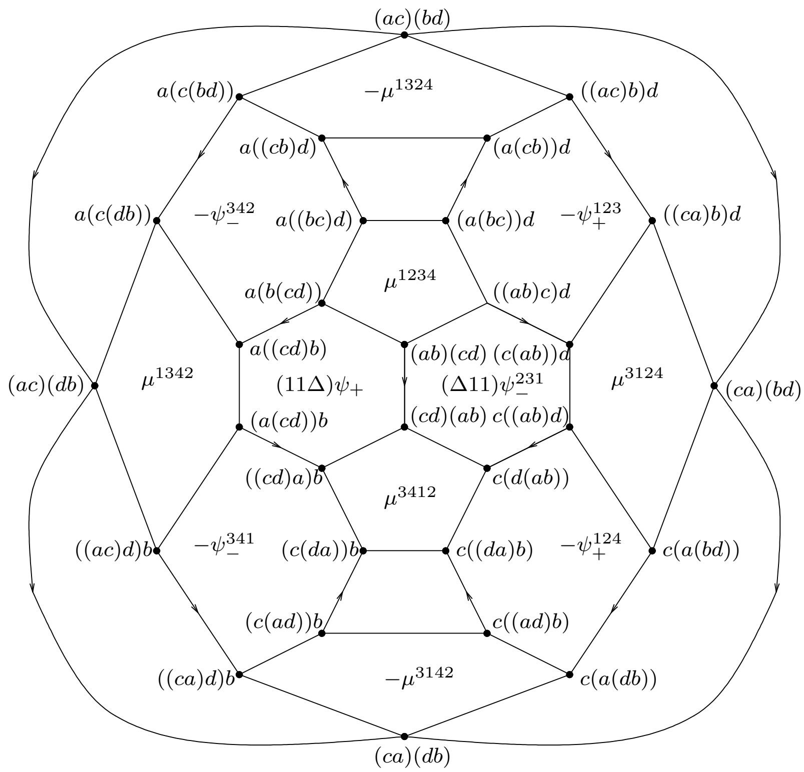


Figure 5. The proof of equation (17)

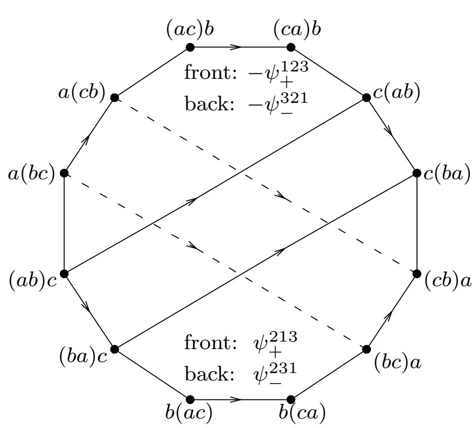


Figure 6. The proof of equation (18). The polyhedron in this figure is a sphere whose equator is a 12-gon, and each of whose hemispheres is made of two hexagons and a square. It is shown from above, with the lower hemisphere ('back') drawn in dashed lines.

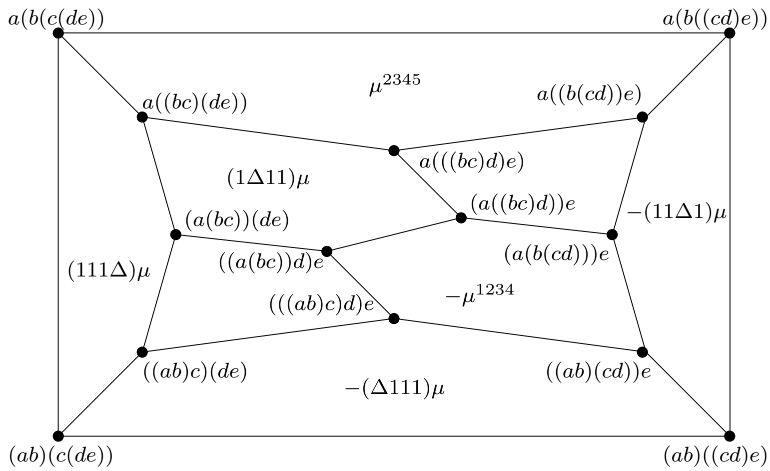


Figure 7. The proof of equation (19)

But the right hand side here is again the pentagon, and so by (10)

$$1 - \mu_i^T = 1 + \mu_i,$$

as required. □

5.2. First step ($i = 1$), renormalizing Φ. The first step is to guarantee (9) for $\Phi_{m,1}$. This is easy: simply set

$$\Phi_{m,1} = \Phi_{m,0} - \frac{||\Phi_{m,0}||^2 - 1}{2}, \tag{20}$$

and leave R unchanged, namely, set $R_{m,1} = R_{m,0}$. Notice that $\Phi_{m,1}$ is still non-degenerate.

5.3. Second step ($i = 2$), fixing R to eliminate $\psi_{\textit{diff}}$.

Lemma 5.3. *$\psi_{\textit{diff},1} \in Z^2_{sym\otimes 1}(\boldsymbol{AP})$.*

Proof. Having guaranteed (9), we can use proposition 5.2, get $\psi_{\textit{diff},1} = \frac{1}{2}\left(\psi_{+,1} + \psi_{+,1}^{T\otimes 1}\right)$, and thus $\psi_{\textit{diff},1} \in C^2_{sym\otimes 1}$. Furthermore,

$$\begin{aligned}(d\otimes 1)\psi_{\textit{diff},1} &= \frac{1}{2}\left((d\otimes 1)\psi_{+,1} - ((d\otimes 1)\psi_{+,1})^{T\otimes 1}\right)\\ &= \frac{1}{2}\left(\mu_1^{1234} - \mu_1^{1243} + \mu_1^{1423} - \mu_1^{4123} - \mu_1^{3214} + \mu_1^{3241} - \mu_1^{3421} + \mu_1^{4321}\right)\end{aligned}$$

by (16). Using the second part of proposition 5.2 on the last four terms of the above sum, we see that they cancel against the first four. □

Now use proposition 4.5 to find a non-degenerate r_2 for which $\psi_{\mathit{diff},1} + (d\otimes 1)r_2 = 0$. (It is easy to check that $\psi_{\mathit{diff},1}$ is non-degenerate and thus by remark 4.9 r_2 can be chosen to be non-degenerate). Setting $R_{m,2} = R_{m,1} + r_2$ and $\Phi_{m,2} = \Phi_{m,1}$, we get from (14) that $\psi_{\mathit{diff},2} = 0$.

5.4. Third step ($i = 3$), symmetrizing R. Unfortunately, our choice of r_2 may have ruined the symmetry of R_m; we need to have some control of the antisymmetric part of r_2:

Lemma 5.4. $(d \otimes d)(r_2^{21} - r_2) = 0$.

Proof.
$$\begin{aligned}(d \otimes d)(r_2^{21} - r_2) &= (d \otimes 1 \otimes 1)(1 \otimes d)r_2^{21} - (1 \otimes 1 \otimes d)(d \otimes 1)r_2 \\ &= (d \otimes 1 \otimes 1)\psi^{321}_{\mathit{diff},1} - (1 \otimes 1 \otimes d)\psi_{\mathit{diff},1} \\ &= \frac{1}{2}(d \otimes 1 \otimes 1)(\psi^{321}_{-,1} + \psi^{231}_{-,1}) + \frac{1}{2}(1 \otimes 1 \otimes d)(\psi_{+,1} + \psi^{213}_{+,1}) \\ &= \frac{1}{2}\,\omega_{22}\,(\mu_1) + \frac{1}{2}\,(\omega_{22}(\mu_1))^{2143} \qquad \text{by (17)} \\ &= 0 \qquad\qquad \text{by proposition 5.2.} \quad \square\end{aligned}$$

Now use proposition 4.11 on $\rho = r_2^{21} - r_2$ to find a non-degenerate $r_3 \in \ker d \otimes 1 \subset \mathbf{AP}_2$ for which $r_2^{21} - r_2 = r_3 - r_3^{21}$. Setting $R_{m,3} = R_{m,2} + r_3$ and $\Phi_{m,3} = \Phi_{m,2}$ we find that $R_{m,3}$ is symmetric, and $r_3 \in \ker d \otimes 1$ together with (14) and $(d \otimes 1)^2 = 0$ show that we didn't spoil the vanishing of ψ_{diff}.

5.5. Fourth step ($i = 4$), solving the hexagons.

Lemma 5.5. *ψ_3 is totally antisymmetric: if σ is a permutation of $\{1, 2, 3\}$, then $\psi_3^\sigma = (-1)^\sigma \psi_3$.*

Proof. The vanishing of $\psi_{\mathit{diff},3}$ implies that $\psi_3 = \psi_{+,3} = \psi_{-,3}$ and thus proposition 5.2 implies that $\psi_3^{213} = -\psi_3$, and it only remains to show that $\psi_3^\sigma = \psi_3$ for some 3-cycle σ. Rewrite equation (18) in terms of ψ_3, and use $\psi_3^{213} = -\psi_3$ on the middle two terms. The resulting equation is $2\psi_3 - 2\psi_3^{231} = 0$. $\square$

Now set $r_4 = 0$ and $\varphi_4 = -\psi_3/3$; namely, $R_{m,4} = R_{m,3}$ and $\Phi_{m,4} = \Phi_{m,3} - \psi_3/3$. Equation (14) and the total antisymmetry of ψ_3 imply that $\psi_4 = 0$. We did not touch ψ_{diff}, and so $\psi_{\pm,4} = 0$; namely, the hexagons $\hexagon_\pm$ hold for $(R_{m,4}, \Phi_{m,4})$ (modulo $\mathcal{R}_m\mathbf{AP}_3$). The antisymmetry of ψ_3 also implies $||\Phi_{m,4}||^2 = ||\Phi_{m,3}||^2 = 1$ (modulo $\mathcal{R}_m\mathbf{AP}_3$). Notice that $\Phi_{m,1}$ is still non-degenerate.

5.6. Fifth step ($i = 5$), solving the pentagon.

Lemma 5.6. $\mu_4 \in Z^4_{Harr}$.

Proof. Now that $\psi_{\pm,4}$ vanish, equations (16) and (17) say that $\text{ш}_{31}(\mu_4) = \text{ш}_{22}(\mu_4) = 0$. $\text{ш}_{31}(\mu_4) = 0$ and $\mu_4 + \mu_4^T = 0$ together imply that $\text{ш}_{13}(\mu_4)$ is also 0, and so $\mu_4 \in C^4_{Harr}$. It only remains to recall equation (19): $d\mu_4 = 0$. $\square$

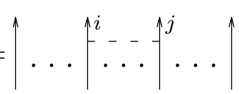
Now use theorem 5 and remark 4.9 to find a non-degenerate $\varphi_5 \in C^3_{Harr}$ with $d\varphi_5 = -\mu_4$, and set $R_{m,5} = R_{m,4}$ and $\Phi_{m,5} = \Phi_{m,4} + \varphi_5$. Equation (12) implies that the pentagon $\pentagon$ now holds, and equation (14) together with $\text{ш}_{21}\varphi_5 = 0$ imply that the hexagons still hold. Finally, as both hexagons now hold up to and including degree m, proposition 3.7 implies that $||\Phi_{m,5}||^2 = 1 \mod \mathcal{R}_m\mathbf{AP}_3$, and we are ready to go back to step 1 with m replaced by $m + 1$.

6. Odds and ends

6.1. Horizontal chords. The original R and Φ, constructed from the KZ equation as indicated in section 1, contain only horizontal chords. One may hope to have an inductive procedure, similar to the one in section 5, to combinatorially construct such horizontal-chords-only R and Φ. The first step is clear:

Definition 6.1. Let $\mathbf{AP}^{hor}_n$ be the associative algebra generated by formal symbols t^{ij}, $1 \le i, j \le n$, modulo the relations $t^{ij} = t^{ji}$, $[t^{ij}, t^{kl}] = 0$ when $|\{i, j, k, l\}| = 4$ and $[t^{ij} + t^{ik}, t^{jk}] = 0$ when $|\{i, j, k\}| = 3$.

Thinking of t^{ij} as a horizontal chord connecting the ith and the jth vertical strands,

$t^{ij} =$ 

the relations between the t^{ij} become the $4T$ relations of [2], and so the natural maps $\mathbf{AP}^{hor}_n \to \mathbf{AP}_n$ are well defined. Both locality in space and in scale still hold in $\mathbf{AP}^{hor}$: the former follows from the relation $[t^{ij}, t^{kl}] = 0$ (for appropriate i–l), and the latter from the same argument as in the proof of lemma 3.4, only this time using the $4T$ relation instead of IHX and STU. The algebras $\mathbf{AP}^{hor}_n$ are graded, and so we may hope that the same inductive procedure as in section 5 would work for $\mathbf{AP}^{hor}$ as well. It does, with the only exception being that we don't know how to prove theorem 5 in this case.

Conjecture 1. *(Suggested by Drinfel'd's* [9, remark 2 following proposition 5.8]*) With the obvious definition, $\mathcal{G}_m H^4_{Harr}(\boldsymbol{AP}^{hor}) = 0$ for all m.*

Another variation is to define $\mathbf{AP}_n^{lie}$ in the same way as $\mathbf{AP}_n^{hor}$, only replacing the words "associative algebra" by "lie algebra", and to define $\mathbf{AP}_n^{Lie}$ to be the group generated by the exponentials of elements of $\mathbf{AP}_n^{lie}$. It is easy to check that everything carries through in section 5, and so if we only knew that $\mathcal{G}_m H^4_{Harr}(\mathbf{AP}^{lie}) = 0$ for all m, we could combinatorially construct a solution (R, Φ) in $\mathbf{AP}^{Lie}$. The KZ pair (R, Φ) has that property.

I was able to check conjecture 1 for $m \leq 5$ on a computer, and (as section 7 shows) compute a pair (R_7, Φ_7) in $\mathbf{AP}^{Lie}$ up to degree 7. Using transcendental methods (i.e., the KZ equation), Drinfel'd [9] shows that it should be possible to extend my (or any other partial solution) (R_7, Φ_7) to a solution that works in all degrees.

Notice that if R is in $\mathbf{AP}_2^{hor}$, life is made considerably simpler; step 3 in section 5 becomes unnecessary, and if one reduces modulo diagrams having an isolated chord (the additional relation in $\mathcal{A}^r$), C (of equation (7)) becomes the identity 1, and the reduced $\mathbf{Z^{PFT}}$ satisfies (R9) automatically.

6.2. The Commuto-Associahedrons. The polyhedra displayed in figures 4–7 make it tempting to make the following definition:

Definition 6.2. The nth Commuto-Associahedron CA_n is the two dimensional CW complex made of the following cells:

0-cells: All possible products of n elements $a_1, \dots, a_n$ in a non associative non commutative algebra. In other words, all parenthesized strings on the alphabet $\{1, \dots, n\}$ in which every letter appears exactly once.

1-cells: All basic associativities and commutativities. These correspond to generating morphisms of **NAB** in which the orientations of strands and the signs of crossings are disregarded.

2-cells: – A pentagon sealing every pentagon of the kind appearing in relation (R2).
- A hexagon sealing every hexagon of the kind appearing in relation (R6), just disregarding the difference between overcrossings and undercrossings.
- A square sealing every elementary locality relation of the type considered in (R3) and (R4) (again disregarding the difference between overcrossings and undercrossings). Notice that every elementary locality relation can be written as a product of four morphisms, and so it corresponds to a square in the 1-skeleton of CA_n.

For an example, see figure 8.

By the Mac Lane coherence theorem, the commuto-associahedrons are all connected and simply connected, and thus their 2^{nd} homology can be computed by simply counting cells and computing the Euler characteristic. This done, one finds that that CA_{3-5} have no 2^{nd} homology other than that displayed in figures 4–7 (with $a, b, c, \dots$ an arbitrary permutation of $1, 2, 3, \dots$) and other than classes that look like the shell of a prism with basis a square, a pentagon, or a hexagon. The latter

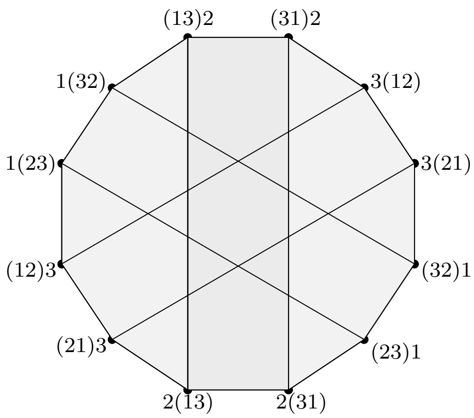


Figure 8. The third commuto-associahedron CA_3. It is made by gluing three sets of a square and two hexagons each into a 12-gon. Only one of these sets is shaded in the figure; the other two are obtained from it by rotations by 60° and 120° respectively. Topologically, the result is a circle with three disks glued in, and has the homotopy type of a wedge of two spheres.

classes can also be used to obtain relations between μ and ψ, but the relations thus obtained turn out to be not interesting — they all turn out to be consequences of lemma 3.4. In other words, in section 5 we've used *all* the relations between μ and ψ that we could generate using our technique.

Notice, however, that figures 4–7 contain a little more information than just a class in $H_2(CA_n)$ for some n; they also contain information about the the orientations of the crossings involved (disguised as the orientations of the 1-cells marked by a variant of R). It is therefore not a-priori clear that every class in $H_2(CA_n)$ corresponds to a relation between μ and ψ; even the distinction between ψ_+ and ψ_- is lost in the commuto-associahedrons.

Notice also that our definition is somewhat different than the 2-skeletons of Kapranov's permutoassociahedrons [11]. For example, his KP_3 is just a planar 12-gon, as opposed to our CA_3 which is a wedge of two spheres. I do not know if there is a natural way to add higher dimensional cells to our CA_n in a way similar to [11].

7. Some computations

Starting up mathematica [29], loading a definitions file (available at [4]) and testing the $4T$ relation:

```
Mathematica 2.2 for SPARC
Copyright 1988-93 Wolfram Research, Inc.
 -- Open Look graphics initialized --
```

In[1]:= `<< NAT.m`

In[2]:= `t[1,3]**t[1,2] + t[1,3]**t[2,3] - t[2,3]**t[1,3]`

Out[2]= $t^{12}t^{13}$

Basing the induction and setting $\Phi_{m,1}$ to be as in (20):

In[3]:= `{R[1]=1+t[1,2]/2, Phi[1]=1}`

Out[3]= $\{1 + \frac{t^{12}}{2}, 1\}$

In[4]:= `R[m_,0] := R[m-1]; Phi[m_,0] := Phi[m-1]`

In[5]:=
```
R[m_,1] := R[m,0]; Phi[m_,1] := Phi[m,1] =
  ModDegree[m+1,Phi[m,0]-Expand[(-1+NormSquared[Phi[m,0]])/2]]
```

Next, define R^{-1}, Φ^{-1}, $\psi_\pm$, ψ, ψ_{diff} as in section 5. Notice that `Act[{1,2},3][R]` means $(\Delta \otimes 1)(R)$ ("the first strand becomes the pair `{1,2}` and the second becomes number `3`"), and that `Act[3,1,2][Phi]` means Φ^{312} ("first becomes number `3`, second becomes number `1`, third becomes number `2`"):

In[6]:= `RInverse[m_,i___] := RInverse[m,i]=ModDegree[m+1,Invert[R[m,i]]]`

In[7]:= `PhiInverse[m_,i___] := PhiInverse[m,i]=ModDegree[m+1,Invert[Phi[m,i]]]`

In[8]:=
```
psip[m_,i_] := ModDegree[m+1, -1 + Act[{1,2},3][RInverse[m,i]]**
  Phi[m,i]**Act[2,3][R[m,i]]**Act[1,3,2][PhiInverse[m,i]]**
  Act[1,3][R[m,i]]**Act[3,1,2][Phi[m,i]]]
```

In[9]:=
```
psim[m_,i_] := ModDegree[m+1, -1 + Act[{1,2},3][R[m,i]]**Phi[m,i]**
  Act[2,3][RInverse[m,i]]**Act[1,3,2][PhiInverse[m,i]]**
  Act[1,3][RInverse[m,i]]**Act[3,1,2][Phi[m,i]]]
```

In[10]:= `psi[m_,i_]:=Expand[(psip[m,i]+psim[m,i])/2]`

In[11]:= `psidiff[m_,i_]:=Expand[(psip[m,i]-psim[m,i])/2]`

Setting $R_{m,2}$ as in section 5.3, and computing $R_{2,2}$ and $\Phi_{2,2}$: (the `OX` here means $\otimes$)

In[12]:=
```
R[m_,2] := R[m,2]=ModDegree[m+1, R[m,1] - {APPower[t[1,2],m]}.
  (psidiff[m,1]~InTermsOf~{(d[1]~OX~1)[APPower[t[1,2],m]]})]
```

In[13]:= `Phi[m_,2] := Phi[m,1]`

In[14]:= `{R[2,2],Phi[2,2]}`

Out[14]= $\{1 + \frac{t^{12}t^{12}}{8} + \frac{t^{12}}{2}, 1\}$

As we are dealing with diagrams that have only horizontal chords, R is always symmetric and step 3 is superfluous. Here we follow step 4 as in section 5.5 and compute $R_{2,4}$ and $\Phi_{2,4}$:

In[15]:= `R[m_,3] := R[m,2]; Phi[m_,3] := Phi[m,2]`

In[16]:= `R[m_,4] := R[m,3]; Phi[m_,4] := Phi[m,4]=Phi[m,3]-Expand[psi[m,3]/3]`

In[17]:= `{R[2,4],Phi[2,4]}`

Out[17]= $\{1 + \frac{t^{12}t^{12}}{8} + \frac{t^{12}}{2}, 1 + \frac{t^{13}t^{23}}{24} - \frac{t^{23}t^{13}}{24}\}$

Defining μ as in (10), $\Phi_{m,5}$ as in section 5.6, and computing (R_3, Φ_3):

In[18]:= `mu[m_,i_] := mu[m,i]=ModDegree[m+1, -1 + Phi[m,i]**`

```
  Act[1,{2,3},4][Phi[m,i]]**Act[2,3,4][Phi[m,i]]**
  Act[1,2,{3,4}][PhiInverse[m,i]]**Act[{1,2},3,4][PhiInverse[m,i]]]
```

In[19]:=
```
R[m_,5] := R[m,4]; Phi[m_,5] := Phi[m,5]=Phi[m,4] + Expand[
  HarrisonBasis[3,m].((-mu[m,4])~InTermsOf~(d[3] /@ HarrisonBasis[3,m]))]
```

In[20]:=
```
R[m_] := R[m,5]; Phi[m_] := Phi[m,5]
```

In[21]:=
```
{R[3],Phi[3]}
```

Out[21]= $\{1 + \frac{t^{12}t^{12}}{8} + \frac{t^{12}t^{12}t^{12}}{48} + \frac{t^{12}}{2}, 1 + \frac{t^{13}t^{23}}{24} - \frac{t^{23}t^{13}}{24}\}$

Just for fun, checking that $\mu_4 \neq 0$ but $d^4\mu_4 = 0$ in degree 4:

In[22]:=
```
{mu[4,4]===0, d[4][mu[4,4]]===0}
```

Out[22]= $\{False, True\}$

Note that $R = \exp \uparrow\!\!-\!\!\uparrow/2 = \exp t^{12}/2$, at least up to degree 4:

In[23]:=
```
ModDegree[5,FormalLog[R[4]]]
```

Out[23]= $\frac{t^{12}}{2}$

$\log \Phi_4$ is nothing as simple. To avoid ugly denominators, we need to multiply it by 5760:

In[24]:=
```
Expand[5760(LogPhi4=ModDegree[5,FormalLog[Phi[4]]])]
```

Out[24]= $240\,t^{13}t^{23} - 240\,t^{23}t^{13} - 4\,t^{13}t^{13}t^{13}t^{23} + 12\,t^{13}t^{13}t^{23}t^{13} + 7\,t^{13}t^{13}t^{23}t^{23} - 12\,t^{13}t^{23}t^{13}t^{13} - 14\,t^{13}t^{23}t^{13}t^{23} - 7\,t^{13}t^{23}t^{23}t^{23} + 4\,t^{23}t^{13}t^{13}t^{13} + 14\,t^{23}t^{13}t^{23}t^{13} + 21\,t^{23}t^{13}t^{23}t^{23} - 7\,t^{23}t^{23}t^{13}t^{13} - 21\,t^{23}t^{23}t^{13}t^{23} + 7\,t^{23}t^{23}t^{23}t^{13}$

But it belongs to the Lie algebra generated by t^{13} and t^{23}, and has a reasonable formula in terms of commutators:

In[25]:=
```
basis=EvenLieBasis[4,{t[1,3],t[2,3]}]
```

Out[25]= $\{[t^{13}, t^{23}], [t^{13}, [t^{13}, [t^{13}, t^{23}]]], [t^{13}, [t^{23}, [t^{13}, t^{23}]]], [t^{23}, [t^{23}, [t^{13}, t^{23}]]]\}$

In[26]:=
```
(LogPhi4~InTermsOf~LieExpand[basis]).basis
```

Out[26]= $\frac{-[t^{13}, [t^{13}, [t^{13}, t^{23}]]]}{1440} - \frac{7\,[t^{13}, [t^{23}, [t^{13}, t^{23}]]]}{5760} + \frac{[t^{13}, t^{23}]}{24} - \frac{7\,[t^{23}, [t^{23}, [t^{13}, t^{23}]]]}{5760}$

Notice that we did not make any effort to find a *non-degenerate* Φ, but luckily, it just came out that way.

Define the correction term $Z(\infty)$ as in section 3.3, define its inverse, and compute the invariant of the unknot (in $\mathcal{A}^r$):

In[27]:=
```
?AP2CD
```

AP2CD[ord,expr] takes an element of the algebra AP_n and closes it to a linear combination of chord diagrams in A^r by traveling around the strands of

AP_n in the order specified by ord, going up at first and then down, up, down, ...

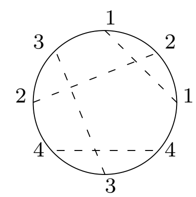
In[27]:= `ZInfinity[m_] := ZInfinity[m] = AP2CD[{3,2,1},Phi[m]]`

In[28]:= `ZInfinityInverse[m_] :=`
`ZInfinityInverse[m]=ModDegree[m+1,Invert[ZInfinity[m]]]`

In[29]:= `ZInfinityInverse[4]`

Out[29]= $$1-\frac{[1212]}{24}+\frac{[12123434]}{5760}+\frac{[12132434]}{960}-\frac{[12314234]}{5760}$$

The notation [12132434] means: write the sequence $1, 2, 1, 3, 2, 4, 3, 4$ counterclockwise around a circle and connect by chords any pair of equal numbers, as shown on the right.



Just for fun, let's evaluate the result for the unknot using the HOMFLY weight system (which, using standard notation, depends on the N of $SL(N)$ and on $h = \log q$), and let us compare it with the well known value of the HOMFLY polynomial on the unknot:

In[30]:= `Expand[WHOMFLY[N,h][ZInfinityInverse[4]]]`

Out[30]= $$N-\frac{h^2\,N}{24}+\frac{7\,h^4\,N}{5760}+\frac{h^2\,N^3}{24}-\frac{h^4\,N^3}{576}+\frac{h^4\,N^5}{1920}$$

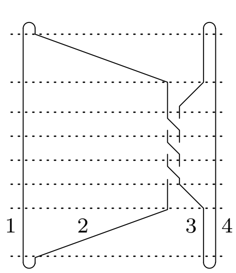
In[31]:= `z=(Exp[h N/2]-Exp[-h N/2])/(Exp[h/2]-Exp[-h/2])`

Out[31]= $$\frac{-e^{\frac{-(h\,N)}{2}}+e^{\frac{h\,N}{2}}}{-e^{\frac{-h}{2}}+e^{\frac{h}{2}}}$$

In[32]:= `Series[z,{h,0,4}]`

Out[32]= $$N+\left(\frac{-N}{24}+\frac{N^3}{24}\right)\,h^2+\left(\frac{7\,N}{5760}-\frac{N^3}{576}+\frac{N^5}{1920}\right)\,h^4+O(h)^5$$

Computing the invariant of the trefoil:



In[33]:=
```
ModDegree[5,AP2CD[{1,3,4,2},
  Act[1,2,{3,4}][Phi[4]]**
  Act[2,3,4][PhiInverse[4]]**
  Act[2,3][RInverse[4]]**
  Act[2,3][RInverse[4]]**
  Act[2,3][RInverse[4]]**
  Act[3,2,4][Phi[4]]**
  Act[1,3,{2,4}][PhiInverse[4]]]**
  ZInfinityInverse[4]**ZInfinityInverse[4]]
```

Notice the conflicting conventions — tangles are composed from bottom to top, while mathematica multiplies from top to bottom.

Out[33]= $$1+\frac{23\,[1212]}{24}-[121323]-\frac{1199\,[12123434]}{5760}+\frac{241\,[12132434]}{960}+\frac{1199\,[12314234]}{5760}$$

Let us now observe that this paper does not prove the inconsistency of mathematics. Recomputing the invariant of the trefoil, this time using the presentation in page 4, we get the same answer as before:

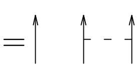
In[34]:=
```
ModDegree[5,AP2CD[{1,3,2,4}, PhiInverse[4]**Act[{1,2},3,4][Phi[4]]**
  RInverse[4]**Act[3,4][RInverse[4]]**Act[2,1,{3,4}][Phi[4]]**
  Act[1,4,3][PhiInverse[4]]**Act[1,4][R[4]]**Act[4,1,3][Phi[4]]]**
  ZInfinityInverse[4]**ZInfinityInverse[4]]
```

Out[34]= $$1+\frac{23\,[1212]}{24}-[121323]-\frac{1199\,[12123434]}{5760}+\frac{241\,[12132434]}{960}+\frac{1199\,[12314234]}{5760}$$

Finally, typing `ModDegree[8,{FormalLog[R[7]],FormalLog[Phi[7]]}]`, waiting a long time and making some effort to typeset the result in a reasonable form, we find that $\log R = \frac{t^{12}}{2}$ up to degree 7, and that up to the same degree,

$$\log\Phi = \left(\frac{[ab]}{48} - \frac{8[aaab]+[abab]}{11520} + \frac{96[aaaaab]+4[aaabab]+65[aabbab]+68[abaaab]+4[ababab]}{5806080}\right) - (\text{interchange } a \leftrightarrow b)\,,$$

where $[a_1 \dots a_m]$ is a short for the iterated bracket $[a_1,[a_2,\dots,[a_{m-1},a_m]\dots]]$, and

$a = t^{12} =$  ; $b = t^{23} =$  .

References

1. D. Altschuler and A. Coste, *Quasi-quantum groups, knots, three-manifolds, and topological field theory*, Commun. Math. Phys. **150** (1992) 83–107.
2. D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.
3. ———, *Vassiliev homotopy string link invariants*, Jour. of Knot Theory and its ramifications **4** (1995) 13–32.
4. ———, *Computer data files*, available via anonymous file transfer from `ftp.math.harvard.edu`, user name `ftp`, subdirectory `dror`. Read the file `README` first. For easier access, point your WWW browser at `http://www.math.harvard.edu/HTML/Individuals/Dror\_Bar-Natan.html`.
5. N. Bergeron, *What is new about Hochschild homology*, Harvard Univ. preprint, November 1993.
6. J. S. Birman and X-S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993) 225–270.
7. P. Cartier, *Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds*, C. R. Acad. Sci. Paris **316** Série I (1993) 1205–1210.
8. V. G. Drinfel'd, *Quasi-Hopf algebras*, Leningrad Math. J. **1** (1990) 1419–1457.
9. ———, *On quasitriangular Quasi-Hopf algebras and a group closely connected with Gal*($\bar{\mathbb{Q}}/\mathbb{Q}$), Leningrad Math. J. **2** (1991) 829–860.

10. D. K. Harrison, *Commutative algebras and cohomology*, Trans. Amer. Math. Soc. **104** (1962) 191–204.
11. M. M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure and Appl. Alg., **85** (1993) 119–142.
12. C Kassel, *Quantum groups*, Springer-Verlag GTM **155**, New York 1994.
13. L. H. Kauffman, *On knots*, Princeton Univ. Press, Princeton, 1987.
14. M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., **16(2)** (1993), 137–150.
15. T. Q. T. Le and J. Murakami, *On Kontsevich's integral for the HOMFLY polynomial and relations of multiple ζ-numbers*, Top. and its Appl., to appear.
16. ———, *Kontsevich integral for Kauffman polynomial*, Max-Planck-Institut Bonn preprint 93-33, March 1993.
17. ———, *Representation of tangles and Kontsevich's integral*, Max-Planck-Institut Bonn preprint, April 1993.
18. ———, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Max-Planck-Institut Bonn preprint, December 1993.
19. J. L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives* Inv. Math., **96** (1989) 205–230.
20. ———, *Cyclic Homology*, Springer-Verlag, New-York 1992.
21. S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag GTM **5**, New-York 1971.
22. M. Markl and J. D. Stasheff, *Deformation theory via deviations*, preprint.
23. N. Yu. Reshetikhin, *Quasitriangle Hopf algebras and invariants of tangles*, Leningrad Math. J. **1** (1990) 491–513.
24. S. Piunikhin, *Combinatorial expression for universal Vassiliev link invariant*, Commun. Math. Phys. **168-1** 1–22.
25. ———, *Addendum to the paper "Combinatorial expression for universal Vassiliev link invariant"*, November 1993, preprint.
26. S. Shnider and S. Sternberg, *Quantum groups — from coalgebras to Drinfel'd algebras*, International Press, Cambridge MA 1994.
27. V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (Providence) (V. I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.
28. ———, *Complements of discriminants of smooth maps: topology and applications*, Trans. of Math. Mono. **98**, Amer. Math. Soc., Providence, 1992.
29. S. Wolfram, *Mathematica — a system for doing mathematics by computer*, Addison-Wesley, 1989.

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: dror@math.harvard.edu