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PERTURBATIVE EXPANSION OF CHERN-SIMONS THEORY WITH NON-COMPACT GAUGE GROUP

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ABSTRACT

Naive imitation of the usual formulas for compact gauge group in quantizing three dimensional Chern-Simons gauge theory with non-compact gauge group leads to formulas that are wrong or unilluminating. In this paper, an appropriate modification is described, which puts the perturbative expansion in a standard manifestly "unitary" format. The one loop contributions (which differ from naive extrapolation from the case of compact gauge group) are computed, and their topological invariance is verified.

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1. Introduction

In evaluating Feynman diagrams in gauge theories, one encounters the Casimir invariants of the gauge group G and of whatever matter representations may be present. In conventional Yang-Mills theory with the usual F^2 action, the Feynman diagrams depend on G only through the values of these Casimirs. One might expect that the same would be true in three dimensional gauge theory with the pure Chern-Simons action. We consider a G bundle E, with connection A, over an oriented manifold M. The Chern-Simons functional is†

$$I(A) = \frac{1}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{1.1}$$

and the Lagrangian is

$$L = -ikI(A) \tag{1.2}$$

with k an integer.‡ For compact G, this theory is closely related to two dimensional current algebra [1]. As long as G is compact, it is quite true that in the perturbative expansion of (1.2), one "sees" only the Casimir invariants of G. One is also interested, however, in Chern-Simons theory for non-compact G, in part because of the relation to three dimensional quantum general relativity [3,4]. In this paper, we will actually only consider the case of semi-simple G.

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† For $G = SU(N)$ or any other real form thereof, we take Tr to be the trace in the N dimensional representation. For any simple and simply connected G, we let Tr be the smallest positive multiple of the trace in the adjoint representation such that the right hand side of the following expression, at $k = 1$, is well-defined as a map to $\mathrm{R}/2\pi\mathrm{Z}$. One uses this definition of Tr as a quadratic form on the Lie algebra $\mathcal{G}$ for any group, not necessarily simply connected, with this Lie algebra, and also for non-compact forms of such groups. See [2] for a more critical discussion of the definition of the action for non-simple and non-simply-connected G.

‡ In Euclidean quantum field theory, which is our point of view in this paper, L must be mapped to $\overline{L}$ under reversal of orientation. This is the reason for the i in L. The argument of the Feynman path integral is then e^{-L}. In Lorentzian quantum field theory, which was the viewpoint of [1] because of the emphasis on the canonical formulation in that paper, L is real and the argument of the path integral is e^{iL}. In Chern-Simons theory, because of the absence of dependence on a metric, the two viewpoints are obviously equivalent.

It is easy to see, on a variety of grounds, that if G is not compact, the perturbative expansion of (1.2) cannot be obtained by simply borrowing the answers one obtains for compact groups and plugging in the appropriate values of the Casimirs. We will explain this qualitatively here and then more precisely in §3-4. In the relation of (1.2) to two dimensional physics, one gets right-moving G current algebra in two dimensions if k is positive and left-moving G current algebra if k is negative.§ This can be better expressed by saying that if the quadratic form $-k \cdot \mathrm{Tr}$ on $\mathcal{G}$ is positive, one gets right-movers from (1.2), while if it is negative, one gets left-movers.¶ What can be the generalization of this for non-compact groups, where $-k \cdot \mathrm{Tr}$ is indefinite? Obviously, components of A for which the quadratic form is positive must correspond to right-movers in two dimensions and components for which it is negative must correspond to left-movers.

For instance, in the one loop computation with compact gauge groups, one gets a "framing anomaly" [1], which will be recalled in §3-4, which is proportional to $\dim G$ if k is positive and to $-\dim G$ if k is negative. (The framing anomaly is proportional to the *difference* of the right- and left-moving central charges of the associated two dimensional conformal field theory.) The function which (for compact groups) is $\dim G$ for $k > 0$ and $-\dim G$ for $k < 0$ is the *signature* of the quadratic form $-k \cdot \mathrm{Tr}$, and one must expect this signature to arise in the general case of a non-compact gauge group. If we write $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, where $\mathcal{K}$ is the Lie algebra of a maximal compact subgroup K of G and $\mathcal{P}$ is the orthocomplement, then the signature of $-k\,\mathrm{Tr}$ (for $k > 0$) is $\dim \mathcal{K} - \dim \mathcal{P}$. This is $N^2 - 1$ for $SU(N)$ and $1 - N$ for the alternative real form $SL(N, \mathrm{R})$, showing that one cannot expect to get the correct answers for the non-compact groups by simple analytic continuation.

Similarly, (as will be reviewed in §3-4) the one loop contributions shift the effective value of k in certain expressions from k to $k + h$ if k is positive but to

§ The notions of "right" and "left" depend on a choice of orientation, of course; such a choice has been made in order to integrate the three-form in (1.1).

¶ We consider the Lie algebra generators for compact groups to be skew symmetric matrices, so that the quadratic form $(a, b) = -\,\mathrm{Tr}\; ab$ is positive definite for compact groups.

$k - h$ if k is negative,* where $h = c_2(\mathcal{G})/2$.** As right- and left-movers evidently make positive and negative contributions, respectively, to the shift in k, one must expect that the generalization of $c_2(\mathcal{G})/2$ for a non-compact semi-simple gauge group is $(c_2(\mathcal{K}) - c_2(\mathcal{P}))/2$ (where $\mathcal{K}$ and $\mathcal{P}$ are regarded as representations of K). This is N for $SU(N)$ and -2 for the real form $SL(N, \mathrm{R})$, showing again that the correct results for the non-compact groups cannot be obtained from those for the compact groups by simple analytic continuation.

The above arguments may seem heuristic, but the conclusions can be verified in case G is a complex Lie group, say $G = K_{\mathrm{C}}$ is the complexification of its maximal compact subgroup K, by comparing to the Hamiltonian formulation of the theory [5]. The framing anomaly shows up in that formulation as the central curvature (of the projectively flat quantum connection that plays a pivotal role in the Hamiltonian formulation), and the large k limit of this vanishes for complex Lie groups according to equation (4.24) or (4.42) of [5]. The shift in the effective value of k shows up in the Hamiltonian formulation as a deviation from the classical value of the coefficient of the leading (second order) part of the connection form, and this vanishes according to equation (4.9) of [5]. These facts are in agreement with the above heuristic considerations, since for complex Lie groups one has $\dim \mathcal{K} = \dim \mathcal{P}$ (so that the signature of $-k\,\mathrm{Tr}$ is zero) and $c_2(\mathcal{K}) = c_2(\mathcal{P})$ (since $\mathcal{K}$ and $\mathcal{P}$ are isomorphic as representations of K). It is not at present possible to compare the results of this paper to the Hamiltonian formulation for non-compact groups other than the complex Lie groups, since the Hamiltonian formulation for such groups is not adequately understood. Part of the motivation for the present paper is precisely to obtain the perturbative formulas to which an eventual Hamiltonian

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* To avoid a frequent misunderstanding, let us note that k is defined in the physical, renormalized theory by saying that the physical wave functions are sections of the k^{th} power of a certain line bundle over the classical phase space. With this definition of the meaning of k, the one loop corrections shift the effective value of k in certain expressions from k to $k \pm h$.

** c_2 is the quadratic Casimir operator, normalized so that for any representation $\mathcal{R}$, and any $a, b \in \mathcal{G}$, $c_2(\mathcal{R}) = \mathrm{Tr}_{\mathcal{R}}(ab)/\,\mathrm{Tr}(ab)$. Here $\mathrm{Tr}_{\mathcal{R}}$ is the trace in the $\mathcal{R}$ representation, while Tr is the trace in the N dimensional representation for $G = SU(N)$ and in general is defined in a footnote above.

treatment can be compared.

2. The Standard Gauge Fixing

Before tackling more exotic cases, we first recall the standard gauge fixing in the case of a compact gauge group. The Fadde'ev-Popov ghost is an anticommuting zero form c in the adjoint representation of the gauge group. We write $c = \sum_a c^a T_a$, where T_a is a basis of the Lie algebra with $[T_b, T_c] = f^a{}_{bc} T_a$. The standard BRST transformation laws are

$$\delta A_i{}^a = -D_i c^a = -\left(\partial_i c^a + f^a{}_{bc} A_i{}^b c^c\right), \qquad \delta c^a = \frac{1}{2} f^a_{bc} c^b c^c. \tag{2.1}$$

One also introduces antighosts and auxiliary fields, which are adjoint-valued three forms $\bar{c}$ and ϕ, respectively anticommuting and commuting, with

$$\delta \bar{c} = i\phi, \;\; \delta\phi = 0. \tag{2.2}$$

The gauge-fixed Lagrangian is

$$L' = L - \delta V, \tag{2.3}$$

where L is the classical Lagrangian of (1.2), and V is any judiciously chosen functional so that (2.3) is non-degenerate. In constructing perturbation theory, one wishes to expand about some background connection $A^{(0)}$, which is naturally taken to be a flat connection, that is, a solution of the classical equations of motion, or in other words a critical point of the functional integral. To simplify life, we will assume that $A^{(0)}$ is isolated and irreducible, so that we need not introduce collective coordinates and the operators encountered below have no zero modes. The exterior derivative twisted by an arbitrary background connection A will be called

D, and that twisted by $A^{(0)}$ will be called $D^{(0)}$. V is permitted to depend on the choice of $A^{(0)}$. We write

$$A = A^{(0)} + B, \tag{2.4}$$

where B is a one form with values in the adjoint representation. We will work out the leading perturbative approximation to the contribution of a particular critical point to the path integral; the final result is obtained by summing over critical points.

Much of the characteristic flavor of the subject comes from the fact that (as is clear from the existence of the framing anomaly that we will soon recall) even after fixing $A^{(0)}$ there is no choice of V that preserves the symmetries of the problem. The standard procedure involves picking a metric g on M, which determines a Hodge duality operator $*$[⋆] and a Riemannian measure μ, and setting

$$V = \frac{k}{2\pi}\int\limits_M d\mu \;\; \mathrm{Tr}\, \overline{c} * D^{(0)} * B. \tag{2.5}$$

This leads to

$$\delta V = \frac{k}{2\pi}\int\limits_M \;\mathrm{Tr}\left(i\phi * D^{(0)} * B - \bar{c} * D^{(0)} * Dc\right). \tag{2.6}$$

We want to study the one loop approximation to the path integral. To this end, we need only the terms in L' that are at most quadratic in $B, c, \overline{c}$, and ϕ. As $A^{(0)}$ is a critical point, there is no linear term, but there is a zeroth order term and a quadratic term. These are

$$-ikI(A^{(0)}) + \frac{k}{2\pi}\int\limits_M \;\mathrm{Tr}\;\left(-\frac{i}{2}B \wedge D^{(0)}B - i\phi * D^{(0)} * B + \bar{c} * D^{(0)} * D^{(0)}c\right), \tag{2.7}$$

⋆ $*$ is the usual operator mapping q forms to $3-q$ forms, with sign conventions fixed by requiring that the metric of equation (2.9) below is positive. It obeys $*^2 = 1$.

or equivalently

$$-ikI(A^{(0)}) - \frac{1}{2}\int\limits_M \mathrm{Tr}\ \left(iB' \wedge D^{(0)}B' + 2i\phi' * D^{(0)} * B'\right) + \int\limits_M \mathrm{Tr}\left(\vec{c}' D^0 * D^{(0)} c'\right), \tag{2.8}$$

where $B' = B\sqrt{k/2\pi}$, $c' = c\sqrt{k/2\pi}$, $\phi' = \phi\sqrt{k/2\pi}$, and $\vec{c}' = *\bar{c}\sqrt{k/2\pi}$. (Thus, $\vec{c}'$ is an adjoint-valued zero form.)

The path integral can be carried out in a fairly standard way. Let Ω^q be the space of adjoint-valued q forms. On Ω^q one introduces the metric

$$(u, v) = -\int\limits_M d\mu\, \mathrm{Tr}\, \bar{u} \wedge *v. \tag{2.9}$$

Let $\Omega_+ = \Omega^0 \oplus \Omega^2$ and $\Omega_- = \Omega^1 \oplus \Omega^3$. We can combine B' and ϕ' to a field $H = (B', \phi')$ valued in Ω_-, while $c', \vec{c}'$ are valued in Ω^0. The quadratic part of the action (2.8) can be written

$$-ikI(A^{(0)}) + \frac{i}{2}(H, L_- H) - (\vec{c}', \Delta_0 c') \tag{2.10}$$

where $\Delta_0 = *D^{(0)} * D^{(0)}$ is the standard Laplacian on adjoint-valued zero forms, and $L_- = (*D^{(0)} + D^{(0)}*)J$ is a standard elliptic operator on Ω_- that enters in the Atiyah-Patodi-Singer theorem about the eta invariant [6]. Here $J\phi$ is defined to be $-\phi$ if ϕ is a zero- or a three-form, while $J\phi = \phi$ if ϕ is a one- or a two-form.

We wish to carry out the path integral

$$\int DH\, D\vec{c}'\, Dc'\ \exp\left(-\frac{i}{2}(H, L_- H) + (\vec{c}', \Delta_0 c')\right). \tag{2.11}$$

This may be done in a standard way, introducing the orthonormal eigenfunctions of Δ_0 and L_-,

$$\Delta_0 \psi_i = u_i \psi_i, \quad (\psi_i, \psi_j) = \delta_{ij} \tag{2.12}$$

and similarly

$$L_- \chi_i = v_i \chi, \quad (\chi_i, \chi_j) = \delta_{ij}. \tag{2.13}$$

We write $c' = \sum_i c_i \psi_i$, $\bar{c}' = \sum_i \bar{c}_i \overline{\psi}_i$, and $H = \sum_i h_i \chi_i$, and we interpret the path integral measures to be

$$D\bar{c}' Dc' = \prod_i d\bar{c}_i dc_i \tag{2.14}$$

and

$$DH = \prod_i \frac{dh_i}{\sqrt{2\pi}}. \tag{2.15}$$

The one loop approximation to the path integral is hence⋆

$$Z = \sum_{A^{(0)}} \frac{1}{\#Z(G)} e^{ikI(A^{(0)})} \int \prod_i \frac{dh_i}{\sqrt{2\pi}} \prod_j d\bar{c}_j dc_j \exp\left(\frac{-i}{2} \sum_i v_i h_i^2 + \sum_j u_j \bar{c}_j c_j \right) \tag{2.16}$$

where we write explicitly the sum over critical points. Using the basic Fermi integral

$$\int d\bar{c} dc \exp(u\bar{c}c) = u, \tag{2.17}$$

and the analogous Bose integral†

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ivx^2/2} = \frac{e^{-i\pi \operatorname{sign}(v)/4}}{\sqrt{|v|}}, \tag{2.18}$$

⋆ We include here a factor of $\#Z(G)$, the order of the center of G. This is the order of the stabilizer of an irreducible flat connection $A^{(0)}$. The origin of this factor is explained in §2.2 of [7]. This factor also appeared in the calculations of [8].

† The following oscillatory integral is not absolutely convergent. The result claimed arises if one includes a smooth convergence factor, such as $e^{-\epsilon x^2}$, and then takes the limit of $\epsilon \to 0$.

we get formally for the one loop approximation to the path integral

$$Z = \frac{1}{\#Z(G)} e^{ikI(A^{(0)})} \cdot \frac{\det \Delta_0 \cdot e^{-i\pi\eta(L_-)/4}}{\sqrt{|\det L_-|}}. \tag{2.19}$$

Here formally $\det \Delta_0 = \prod_i u_i$ and $|\det L_-| = \prod_i |v_i|$, while $\eta(L_-) = \sum_j \text{sign } v_j$ is the signature of the operator L_- (or more exactly of the quadratic form $(H, L_- H)$). The determinants can be conveniently regularized with the zeta function regularization of Ray and Singer [9]. For instance, one defines

$$\zeta(s) = \sum_i u_i{}^{-s} \tag{2.20}$$

(the sum converges for $\text{Re}\, s$ large enough, and the function is then shown to have a meromorphic continuation throughout the s plane) and $\det \Delta_0 = \exp(-\zeta'(0))$. A regularized version of the signature is similarly defined [6] by‡ introducing

$$\eta(L_-, s) = \sum_j \text{sign} v_j \cdot |v_j|^{-s}, \tag{2.21}$$

a sum which converges for $\text{Re}\, s$ large enough, and setting $\eta(L_-) = \eta(L_-, 0)$.

Now the question arises of whether the one loop expression (2.19) is a topological invariant, that is, of whether it is independent of the metric g that was used to define the gauge fixing. At this point we have to formulate what should be regarded as an affirmative answer. Since we have violated the topological invariance in the method of quantization, standard precepts of renormalization theory tell us that we should be prepared to add to the Lagrangian density local counterterms depending on the data used in the quantization (namely the connection A and the metric g) in order to recover topological invariance.§ In the case at hand,

‡ The following (standard) definition of η is larger by a factor of two from the definition used in [1].

§ Such counterterms might not exist, in which case the theory is anomalous. We will explain the absence of anomalies in Chern-Simons theory in §5.

the absolute value of the one loop integral is (apart from the elementary factor of $1/\#Z(G)$)

$$\tau^{1/2} = \frac{\det \Delta_0}{\sqrt{|\det L_-|}}, \tag{2.22}$$

and this is a topological invariant, the square root of the Ray-Singer analytic torsion [9], as was originally explained in the context of abelian Chern-Simons theory by A. Schwarz [10].

As for the η invariant that appears in the phase of the one loop integral, this is not a topological invariant. Rather, according to [6], its variation with respect to a change in A or g is given by a local formula (of a type whose derivation will be recalled in §4). We will use the notation $\eta(A)$ to denote the η invariant of the operator $L_- = (*D + D*)J$ coupled to an arbitrary connection A (not necessarily a flat connection such as was natural in arriving at (2.19)). If we assume that we are working on a trivial G bundle so that one has the trivial connection $A = 0$, the local formula, whose origin will be recalled in §4, implies

$$\eta(A) = \eta(0) - \frac{4h}{\pi} I(A) \bmod 2. \tag{2.23}$$

Here $h = c_2(\mathcal{G})/2$ is the dual Coxeter number (N for $SU(N)$). The term proportional to h shifts the effective value of k in (2.19) to $k+h$. This shift was mentioned in the introduction together with a heuristic explanation of what the analog should be for non-compact groups. The mod 2 term in (2.23), which has been carefully described in [8], arises because η jumps by 2 when an eigenvalue passes through 0, and contributes a fourth root of unity in (2.19).

We still have to understand the role of the $\eta(0)$ term in (2.23). For $A = 0$, L_- is the direct sum of $\dim G$ copies of the operator $D_- = (*d + d*)J$ acting on real-valued $1 \oplus 3$ forms. We set $\eta_{\rm grav} = \eta(D_-)$, so $\eta(0) = \dim G \cdot \eta_{\rm grav}$. $\eta_{\rm grav}$ is not a topological invariant, and it is at this point that we must introduce a counterterm of the sort anticipated by renormalization theory and that a framing anomaly appears. If ω is the Levi-Civita connection on the Riemannian manifold M, then the gravitational Chern-Simons functional is defined as

$$I(g) = \frac{1}{4\pi} \int\limits_M \operatorname{Tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \tag{2.24}$$

$I(g)$ is well-defined as a real-valued functional on a *framed* three manifold; if the framing is shifted by s units, one has $I(g) \to I(g) + 2\pi s$. The local variational formula says that

$$-\frac{1}{4}\eta_{\rm grav} + \frac{I(g)}{24\pi} \tag{2.25}$$

is a topological invariant. The counterterm that must be added to the Lagrangian to cancel the dependence on the metric is thus

$$\Delta L = \dim G \cdot \frac{-iI(g)}{24}, \tag{2.26}$$

and the dependence of this on the framing is the framing anomaly. In the introduction, we heuristically explained that for non-compact semi-simple groups, the analog of the $\dim G$ factor should be $\dim\mathcal{K} - \dim\mathcal{P}$, and we will verify this in §4.

3. Naive Considerations Involving Non-Compact Groups

So far none of this is new. We now want to consider how the discussion is modified if the gauge group G is a non-compact semi-simple group. One can consider the same gauge fixing term (2.5) leading to the same gauge fixed Lagrangian. The changes come because the hermitean metric (2.9) is no longer positive definite (as the quadratic form Tr is indefinite). As a result, the operators Δ_0 and L_- that appear in the gauge-fixed action are no longer self-adjoint operators in a Hilbert space. They are self-adjoint in the indefinite hermitean metric (2.9). One can introduce an unnatural positive metric in the problem (as we will actually do later), but in such a metric these operators are not self-adjoint.* Nonetheless, they are elliptic differential operators, and it is possible to make sense of eigenvalues and Green's functions for non-self-adjoint elliptic operators. So one might expect that (though it might be ugly) one could proceed with the standard analysis. Let us see what happens if one tries to do that.

Suppose we find an eigenfunction of Δ_0 or L_-, say

$$\Delta_0\psi = u\psi. \tag{3.1}$$

Then as Δ_0 is naturally a real operator, even for non-compact G, one has

$$\Delta_0\overline{\psi} = \overline{u}\overline{\psi}. \tag{3.2}$$

Then

$$\overline{u}(\psi,\psi) = (\Delta_0\psi,\psi) = (\psi,\Delta_0\psi) = u(\psi,\psi). \tag{3.3}$$

There are therefore two possibilities: either $u = \overline{u}$ and u is real (but not necessarily positive, as for compact groups), or ψ is a null vector, $(\psi,\psi) = 0$, and then u may be an arbitrary complex number. It is easy to see in simple examples that such null vectors with complex eigenvalues can indeed occur. It is not clear how they are supposed to be treated in the path integral.

Even if there are no null vectors (or if, like null vectors in conformal field theory, they can be thrown away), the analysis will diverge in the following crucial respect from the treatment of compact groups. The orthonormality conditions in (2.12) and (2.13) cannot be imposed, since the hermitean structure is indefinite

* One way to do the computation in this gauge would be to introduce a positive definite but not gauge invariant metric $\langle\ ,\ \rangle$, and replace L_- and Δ_0 by the self-adjoint operators $\widetilde{L}_- = (L_- + L_-{}^\dagger)/2$ and $\widetilde{\Delta}_0 = (\Delta_0 + \Delta_0{}^\dagger)/2$ ($\dagger$ is the adjoint with respect to $\langle\ ,\ \rangle$), which determine the same quadratic forms. This would avoid the ugly features that we are about to find, but the use of the more complicated operators $\widetilde{L}_-$ and $\widetilde{\Delta}_0$ would lead to different results from the naive extrapolation from compact gauge groups, as we will find in §4 by a different method that seems more elegant.

and $(\psi,\psi)<0$ for some (and in fact for infinitely many) ψ. Instead of (2.12) and (2.13), the best that one can do is

$$(\psi_i,\psi_j)=\delta_{ij}\epsilon_i, \qquad (\chi_i,\chi_j)=\delta_{ij}\rho_i, \tag{3.4}$$

with the ϵ_i and ρ_i all being ± 1. The analogue of (2.16) is hence

$$Z=\frac{1}{\#Z(G)}\sum_{A^{(0)}}e^{ikI(A^{(0)})}\int\prod_i\frac{dh_i}{\sqrt{2\pi}}\prod_j d\bar{c}_j dc_j \exp\left(-\frac{i}{2}\sum_i v_i\rho_i h_i^2+\sum_j u_j\epsilon_j\bar{c}_j c_j\right). \tag{3.5}$$

Considering the Bose path integral, define an operator $\rho:\Omega_-\to\Omega_-$ by $\rho\chi_i=\rho_i\chi_i$. The factors of ρ_i in (3.5) mean that the phase of the integral over B is not $e^{-i\pi\eta(L_-)/4}$ but $e^{-i\pi\eta(L_-\rho)/4}$. Thus, even if one can push through for the non-compact groups a direct analog of the APS formula for the variation of $\eta(L_-)$, this would not give the correct result in the Chern-Simons quantum field theory. What is wanted there is $\eta(L_-\rho)$, and this is quite a different kettle of fish. It is conceivable that the variation of $\eta(L_-)$ is governed by the same APS formula as for compact groups, after plugging in the right values of the Casimir operators. This is not the case for $\eta(L_-\rho)$ as one can see directly in the case that the flat connection $A^{(0)}$ about which we are expanding has a structure group that reduces to the maximal compact subgroup K of G. (In that case $L_-\rho$ coincides with an operator that we will meet and analyze in the next section.) Similarly, the factors of ϵ_i in (3.5) mean that the Fermi path integral gives not just $\det\Delta_0$. We will not try to analyze what it does give.

In conclusion, if in Chern-Simons gauge theory with non-compact gauge group one naively imitates the standard computation, new (and seemingly ugly) features appear which ensure that the physical results that emerge cannot be naively extrapolated from the case of compact groups. We have no reason to doubt that the correct results can be obtained by further calculations along these lines, but they will not be a simple imitation of the results in the compact case. Actually, Wodjicki [11] has carried out some of the analysis that would be needed to treat this gauge correctly. We will not go down that road, but instead in the next section we consider a procedure that makes the analysis standard.

4. Unitary Gauge Fixing For Non-Compact Groups

In this section we will describe another gauge which in some sense is manifestly unitary and avoids the pathologies described in the last subsection. This gauge condition is similar to the equations considered by Hitchin [12] in describing the classical moduli spaces of these theories (and is related to considerations involving index theory for non-unitary connections [13,14]). Hopefully, it will eventually be possible to understand the Hamiltonian formulation of Chern-Simons theory for non-compact groups in a way that will make contact both with Hitchin's results and with the perturbative treatment that we are about to describe.

We should also note that Schwarz's treatment of abelian Chern-Simons theory [10] was formulated broadly enough to include the type of gauge fixing that we will use, and therefore our discussion of the torsion below is not essentially new. The discussion of the η invariant and the application to non-abelian Chern-Simons theory are new.

Since a semi-simple group G is contractible to its maximal compact subgroup K, the structure group of any G bundle E over a three manifold M can be reduced to K. Such a reduction determines an endomorphism T of the adjoint bundle $\mathrm{ad}(E)$ which fiberwise is $+1$ on $\mathcal{K}$ and -1 on the orthocomplement $\mathcal{P}$. (Moreover, the space of such reductions is contractible, so later to discuss independence of the choice of T, we only need to consider infinitesimal variations.) Instead of (2.5), we take for the gauge fixing term

$$V = \frac{k}{2\pi} \int\limits_M d\mu \;\; \mathrm{Tr}\, \overline{c} * D^{(0)} * TB. \tag{4.1}$$

This leads to the quadratic part of the gauge fixed Lagrangian being

$$-ikI(A^{(0)}) + \frac{k}{2\pi}\int\limits_M \mathrm{Tr}\ \left(-\frac{i}{2}B\wedge D^{(0)}B - i\phi * D^{(0)}T * B + \bar{c} * D^{(0)}T * D^{(0)}c\right), \tag{4.2}$$

or equivalently

$$-ikI(A^{(0)}) - \frac{1}{2}\int\limits_M \mathrm{Tr}\ \left(iB'\wedge D^{(0)}B' + 2i\phi' * TD^{(0)} * TB'\right) + \int\limits_M \mathrm{Tr}\ \left(\bar{c}'D^0 * TD^{(0)}c'\right), \tag{4.3}$$

where $B' = B\sqrt{k/2\pi}$, $c' = c\sqrt{k/2\pi}$, $\phi' = T\phi\sqrt{k/2\pi}$, and $\bar{c}' = *\bar{c}\sqrt{k/\pi}$.

We can now continue with the discussion of §2, except that $\widehat{*} = *T$ appears in most of the formulas instead of $*$. As $\widehat{*}$ shares the key properties of $*$ (it is a differential operator of order zero, it maps q forms to $3-q$ forms, and $\widehat{*}^2 = 1$), the analysis required is very standard, though the results will not coincide with those for compact groups. The replacement of $*$ by $\widehat{*}$ in fact precisely isolates what is different in the theory with non-compact gauge group. The natural metric to use on the space Ω^q of adjoint-valued q-forms is not (2.9) but

$$(u,v) = -\int\limits_M \mathrm{Tr}\,\bar{u}\wedge\widehat{*}v = -\int\limits_M \mathrm{Tr}\,\bar{u}\wedge *Tv, \tag{4.4}$$

and *this metric is positive definite*, since the negative eigenvalues of T precisely compensate for the indefiniteness of the quadratic form $(a,b) = -\,\mathrm{Tr}\ ab$ on the Lie algebra of G. The Bose and Fermi kinetic operators are $\widehat{L}_- = (\widehat{*}D + D\widehat{*})J$ and $\widehat{\Delta}_0 = \widehat{*}D\widehat{*}D$. Combining the bosons to a $1\oplus 3$ form $H = (B',\phi')$, we have the analog of (2.10), namely

$$-ikI(A^{(0)}) + \frac{i}{2}(H,\widehat{L}_-H) - (\bar{c}',\widehat{\Delta}_0 c'). \tag{4.5}$$

We therefore get for the one loop approximation to the expansion of the path integral about $A^{(0)}$ the analog of (2.19):

$$Z = \sum_{A^{(0)}} \frac{1}{\#Z(G)} e^{ikI(A^{(0)})} \cdot \frac{\det \widehat{\Delta}_0 \cdot e^{-i\pi\eta(\widehat{L}_-)/4}}{\sqrt{|\det \widehat{L}_-|}}. \tag{4.6}$$

Since the operators $\widehat{\Delta}_0$ and $\widehat{L}_-$ are self-adjoint operators in a Hilbert space (with inner product defined in (4.4)) the analysis proceeds along standard lines, and there is no analog of the null vectors or the operator ρ encountered in §3. It remains to analyze the topological invariance of (4.6) and verify that the behavior agrees with what was predicted heuristically in §1. The absolute value of the contribution of a given flat connection is easy to dispose of. Apart from the elementary factor of $1/\#Z(G)$, it is

$$\tau^{1/2} = \frac{\det \widehat{\Delta}_0}{\sqrt{|\det \widehat{L}_-|}}. \tag{4.7}$$

We might call τ the analytic torsion of the non-unitary flat connection $A^{(0)}$, by analogy with [9]. Since $\widehat{*}$ shares all of the key properties of $*$, the proof of topological invariance of the analytic torsion in [9] can be copied line by line, to prove that τ is a topological invariant also in the non-compact case.

It is interesting to note that the definition of the purely combinatorial Reidemeister-Franz torsion is not limited to compact groups. (It applies to arbitrary flat bundles with "unimodular" structure group, a condition that holds for the adjoint bundles arising in the present discussion.) It is plausible that the analytic torsion as defined in (4.7) is equal to the combinatorial torsion, generalizing the conjecture of [9] (later proved by Cheeger and Müller [15,16]) to non-compact groups.

It remains to discuss $\eta(\widehat{L}_-)$ and to compare to the claims in the introduction. First we will dispose of the framing anomaly. In this discussion, we ignore possible jumps in η when an eigenvalue passes through zero; these shift the path integral by a fourth root of unity, and do not affect the framing anomaly.

The adjoint bundle $\mathrm{ad}(E)$ has a decomposition $\mathrm{ad}(E) = \mathrm{ad}(E)_0 \oplus \mathrm{ad}(E)_\perp$, where T acts as $+1$ and -1, respectively, on $\mathrm{ad}(E)_0$ and $\mathrm{ad}(E)_\perp$. $\mathrm{ad}(E)_0$ and $\mathrm{ad}(E)_\perp$ are vector bundles of dimensions dim $\mathcal{K}$ and dim $\mathcal{P}$, respectively. The local variational formula for η, whose origin will be recalled soon, shows that in three dimensions the variation of $\eta(\widehat{L}_-)$ with respect to the Riemannian metric of M is independent of the connection A. Therefore, we can consider the case in which the structure group of A reduces to the maximal compact subgroup K. In this case, the decomposition $\mathrm{ad}(E) = \mathrm{ad}(E)_0 \oplus \mathrm{ad}(E)_\perp$ is invariant under parallel transport by A. We denote as Q_0 and $Q_\perp$ the restrictions of $(*D + D*)J$ to $\mathrm{ad}(E)_0$ and $\mathrm{ad}(E)_\perp$. We have in view of the definition $\widehat{L}_- = (T * D + D * T)J$ the formula

$$\widehat{L}_- = Q_0 \oplus (-Q_\perp) \tag{4.8}$$

for this situation in which D and T commute. Hence

$$\eta(L_-) = \eta(Q_0) + \eta(-Q_\perp) = \eta(Q_0) - \eta(Q_\perp). \tag{4.9}$$

We recall that at the end of §2, we introduced the operator $D_- = (*d + d*)J$ acting on real-valued differential forms (not twisted by any vector bundle) and the corresponding eta invariant $\eta_{\mathrm{grav}} = \eta(D_-)$. If the bundle E is trivial, we can consider the case in which the connection A is zero, and then Q_0 and $Q_\perp$ are direct sums, respectively, of dim $\mathcal{K}$ or dim $\mathcal{P}$ copies of D_-. Even if E is not trivial, it follows from the local formula for the variation of η that as regards the dependence on the metric of the η invariant, Q_0 or $Q_\perp$ can be replaced by dim $\mathcal{K}$ or dim $\mathcal{P}$ copies of D_-.⋆ In view of (4.9), therefore,

$$-\eta(\widehat{L}_-) = (\mathrm{dim}\mathcal{K} - \mathrm{dim}\mathcal{P}) \cdot \eta_{\mathrm{grav}} + \dots \tag{4.10}$$

⋆ This follows from just from the existence of a local formula for $\delta\eta$ – irrespective of its details – together with dimensional analysis. Indeed, the existence of a local formula for the gravitational variation of η means $\delta\eta = \int_M d\mu \;\; \delta g^{ij} N_{ij}$, where N_{ij} is a locally constructed tensor of scaling dimension three. For G a semi-simple group, there are no such tensors with a non-trivial dependence on A. Hence, the gravitational variation of η is independent of A, and, since it can be computed locally by a universal formula, the formula is the same as if the bundle were trivial and one can set $A = 0$.

where ... are terms independent of the metric. This gives us then the generalization of (2.26): the metric dependent counterterm needed to achieve metric independence of the path integral is

$$\Delta L = (\dim \mathcal{K} - \dim \mathcal{P}) \cdot \frac{-iI(g)}{24}, \tag{4.11}$$

and this gives us the coefficient of the framing anomaly in the large k limit, in accord with the heuristic claim in the introduction.

We now wish to analyze the A dependence of $\eta(\widehat{L}_-)$. This will be more involved than what we have said until this point, since we will have to work out the local formula for the variation of η in a not quite standard situation.

First, following [17] and more rigorously [18], we recall how local formulas for the variation of η are obtained. One starts with the fact that for real non-zero x,

$$x^{-s}\mathrm{sign}\, x = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty dy\ y^s\, x e^{-x^2y^2}. \tag{4.12}$$

Hence

$$\eta(L_-, s) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty dy\ y^s\ \mathrm{Tr}\left(L_- e^{-y^2 L_-{}^2}\right). \tag{4.13}$$

In favorable cases, including the three dimensional cases we are considering, the integral in (4.13) converges down to $\mathrm{Re}\, s = 0$, and one can simply set

$$\eta(L_-) = \frac{2}{\sqrt{\pi}} \lim_{s\to 0} \int_0^\infty dy\ y^s\ \ \mathrm{Tr}\left(L_- e^{-y^2 L_-{}^2}\right). \tag{4.14}$$

Under an arbitrary variation of A, g, and T, we therefore have

$$\begin{aligned}\delta\eta(L_-) =& \frac{2}{\sqrt{\pi}} \lim_{s\to 0} \int_0^\infty dy\ y^s\ \ \mathrm{Tr}\left((\delta L_- - 2y^2 \delta L_- \cdot L_-{}^2) e^{-y^2 L_-{}^2}\right) \\ =& \frac{2}{\sqrt{\pi}} \lim_{s\to 0} \int_0^\infty dy\ y^s\ \frac{d}{dy}\left(y\, \mathrm{Tr}\, \delta L_- e^{-y^2 L_-{}^2}\right).\end{aligned} \tag{4.15}$$

Integrating by parts, taking $s \to 0$, and picking up the boundary term at $y = 0$, we get

$$\delta\eta(L_-) = -\frac{2}{\sqrt{\pi}} \lim_{y \to 0} \left(y \,\mathrm{Tr}\, \delta L_- e^{-y^2 {L_-}^2} \right), \tag{4.16}$$

if, as in this case, the limit exists. (The more general story, as explained in [18], involves picking out the coefficient of y^{-1} in the small y expansion of $\mathrm{Tr}\, e^{-y^2 {L_-}^2}$.)

To proceed further, we need some calculations. For local coordinates x^i on M, we denote the operator $a \to dx^i \wedge a$ on differential forms as ψ^i. We denote the opposite operation of contraction with the dual vector field as χ^i. Thus

$$*\psi^i * = (-1)^F \chi^i, \quad *\chi^i * = \psi^i (-1)^F \tag{4.17}$$

and

$$\{\psi^i, \psi^j\} = \{\chi^i, \chi^j\} = 0, \quad \{\psi^i, \chi^j\} = g^{ij}. \tag{4.18}$$

Also,

$$D = \sum_i \psi^i D_i, \quad *D* = \sum_i (-1)^F \chi^j D_j. \tag{4.19}$$

One computes

$$D \widehat{*} J \widehat{*} D J = -\frac{1}{2} \psi^i \psi^j [D_i, D_j] (-1)^F, \tag{4.20}$$

and similarly, if we set $\overline{D} = TDT$, then

$$\widehat{*} D J D \widehat{*} J = -\frac{1}{2} \chi^i \chi^j [\overline{D}_i, \overline{D}_j] (-1)^F. \tag{4.21}$$

One likewise computes

$$D \widehat{*} J D \widehat{*} J + \widehat{*} D J \widehat{*} D J = -\frac{1}{2} (\overline{D}_i D^i + D_i \overline{D}^i) - \frac{1}{2} (\chi^i \psi^j - \psi^j \chi^i) [\overline{D}_i, D_j]. \tag{4.22}$$

Hence

$${L_-}^2 = -(\Delta + X), \tag{4.23}$$

with

$$\Delta = \frac{1}{2}(D_i\overline{D}^i + \overline{D}_i D^i) \tag{4.24}$$

and

$$X = \frac{1}{2}\left(\psi^i\psi^j[D_i, D_j](-1)^F + \chi^i\chi^j[\overline{D}_i, \overline{D}_j](-1)^F + (\chi^i\psi^j - \psi^j\chi^i)[\overline{D}_i, D_j]\right). \tag{4.25}$$

If one expands

$$e^{y^2(\Delta+X)} = e^{y^2\Delta} + \int_0^{y^2} ds\ e^{s\Delta}\ Xe^{(y^2-s)\Delta} + \dots, \tag{4.26}$$

then the only term that survives when substituted in (4.16) is the second term. (Other terms vanish simply upon taking the limit of $y \to 0$ or upon evaluating the trace in (4.16).) Moreover, we can manipulate the second term in (4.26) as if Δ and X commuted, since terms resulting from their failure to commute are of higher order in y. Thus inside (4.16) we can make the substitution

$$e^{y^2(\Delta+X)} \to y^2 X e^{y^2\Delta}. \tag{4.27}$$

As three powers of y are now visible (one in (4.16) and two more in (4.27)), to evaluate the trace in (4.16), we need only terms in the diagonal matrix elements of $e^{y^2\Delta}$ that are at least as singular as y^{-3}.

The leading small y behavior is the standard heat kernel, with precisely such a behavior, $e^{y^2\Delta} \sim (4\pi)^{-3/2}y^{-3}$. Putting the pieces together, the powers of y cancel out in (4.16), and we get

$$\delta\eta = -\frac{1}{4\pi^2}\,\mathrm{Tr}\,(\delta L_- \cdot X)\,. \tag{4.28}$$

This formula can be used to study the variation of η under a change in the metric (in which case one would recover standard formulas and justify some assertions made above) or the gauge fields. We will consider only the latter case. That being so, we can take $\delta\widehat{*}=0$, so

$$\delta\eta=-\frac{1}{4\pi^2}\,\mathrm{Tr}\,((\widehat{*}\delta D+\delta D\widehat{*})J\cdot X)\,. \tag{4.29}$$

Now in (4.29) and all our previous formulas, the trace of course is taken in the space Ω_- of differential forms of odd order, since this is the space in which L_- is defined. However, using the fact that $\widehat{*}$ maps $\Omega_\pm$ to $\Omega_\mp$ and commutes with J and X, (4.29) is equivalent to

$$\delta\eta=-\frac{1}{4\pi^2}\int\limits_M \widehat{\mathrm{Tr}}\,(\widehat{*}\delta DJ\cdot X)\,, \tag{4.30}$$

with $\widehat{\mathrm{Tr}}$ being the trace in the full de Rham complex $\Omega^*=\Omega_+\oplus\Omega_-$.

To evaluate the trace explicitly, note that $\delta D=\psi^i\delta A_i$, and that if each γ^i is ψ^i or χ^i, then

$$\widehat{\mathrm{Tr}}*\psi^iJ\gamma^j\gamma^k=\epsilon^{ijk}, \tag{4.31}$$

with ϵ^{ijk} being the Levi-Civita antisymmetric tensor. Using the explicit formula (4.25) for X and computing the trace, we get

$$\delta\eta=-\frac{1}{8\pi^2}\int\limits_M \epsilon^{ijk}\,\mathrm{Tr}\,T\delta A_i[D_j+\overline{D}_j,D_k+\overline{D}_k]. \tag{4.32}$$

If $A=a+b$ where a is even under T and b is odd, this is equivalent to

$$\delta\eta=-\frac{1}{\pi^2}\int\limits_M \mathrm{Tr}\,T(\delta a\wedge(da+a\wedge a)). \tag{4.33}$$

This formula can be integrated, to show that the A dependent part $\widetilde{\eta}$ of η (modulo the usual mod 2 jumps) is

$$\widetilde{\eta} = -\frac{1}{2\pi^2} \int\limits_M \mathrm{Tr}_{\mathcal{G}}\, T \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right). \tag{4.34}$$

Since a is $\mathcal{K}$-valued, and so commutes with T, and since T is $+1$ on $\mathcal{K}$ and -1 on $\mathcal{P}$, (4.34) is equivalent to

$$\widetilde{\eta} = -\frac{1}{2\pi^2} \int\limits_M \left((\mathrm{Tr}_{\mathcal{K}} - \mathrm{Tr}_{\mathcal{P}})(a \wedge da + \frac{2}{3} a \wedge a \wedge a) \right). \tag{4.35}$$

The definition of the quadratic Casimir operator in §1 was that for any representation $\mathcal{R}$, $\mathrm{Tr}_{\mathcal{R}} = c_2(\mathcal{R}) \cdot \mathrm{Tr}$ (where Tr is the fundamental trace defined in the footnote before (1.1)), so

$$\widetilde{\eta} = -\frac{1}{2\pi^2} \left(c_2(\mathcal{K}) - c_2(\mathcal{P}) \right) \cdot \int\limits_M \mathrm{Tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right). \tag{4.36}$$

This contribution to the effective action is thus

$$-\frac{i\pi\widetilde{\eta}}{4} = \frac{i(c_2(\mathcal{K}) - c_2(\mathcal{P}))}{8\pi} \int\limits_M \mathrm{Tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right) = i \left(\frac{c_2(\mathcal{K}) - c_2(\mathcal{P})}{2} \right) \cdot I(a) \tag{4.37}$$

(where of course I is the Chern-Simons functional (1.1)).

If one ignores the difference between a and A, (4.37) corresponds to a shift in the effective value of k from k to $k + (c_2(\mathcal{K}) - c_2(\mathcal{P}))/2$, as was anticipated heuristically in the introduction. We are not entitled to ignore this difference, however, and on the contrary at first sight it is disturbing that (4.37) is not gauge invariant. At this point, however, we must recall that although the basic quantum field theory (1.2) is gauge invariant, we have quantized it in a gauge that depends on the choice of g and T. General tenets of renormalization theory indicate, therefore, that in the absence of a favorable regularization, to restore the g and T independence, we must be prepared to add to the Lagrangian local counterterms that depend on A, g, and T. We have indeed already used this freedom to eliminate the g dependence of the one loop effective action, and now we must do the same to eliminate the T dependence. The local counterterm that is needed in this case is

$$\Delta L' = \frac{c_2(\mathcal{K}) - c_2(\mathcal{P})}{8c_2(\mathcal{G})} \int_M \mathrm{Tr}_{\mathcal{G}}[DT, T] \wedge F. \tag{4.38}$$

($F = dA + A \wedge A$ is the curvature.) Indeed one has

$$-\frac{\pi\widetilde{\eta}}{4} + \Delta L' = \frac{1}{2}(c_2(\mathcal{K}) - c_2(\mathcal{P})) \cdot I(A), \tag{4.39}$$

which is the desired gauge invariant result.

5. Absence Of Anomalies

Both in §2 and in §4 we found it necessary to add certain counterterms to compensate for the lack of manifest topological invariance of the gauge fixed quantum field theory. This discussion may have appeared rather *ad hoc*, and the reader may have wondered whether unpleasant surprises would arise in higher orders of perturbation theory. We will now briefly sketch the general situation. (In doing so, we will for simplicity consider the case of compact gauge group in which setting up the quantum theory involves only the introduction of a metric g. Consideration of the endomorphism T that appears in the non-compact case would lead to nothing essentially new.)

Entirely apart from gauge fixing, the Chern-Simons quantum field theory needs to be regularized. Except for certain difficulties with one loop diagrams, the discussion of which we will postpone, the regularization can be accomplished by adding to the Lagrangian a local interaction of higher dimension, for instance the conventional Yang-Mills Lagrangian

$$\Delta_1 L = -\frac{\epsilon}{4}\int\limits_M \sqrt{g} g^{ij} g^{kl} \operatorname{Tr} F_{ik} F_{jl}. \tag{5.1}$$

(It might be useful to add terms of still higher dimension proportional to higher powers of ϵ, but we will use (5.1) for clarity.) This preserves gauge invariance, and hence preserves the BRST invariance of the gauge fixed theory, but because of the metric dependence it spoils the topological invariance. The regularized and gauge fixed Lagrangian is

$$L'' = L + L_{gf} + \Delta_1 L, \tag{5.2}$$

where L is the classical Lagrangian, $L_{gf} = -\delta V$ is the gauge fixing term, and $\Delta_1 L$ is the regulator term. The stress tensor is

$$T_{ij} = \frac{\delta L''}{\delta g^{ij}}. \tag{5.3}$$

Here $\delta L/\delta g^{ij} = 0$ because of topological invariance of the classical Lagrangian. And as $L_{gf} = -\{Q, V\}$ is a BRST commutator, $\delta L_{gf}/\delta g^{ij}$ is also a BRST commutator. Indeed, $\delta L_{gf}/\delta g^{ij} = \{Q, V_{ij}\}$ with $V_{ij} = \delta V/\delta g^{ij}$. So

$$T_{ij} = \{Q, V_{ij}\} - \epsilon W_{ij}, \tag{5.4}$$

where $W_{ij} = \frac{1}{2}\operatorname{Tr}\left(F_{is}F_{jt}g^{jt} - \frac{1}{4}g_{ij}F_{st}F^{st}\right)$ is the conventional Yang-Mills stress tensor.

Let

$$Z = \int DA\ D\phi\ D\bar{c}\ Dc\ \ e^{-L''} \tag{5.5}$$

be the partition function of the quantum field theory, on some three manifold M. First of all, we must discuss whether Z is finite in the limit as $\epsilon \to 0$. On general grounds, divergences in the effective action are of the form $\int M d\mu \mathcal{O}$, where

$\mathcal{O}$ is some local counterterm, that is, some gauge invariant local functional of the data that is of dimension three or less. There is no such functional that can be formed from the connection A alone. Note in particular that the original classical Lagrangian L of equation (1.2) is itself not the integral of a gauge invariant local functional and so cannot appear as a counterterm. This is the essence of why the theory is finite. More generally, one can verify that every BRST invariant gauge invariant local functional of dimension three or less that has a non-trivial dependence on the integration variables A, ϕ, c, and $\bar{c}$ in (5.5) is a BRST commutator. It may be necessary to add such terms to the Lagrangian, with ϵ dependent coefficients, to make the effective action finite. As these terms are BRST commutators, they will not affect the values of gauge invariant observeables. Finally, and crucially, there may be counterterms that do not depend on the integration variables but on the metric only. By dimensional analysis, these must be of the form

$$-\Delta_2 L = \frac{A}{\epsilon^3} \int_M d\mu + \frac{B}{\epsilon} \int_M d\mu R \tag{5.6}$$

for some constants A and B. Upon adding $\Delta_2 L$ to the Lagrangian, and replacing L'' by

$$L''' = L'' + \Delta_2 L, \tag{5.7}$$

one eliminates the infinities and ensures that the partition functions (and the values of gauge invariant observables) are finite.

It remains to consider the question of metric dependence. The metric dependence of Z is

$$\frac{\delta \ln Z}{\delta g^{ij}} = -\langle T_{ij} \rangle, \tag{5.8}$$

where $\langle \quad \rangle$ denotes the expectation value in the ensemble (5.5). So

$$\frac{\delta \ln Z}{\delta g^{ij}} = \langle \{Q, V_{ij}\} \rangle + \epsilon \langle W_{ij} \rangle. \tag{5.9}$$

The first term vanishes because of BRST invariance, so we are left to examine $\epsilon \langle W_{ij} \rangle$.

We only expect to recover metric independence in the limit of $\epsilon \to 0$ (which exists after the counterterms $\Delta_2 L$ have been included), so we must consider the behavior of $\epsilon\langle W_{ij}\rangle$ for $\epsilon \to 0$. Naively this vanishes, but in fact it might happen that matrix elements of W_{ij} diverge as $\epsilon \to 0$. On general grounds, such divergences are proportional to the matrix elements of local, BRST invariant operators of dimension less than the dimension of W_{ij}; this means dimension three or less, as W_{ij} has dimension four. Just as in the discussion of the finiteness of the theory, these conditions mean that (apart from possible BRST commutator terms, whose contributions in (5.9) will vanish) the dangerous part of W_{ij} will be a local functional of the metric only. If we bear in mind that Bose statistics implies that

$$\frac{\delta\langle W_{ij}(x)\rangle}{\delta g^{kl}(y)} = \frac{\delta\langle W_{kl}(y)\rangle}{\delta g^{ij}(x)} \tag{5.10}$$

(this is the statement that the two point function $\langle T_{ij}(x)T_{kl}(y)\rangle$ is symmetric) then the possibilities are

$$\epsilon\langle W_{ij}\rangle = \alpha\epsilon^{-3}g_{ij}+\beta\epsilon^{-1}(R_{ij}-\frac{1}{2}g_{ij}R)+\gamma(\epsilon_{ist}g^{su}g^{tv}D_uR_{vj}+\epsilon_{jst}g^{su}g^{tv}D_uR_{vi})+O(\epsilon). \tag{5.11}$$

This, then, is the general form of $\delta \ln Z/\delta g^{ij}$.

If we do find metric dependence of this form, can it be eliminated? To eliminate it, one must find a locally defined functional Λ of the metric such that

$$\epsilon\langle W_{ij}\rangle = \frac{\delta\Lambda}{\delta g^{ij}} + O(\epsilon). \tag{5.12}$$

Then, upon replacing L'' by $L''+\Lambda$ in the definition (5.5) of the partition function, we finally obtain the right definition of the topological partition function.

Does Λ exist? Not quite. The counterterms needed to eliminate the α and β terms in (5.11) are the ones that have already been considered in (5.8). This is no accident; as the α and β terms in (5.8) are proportional to negative powers of ϵ, they arose in the discussion of the finiteness of the theory. As for the γ term in

(5.12), it is not the variation of any locally defined functional of the metric. It is, however, up to a constant factor, the variation of the gravitational Chern-Simons functional of equation ((2.24)), whose definition requires a choice of framing of M. The γ term, therefore, is the origin of the framing anomaly. Once a framing of M is picked, an appropriate counterterm Λ that removes the metric dependence can be picked, with the coefficient of the gravitational Chern-Simons functional (and hence the framing dependence) being proportional to γ.

This, then, shows that on general grounds the framing anomaly is the only obstruction to defining $Z(M)$ as a topological invariant. The argument leaves open the possibility (which is known from the exact solution of the theory [1] to be realized) that γ receives contributions from all orders of perturbation theory. In this paper, we have essentially computed the one loop contribution to γ, for not necessarily compact groups.

The reader may note that (5.10) means that $\langle W_{ij} \rangle$ can be interpreted as a closed one form on the space of metrics. The local terms in $\epsilon\langle W_{ij} \rangle$ that survive for $\epsilon \to 0$ are closed one forms on the space of metrics that can be defined locally. The existence of a Λ obeying (5.12) would mean that these closed one forms are exact (in the space of local functionals). The framing anomaly is a cohomological obstruction to this; it comes from the existence of a closed one form Θ on the space of metrics which is defined locally but cannot be written as $\Theta = d\Lambda$ where Λ is the integral of a locally defined functional.

Finally, we must acknowledge a technical difficulty in this argument. The type of regularization in (5.1) renders all Feynman diagrams convergent except for one loop diagrams and one loop subdiagrams of more complicated diagrams. (This type of regularization does not improve one loop diagrams because the vertices are worsened to the same extent that the propagators are improved. Diagrams of more than one loop have more propagators than vertices.) On the other hand, the one loop diagrams can be conveniently treated with the zeta function regularization that we have used in this paper (which leads to precisely the framing anomaly predicted by the above argument). The problem arises because one loop diagrams can appear as subdiagrams of more complicated diagrams, and there is no evident way to use (5.1) to regularize a high order diagram while using zeta function regularization to treat the one loop subdiagrams. This difficulty has analogs in renormalization of other quantum field theories, and in other cases leads to only technical complications.

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